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Infinite hyper-regulus Sperner spaces

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Abstract. New constructions of infinite hyper-reguli are given, which produce a variety of new translation planes and Sperner spaces.

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1 Introduction

A hyper-regulus net of degree $(q^n-1)/(q-1)$ and order q^n is a net arising from a partial spread $\mathcal{H}$ in V_{2n}^q of $(q^n-1)/(q-1)$ components, which admits a proper net replacement. The partial spread $\mathcal{H}^*$ corresponding to the net replacement is then also a hyper-regulus as is $\mathcal{H}$. In a series of recent articles, the authors have constructed new hyper-reguli and a variety of sets of hyper-reguli that lead to new translation planes of order q^n (see Jha and Johnson [1, 2, 3]).

In this article, we consider infinite hyper-reguli and show that constructions similar to those in the finite case may be carried out to produce a wide range of new infinite translation planes and Sperner spaces.

2 Infinite hyper-reguli

The ideas of the constructions in both the finite and infinite cases are as follows: Begin with a given Pappian spread Σ coordinatized by a field L and let P denote the prime field of L. Realize the underlying vector space $L \oplus L$ and consider this vector space as a P-space. Determine a P-space T of $L \oplus L$ P-isomorphic to L as a P-space and such that whenever any two components M and N of Σ non-trivially intersect T then the intersections $M \cap T$ and $N \cap T$ are isomorphic P-subspaces. Let K denote the maximum subfield J of L such that T is an J-space. Then since each component M and N of Σ are L-spaces,

this makes $M \cap T$ a K-space. Moreover, if for a subfield S of L, $M \cap T$ is an S-space then T must be an S-space since M is an L-space, forcing S to be a subfield of K. Now take the image of T under the kernel homology group of the Pappian affine plane with spread Σ. The image set will cover M if $M \cap T$ is non-trivial. In this way, a partial spread of components of Σ is covered by images of T under the kernel homology group. A 'switching' of partial spreads then constructs either a new translation plane or a new Sperner space.

We begin with a general definition of a hyper-regulus.

1 Definition. Let V be a vector space over a prime field P. A P-partial spread of $V \oplus V$ is a set of mutually disjoint subspaces P-isomorphic to V. Given a P-partial spread $\mathcal{P}$, with the following properties:

(i) There is another P-partial spread $\mathcal{P}^*$ such that

  (a) each element of $\mathcal{P}^*$ intersects each element of $\mathcal{P}$ in isomorphic P-subspaces and

  (b) $\bigcup \mathcal{P} = \bigcup \mathcal{P}^*$.

(ii) The non-zero vectors of $\mathcal{P}^*$ cover the non-zero vectors of $\mathcal{P}$.

Then then $\mathcal{P}$ is said to be a 'hyper-regulus'.

Note that then $\mathcal{P}^*$ also becomes a hyper-regulus.

Let K denote a maximum field containing P such that elements of both partial spreads $\mathcal{P}$ and $\mathcal{P}^*$ are all K-spaces. Then K shall be called the 'kernel' of the pair of partial spreads.

Of course, reguli in $PG(3, Z)$, for any field Z produce hyper-reguli in the corresponding vector space.

2 Definition. We recall that a 'spread' of $L \oplus L$ considered as a vector space over its prime field P is a set of subspaces each of which is P-isomorphic to L and whose union covers the vector space. We note that any spread produces a translation 'Sperner' space by taking translates of the spread elements ('components') as lines.

If a spread has the property that any two distinct subspaces generate $L \oplus L$, the spread is a said to be a 'congruence partition', and a 'translation plane' is then obtained.

In the following, we will construct either Sperner spaces or translation planes by the replacement of certain subspaces.

3 Definition. Let L be any field, let σ be any non-trivial automorphism of L and assume that $K = \mathrm{Fix}\sigma$. Let π_Σ denote the Pappian translation plane determined by the L-spread Σ (a L-partial spread that covers $L \oplus L$),

$$x = 0, y = xm;\ m \in L\,.$$

(1) If $L \neq L^{\sigma-1}$, we may define the 'André' hyper-reguli as follows: Let A_k be defined as

$$A_k = \left\{ y = xmk; m \in L^{\sigma-1} \backslash \{0\} \right\}, k \text{ a fixed non-zero element of } L\,.$$

It is clear that A_k is a L-partial spread. Let

$$A_k^{\sigma^i} = \left\{ y = x^{\sigma^i} mk; m \in L^{\sigma-1} \backslash \{0\} \right\}\,.$$

If $L^{\sigma-1} = L^{\sigma^i-1}$, then $y = x^{\sigma^i} nk$ and $y = xmk$ has a non-zero solution if and only if

$$x^{\sigma^i-1} = mn^{-1}\,,$$

which is valid since $m = t^{\sigma-1}$ and $n^{-1} = s^{\sigma-1}$.

(2) The partial spread $A_k^{\sigma^i}$ is said to be an 'André replacement' for A_k.

4 Remark. Both A_k and $A_k^{\sigma^i}$ are hyper-reguli with kernel $K^+ = Fix\sigma^i$ and all non-trivial intersections are 1-dimensional K^+-subspaces.

Proof. Let $K^+ = Fix\sigma^i$. Note that $y = x^{\sigma^i} k \cap y = xmk$ for $m \in L^{\sigma-1}$ is a 1-dimensional K^+-subspace. Hence, all non-trivial intersections with $y = x^{\sigma^i} k$ with components of Σ are isomorphic K^+-subspaces. Note that $K \subseteq K^+$, but it is possible that $K \neq K^+$. The remaining elements of the partial spread are images of $y = x^{\sigma^i} k$ under the kernel homology group with elements

$$(x, y) \longmapsto (dx, dy), 0 \neq d \in L.$$

These images are $y = x^{\sigma^i} kd^{1-\sigma^i}$ and since $L^{\sigma-1} = L^{\sigma^i-1}$, the proof follows immediately. QED

The importance of hyper-reguli is suggested by the following construction producing translation planes or translation Sperner spaces.

5 Theorem. *Let Σ be the Pappian spread coordinatized by a field L, let σ be an automorphism of L and let S be a set of André hyper-reguli A_k for $k \in \lambda$, corresponding to σ. Let S^* be any set of corresponding sets $A_k^{\sigma^{i_k}}$, where σ^{i_k} depends only on k, for $k \in \lambda$, and where for each σ^{i_k}, $L^{\sigma-1} = L^{\sigma^{i_k}-1} \neq L$.*

(1) *Then*

$$x = 0, y = x^{\sigma^{i_k}} mk,\ y = xn; m \in L^{\sigma-1}, k \in \lambda,\ n \in L - L^{\sigma-1}$$

is a spread of $L \oplus L$ with kernel $\cap Fix\sigma^{i_k}$.

(2) *This spread defines a translation plane (i.e., is a 'congruence partition') if and only if the following define bijective functions on L:*

$$x \longmapsto x^{\sigma^{i_k}} mk - xn,\ \textit{for each } m \in L^{\sigma-1} \backslash \{0\}, n \in L - L^{\sigma-1}, k \in \lambda,$$

$$x \longmapsto x^{\sigma^{i_k}} mk - x^{\sigma^{i_t}} nt,\ \textit{for each } m \in L^{\sigma-1} \backslash \{0\}, t \neq k; t, k \in \lambda\,.$$

If any one of these functions is not surjective the spread defines a translation Sperner space which is not a translation plane.

PROOF. In order to obtain a translation plane, we need only show that we obtain a net; that distinct lines from distinct parallel classes intersect. This is equivalent to having the direct sum of any two of the subspaces under consideration sum to $L \oplus L$. The conditions given are equivalent to this sum requirement. For example consider two components of the spread $y = x^{\sigma^{i_k}} mk$ and $y = xn$ and consider any translate $y = xn + b$ of $y = xn$. Then $y = x^{\sigma^{i_k}} mk$ intersects $y = xn + b$ in a unique point, if and only if $x \longmapsto x^{\sigma^{i_k}} mk - xn$ defines a bijective function on L. This is equivalent to

$$\left(y = x^{\sigma^{i_k}} mk\right) \oplus (y = xn) = L \oplus L\,.$$

QED

It has been an open question if there are any hyper-reguli that are not André hyper-reguli. One might think that any derivable net which is not a regulus net would not necessarily be an André net. Indeed, it is possible to find infinite derivable nets which are not coordinatized by fields so these would be hyper-reguli which are not André. In general, it is not difficult to show that any regulus can be given the form of an André regulus and by the work of the second author (see Johnson [5]), any derivable net may be coordinatized by a skew field and hence any finite derivable net then becomes an André hyper-regulus. It is actually possible to find unusual replacements for finite André nets such that the replacements are not in themselves André (see Johnson [4]). But the major problem then is whether there are non-André hyper-reguli that lie within a Pappian spread.

Furthermore, the authors have constructed various new classes of translation planes of order q^n and kernel $GF(q)$ by constructing sets of mutually disjoint hyper-regulus nets in Desarguesian affine planes. When $n > 3$, none of these hyper-reguli can be André hyper-reguli, and they lead to translation planes that admit very few central collineations. The question then is whether there are any similar constructions in the infinite case.

In this note, we show that a very general construction is valid for arbitrary fields. We obtain a variety of new hyper-reguli from any automorphism σ of a field L. If $L^{\sigma-1} \neq L$ and σ does not have order 2 or 3, then any such hyper-regulus is never an André hyper-regulus using any automorphism of L. Hence, we obtain a tremendous number of new translation planes or translation Sperner spaces.

6 Theorem. *Let L be a field and let σ denote a non-trivial automorphism, such that $L^{\sigma-1} \neq L$, and let $K = Fix\sigma$. Let Σ denote the Pappian spread*

$$\{x = 0, y = xm; m \in L\} .$$

Let $b \in L - L^{\sigma^{-1}}$ *and let*

$$\mathcal{H}^* = \left\{ y = x^\sigma d^{1-\sigma} + x^{\sigma^{-1}} d^{1-\sigma^{-1}} b; d \in L \backslash \{0\} \right\} .$$

Then $\mathcal{H}^*$ *is a hyper-regulus and the set of non-trivial intersections of* $\mathcal{H}^*$ *with* Σ *is a hyper-regulus* $\mathcal{H}$ *of the Pappian spread* Σ.

Proof. Consider the subspace $y = x^\sigma + x^{\sigma^{-1}} b$ over K. We note that the dimension over K of $y = x^\sigma + x^{\sigma^{-1}} b$ is the dimension of $y = x$ over K, the dimension of L over K. Let B be a basis for L over K. Then it follows directly that $\left\{ e, e^\sigma + e^{\sigma^{-1}} b; e \in B \right\}$ is a basis for $y = x^\sigma + x^{\sigma^{-1}} b$. That is, if $x^\sigma + x^{\sigma^{-1}} b = 0$ and x is not zero, then $b = x^{\sigma - \sigma^{-1}} = x^{\sigma^{-1}(\sigma^2 - 1)} = (x^{\sigma^{-1}(\sigma+1)})^{\sigma-1}$, contrary to our assumptions. Now consider the kernel homology group with elements $(x, y) \longmapsto (xd, yd)$, for $d \in L \backslash \{0\}$. Then $y = x^\sigma + x^{\sigma^{-1}} b$ maps to

$$y = x^\sigma d^{1-\sigma} + x^{\sigma^{-1}} d^{1-\sigma^{-1}} b.$$

Now suppose we could show that these subspaces were mutually disjoint. In this case, we would obtain a partial spread. Consider a non-trivial intersection of $y = x^\sigma + x^{\sigma^{-1}} b$ and $y = xm$. Clearly, the maximum subfield of L that fixes $y = x^\sigma + x^{\sigma^{-1}} b$ is K. If

$$x_i^\sigma + x_i^{\sigma^{-1}} b = x_i m, \text{ for } i = 1, 2$$

then

$$x_1^{\sigma-1} + x_1^{\sigma^{-1}-1} b = x_2^{\sigma-1} + x_2^{\sigma^{-1}-1} b,$$

which implies that

$$\left(x_1^{\sigma-1} - x_2^{\sigma-1}\right) = \left(x_2^{\sigma^{-1}-1} - x_1^{\sigma^{-1}-1}\right) b .$$

Note that

$$\left(x_2^{\sigma^{-1}-1} - x_1^{\sigma^{-1}-1}\right)^\sigma = \frac{\left(x_1^{\sigma-1} - x_2^{\sigma-1}\right)}{(x_1 x_2)^{\sigma-1}}$$

Hence we obtain

$$\left(x_2^{\sigma^{-1}-1} - x_1^{\sigma^{-1}-1}\right)^\sigma (x_1 x_2)^{\sigma-1} = \left(x_2^{\sigma^{-1}-1} - x_1^{\sigma^{-1}-1}\right) b.$$

Since $b \in L - L^{\sigma^{-1}}$, it can only be that $x_1^{\sigma-1} = x_2^{\sigma-1}$, so that $\left(x_1 x_2^{-1}\right)^\sigma = \left(x_1 x_2^{-1}\right)$, implying that $x_1 x_2^{-1} \in K$. Hence, any non-trivial intersection is necessarily of dimension 1 over K.

Then if the original subspace $y = x^{\sigma} + x^{\sigma^{-1}}b$ intersects $y = xm$; $m \in \lambda$, it follows that each such component $y = xm$ is completely covered by mutually disjoint K-subspaces, each of which intersects each $y = xm$ in a 1-dimensional K-subspace.

Observe that

$$x^{\sigma}d^{1-\sigma} + x^{\sigma^{-1}}d^{1-\sigma^{-1}}b = x^{\sigma} + x^{\sigma^{-1}}b$$

for x non-zero, provided that

$$x^{\sigma-\sigma^{-1}}\left(d^{1-\sigma} - 1\right) = b\left(1 - d^{1-\sigma^{-1}}\right).$$

If $d^{1-\sigma} = 1$ then we have the same subspace. Hence, assume that $d^{1-\sigma} \neq 1$, implying that

$$b = x^{\sigma^{-1}(\sigma+1)(\sigma-1)}\frac{\left(d^{1-\sigma} - 1\right)}{\left(1 - d^{1-\sigma^{-1}}\right)} = x^{\sigma^{-1}(\sigma+1)(\sigma-1)}\frac{\left(d^{1-\sigma} - 1\right)}{d^{1-\sigma^{-1}}\left(d^{\sigma^{-1}-1} - 1\right)}.$$

Note that

$$\left(d^{\sigma^{-1}-1} - 1\right)^{\sigma} = d^{1-\sigma} - 1\,.$$

Hence,

$$b = x^{\sigma^{-1}(\sigma+1)(\sigma-1)}\frac{\left(d^{\sigma^{-1}-1} - 1\right)^{\sigma-1}}{d^{1-\sigma^{-1}}}.$$

Now $d^{1-\sigma^{-1}} = d^{\sigma^{-1}(\sigma-1)}$, implying that

$$b = \left(\left(\frac{x^{\sigma^{-1}(\sigma+1)}}{d^{\sigma^{-1}}}\right)\left(d^{\sigma^{-1}-1} - 1\right)\right)^{(\sigma-1)},$$

which is contrary to our assumptions. Hence, we have a partial spread. This completes the proof to the theorem. QED

7 Theorem. *Under the above assumptions, let T be a subgroup of $K\backslash\{0\} = \mathrm{Fix}\sigma\backslash\{0\}$, such that $T \cap L^{\sigma-1} = \langle 1\rangle$ and $L^{\sigma-1}K$ is proper in L. Assume that $b \in L - L^{\sigma-1}T$. For $\alpha \in T$, let*

$$\mathcal{H}^{*}_{\alpha} = \left\{y = x^{\sigma}\alpha d^{1-\sigma} + x^{\sigma^{-1}}\alpha^{-1}d^{1-\sigma^{-1}}b; d \in L\backslash\{0\}\right\}.$$

Then

$$\cup_{\alpha\in T}\mathcal{H}^{*}_{\alpha}$$

is a set of mutually disjoint hyper-reguli.

PROOF. Look at

$$x^{\sigma}\beta + x^{\sigma^{-1}}\beta^{-1}b = x^{\sigma}\alpha d^{1-\sigma} + x^{\sigma^{-1}}\alpha^{-1}d^{1-\sigma^{-1}}b\,.$$

Assume that $\beta = \alpha d^{1-\sigma}$. Then $\beta/\alpha = d^{1-\sigma}$. Since $\beta/\alpha \in T$ and $d^{1-\sigma} \in L^{\sigma-1}$, then $d^{1-\sigma} = 1$ and $\beta = \alpha$. Thus, we may assume that $d^{1-\sigma} \neq 1$ and $\beta \neq \alpha d^{1-\sigma}$ Then the question, just as above, is whether there is a non-zero x satisfying the above equation. If so then

$$x^{\sigma-\sigma^{-1}}\frac{\left(\beta - \alpha d^{1-\sigma}\right)}{\alpha^{-1}\beta^{-1}d^{1-\sigma^{-1}}\left(\beta - \alpha d^{\sigma^{-1}-1}\right)} = b.$$

Note that $\left(\beta - \alpha d^{\sigma^{-1}-1}\right)^{\sigma} = \left(\beta - \alpha d^{1-\sigma}\right)$. Hence, we obtain a general equation of the form:

$$b = e^{(\sigma-1)}\alpha\beta\,,$$

implying that

$$b \in L^{\sigma-1}T\,,$$

a contradiction. QED

We now generalize these constructions as follows:

8 Theorem. *Let L be any field and σ be any automorphism of L. Assume that $L^{\sigma-1} \neq L$. Then $(L\backslash\{0\})^{\sigma-1} = L^{*(\sigma-1)}$ is a proper subgroup of $L\backslash\{0\} = L^*$. Let $\mathcal{B}$ be a coset representative set for $L^{*(\sigma-1)}$.*

Let λ be a subset of $\mathcal{B}$ for which that $\bigcup \omega_i\omega_j L^{*(\sigma-1)} \neq L^*$, for all ω_i, ω_j in λ. For $\omega_i \in \lambda$, let $\mathcal{H}_i^*$

$$\mathcal{H}_i^* = \left\{ y = x^{\sigma}\omega_i d^{1-\sigma} + x^{\sigma^{-1}}\omega_i^{-\sigma^{-1}}d^{1-\sigma^{-1}}b; d \in L^* \right\}.$$

(1) If $b \in L^* - \bigcup \omega_i\omega_j L^{*(\sigma-1)}$ then

$$\cup_{\omega_i \in \lambda}\mathcal{H}_i^*$$

is a set of mutually disjoint hyper-reguli, where

$$y = xm; m \in L, x = 0,$$

defines the corresponding Pappian spread Σ coordinatized by L.

(2) Choose any subset λ^* of λ. Then there is a corresponding Sperner space constructed using these subspaces together with the components of Σ not intersecting the components of λ^* and forming a spread.

If the spread is a congruence partition (for example, in the finite-dimensional or finite case), any such translation plane will have kernel containing the fixed field of σ.

PROOF. We need only show that we obtain a partial spread. So assume that

$$x^{\sigma}\omega_i d^{1-\sigma} + x^{\sigma^{-1}}\omega_i^{-\sigma^{-1}} d^{1-\sigma^{-1}} b = x^{\sigma}\omega_j + x^{\sigma^{-1}}\omega_j^{-\sigma^{-1}} b.$$

If $\omega_i d^{1-\sigma} = \omega_j$ then the two elements ω_i and ω_j are equal since they are in a coset representative set. In this case then $d^{1-\sigma} = 1$ and we have the same subspace. Hence, we may assume that $\omega_i d^{1-\sigma} \neq \omega_j$. Similarly $\omega_i^{-\sigma^{-1}} d^{1-\sigma^{-1}} \neq \omega_j^{-\sigma^{-1}}$. Assume that x is not zero. Then this would require that

$$\begin{aligned} b &= x^{\sigma-\sigma^{-1}} \left(\frac{\omega_i d^{1-\sigma} - \omega_j}{\omega_j^{-\sigma^{-1}} - \omega_i^{-\sigma^{-1}} d^{1-\sigma^{-1}}} \right) \\ &= x^{\sigma-\sigma^{-1}} \left(\frac{\omega_i d^{1-\sigma} - \omega_j}{\omega_j^{-\sigma^{-1}} \omega_i^{-\sigma^{-1}} d^{1-\sigma^{-1}} (\omega_i^{\sigma^{-1}} d^{\sigma^{-1}-1} - \omega_j^{\sigma^{-1}})} \right) \\ &= x^{\sigma-\sigma^{-1}} \left(\frac{\omega_i d^{1-\sigma} - \omega_j}{\omega_j^{-\sigma^{-1}} \omega_i^{-\sigma^{-1}} d^{1-\sigma^{-1}} (\omega_i d^{1-\sigma} - \omega_j)^{\sigma^{-1}}} \right) \\ &= x^{\sigma-\sigma^{-1}} \left(\frac{(\omega_i d^{1-\sigma} - \omega_j)^{1-\sigma^{-1}}}{\omega_j^{-\sigma^{-1}} \omega_i^{-\sigma^{-1}} d^{1-\sigma^{-1}}} \right) \\ &= \left((\omega_i \omega_j) \left(\frac{x^{\sigma^2-1} (\omega_i d^{1-\sigma} - \omega_j)^{\sigma-1}}{d^{\sigma-1}} \right) \right)^{\sigma^{-1}}. \end{aligned}$$

This implies that

$$\begin{aligned} b^{\sigma} &= (\omega_i \omega_j) \left(\frac{x^{\sigma^2-1} (\omega_i d^{1-\sigma} - \omega_j)^{\sigma-1}}{d^{\sigma-1}} \right) \\ &= (\omega_i \omega_j) \left(\frac{x^{\sigma+1} (\omega_i d^{1-\sigma} - \omega_j)}{d} \right)^{(\sigma-1)}. \end{aligned}$$

Hence, $b^{\sigma} \in \bigcup \omega_i \omega_j L^{*(\sigma-1)}$. Note that $\omega_i L^{*(\sigma-1)} = \omega_i^{\sigma} L^{*(\sigma-1)}$. So, $(\omega_i L^{*(\sigma-1)})^{\sigma}$ $= \omega_i L^{*(\sigma-1)}$. Hence, it follows that $b \in \bigcup \omega_i \omega_j L^{*(\sigma-1)}$, a contradiction to our assumptions. Hence, we obtain a partial spread. Now choose any subset of λ, λ^*, and define a set of subspaces of the following form: Let M denote the set of components $y = xm$ not intersected by any of the subspaces of λ^*. Then

$$\{x = 0, y = xm; m \in M\} \cup \cup_{\omega_i \in \lambda^*} \mathcal{H}_i^*$$

defines a Sperner space or a translation plane with kernel containing the fixed field of σ. Every subspace $y = x^{\sigma} t + x^{\sigma^{-1}} t^{-\sigma^{-1}} b$ is a vector space over Fixσ

isomorphic to L over Fixσ. Clearly, $y = x^\sigma t + x^{\sigma^{-1}} t^{-\sigma^{-1}} b$ and $y = xm$ (where this subspace does not intersect the given subspace) generate $L \oplus L$. However, it is not completely clear that two disjoint subspaces $y = x^\sigma t + x^{\sigma^{-1}} t^{-\sigma^{-1}} b$ and $y = x^\sigma \alpha + x^{\sigma^{-1}} \alpha^{-\sigma^{-1}} b$ will generate $L \oplus L$. If this condition is satisfied then we obtain a congruence partition for $L \oplus L$ and a corresponding translation plane. QED

2.1 André hyper-reguli

Previous to Jha and Johnson [2], the only known hyper-reguli were André hyper-reguli and these were only known in the finite case. Here we show that our constructed hyper-reguli are never André hyper-reguli when the associated automorphism does not have order 2 or 3.

9 Theorem. *Let L be a field and σ a non-identity automorphism of L. Assume that the order of σ is not 2 or 3 (note that the order could be finite or infinite). Then any hyper-regulus defined by any subspace of the form $y = x^\sigma t + x^{\sigma^{-1}} t^{-\sigma^{-1}} b$ is never an André hyper-regulus.*

Proof. Let π_Σ denote the associated Pappian plane coordinatized by L with spread Σ. Since $GL(2, L)$ is doubly transitive on the line at infinity, any André hyper-regulus may be defined by the image of a subspace of the form $y = x^\tau m$, where τ is a non-trivial automorphism of L. Hence, if $y = x^\sigma t + x^{\sigma^{-1}} t^{\sigma^{-1}} b$ defines an André hyper-regulus, it must be an image of some $y = x^\tau m$ under $\left[\begin{smallmatrix} a & e \\ c & d \end{smallmatrix}\right]$; $ad - ce \neq 0$. Therefore, we obtain the following condition:

$$(xa + x^\tau mc)^\sigma t + (xa + x^\tau mc)^{\sigma^{-1}} t^{-\sigma^{-1}} b = xe + x^\tau md\,.$$

Then, consider the automorphism set $\{\sigma, \tau\sigma, \sigma^{-1}, \tau\sigma^{-1}, 1, \tau\}$. Assume that this is a set of distinct automorphisms. Then, since such sets are linearly independent, it would follow that $a^\sigma t = (mc)^\sigma t = 0$, so that $a = c = 0$, a contradiction. Hence, it can only be that $\tau = \sigma$ or σ^{-1}, or $\sigma^{-1} = \sigma$. In the latter case, $\sigma^2 = 1$. So, assume that $\tau = \sigma$, implying that $\{\sigma, \tau\sigma, \sigma^{-1}, \tau\sigma^{-1}, 1, \tau\} = \{\sigma, \sigma^2, \sigma^{-1}, 1\}$. Since this is now, by assumption, a distinct set of automorphisms, it follows that the coefficient of x^{σ^2}, namely $(mc)^\sigma t$, is zero. Hence, $c = 0$. This leaves the coefficient of $x^{\sigma^{-1}}$, $(a^{\sigma^{-1}} t^{-\sigma^{-1}} b) = 0$, so $a = 0$, a contradiction. This completes the proof. QED

3 Examples

In this section, we offer just a few of the many fields L that admit automorphisms such that $L \neq L^\sigma - 1$.

10 Theorem. *Let G be any finite group. Let K be any infinite field. Then there is a field extension L of K admitting G as its automorphism group. Let F denote the fixed field of G. We may assume that L is a Galois extension of F with Galois group G. For any automorphism σ of G, we may construct non-André translation planes obtained by the replacement of mutually disjoint hyper-reguli in the Pappian spread coordinatized by L.*

PROOF. We note that L is Galois over $\mathrm{Fix}\sigma \supseteq F$, so it follows that $L^{\sigma-1} \cap F$ has only elements of finite order. Since F is proper in this group, it follows that $L^{\sigma-1} \neq L$. QED

11 Theorem. *Let K be the fixed field of an automorphism σ of L, where σ has finite order, and assume that K is infinite. Assume that K has only a finite number of elements of finite order. For example, let Q_a, the field of rationals, be K and let σ have finite odd order. Then $K^* \cap L^{*(\sigma-1)} = \langle 1 \rangle$ and $KL^{\sigma-1} \neq L$.*

PROOF. If $k = a^{\sigma-1}$, then $k^{1+\sigma+\cdots+\sigma^{n-1}} = k^n = 1$, if $n = |a|$. The rational numbers of finite order are ± 1, so if n is odd, we have $K^* \cap L^{*(\sigma-1)} = \langle 1 \rangle$. Now assume that for each c in L, there exists an element k in K and $d^{\sigma-1}$ in L such that $c = kd^{\sigma-1}$. Then $c^{(\sigma^n-1)/(\sigma-1)} = k^n$. We note that since $K^* \cap L^{*(\sigma-1)} = \langle 1 \rangle$, the representation of elements is unique. Since both groups are normal subgroups, it follows that $L^* \simeq K^* \times L^{*(\sigma-1)}$. QED

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