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Extended André Sperner spaces

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Abstract. The set of André spreads and generalized André spreads obtained using multiple André replacement of order q^{sn} is generalized to produce new constructions of $r-(sn,q)$-spreads, which are called extended André spreads and generalized extended André spreads. These sn-spreads produce new Sperner Spaces.

Keywords: Sperner spaces, sn-spread

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1 Introduction

This article concerns a generalization of the construction of the finite André planes π with kernel containing $GF(q)$ from Desarguesian affine planes Σ_{q^n} of order q^n, where $q=p^r$, for p a prime. The first André planes constructed are those of 'dimension two', that of order q^2 with kernel containing $GF(q)$, or equivalently, with their spreads in $PG(3,q)$. If the spread for the Desarguesian plane Σ_{q^n} is given by

$$x=0, y=xm; m \in GF(q^n),$$

then an 'André partial spread' A_δ is defined by

$$A_\delta : \left\{ y=xm; m^{\frac{(q^n-1)}{(q-1)}}=\delta \right\},$$

$\delta \in GF(q)$. A_δ is a replaceable partial spread with $n-1$ replacement nets $A_\delta^{q^i}$ with partial spread

$$A_\delta^{q^i} : \left\{ y=x^{q^i}m; m^{\frac{(q^n-1)}{(q-1)}}=\delta \right\}.$$

When $n=2$, the André partial spreads are reguli and the replacement partial spread A_δ^q is the opposite regulus to A_δ. In this case, any translation plane obtained from a Desarguesian plane of order q^2 by 'deriving' 'multiply deriving' a set of André partial spreads, is called an André plane (or more precisely, an André plane of dimension two or equivalently with spread in $PG(3, q)$). More generally, any translation plane obtained by replacing a set of André partial spreads by one of the $n-1$ replacements per partial spread is called an 'André plane of order q^n with kernel containing $GF(q)$' (i.e. with spread in $PG(2n-1, q)$). The 'kernel homology group' of the Desarguesian plane Σ_{q^n}, is the group of central collineations with axis the line at infinity and center the zero vector of the associated vector space. This group of order $q^n - 1$ then will act on each André plane but the replacement nets are now in an orbit of length $\frac{(q^n-1)}{(q-1)}$ under the kernel homology group.

In the present article, we generalize the concept of an André plane as arising from a Desarguesian affine plane to analogous structures constructable from what are called 'Desarguesian t-spreads'. Any t-spread of a vector space produces a Sperner space as realized by Barlotti and Cofman [1]. Thus we obtain new Sperner spaces in a similar way that André planes are produced from Desarguesian affine planes. In the present construction, new sn-spreads over $GF(q)$ that we called 'generalized extended André' $r-(sn, q)$-spreads, in a manner analogous to the construction of the generalized André translation planes, are constructed by what we call 'extended André' and 'generalized extended André' replacement. In this way, we obtain a vast variety of new sn-spreads from vector spaces of dimension r over $GF(q^{sn})$. The reader is directed to Biliotti, Jha and Johnson [2] for additional background on André and generalized André planes.

Actually, some of our constructed $r-(sn, q)$-spreads, when $r = s$, have been obtained by other methods. Recently, Ebert and Mellinger [3] construct some new subgeometry partitions of projective spaces. All of these subgeometry partitions 'lift' to rn-spreads. The methods that we employ look first for the spread in the affine setting and then ask what property such spread might have so that it is possible to 'retract' to a subgeometry partition. Indeed, it is this perspective that gives rise to the generalized extended André spreads. In fact, we note that all of the rn-spreads of Ebert and Mellinger may be constructed using our techniques. Furthermore, many of the generalized André spreads 'retract' to a great variety of subgeometry partitions of projective spaces, and this work will be reported in a companion article (see Johnson [4]).

2 $r-(sn, q)$-spreads

Consider a field $GF(q^{rsn})$, where $q = p^z$, for p a prime. Then $GF(q^{rsn})$ is an r-dimensional vector space over $GF(q^{sn})$. More generally, let V be the r-dimensional vector space over $GF(q^{sn})$.

1 Definition. A '1-dimensional r-spread' or 'Desarguesian $r-(sn, q)$-spread' is defined to be a partition of V by set of all 1-dimensional $GF(q^{sn})$-subspaces, where q is a prime power.

In this case, the vectors are represented in the form $(x_1, x_2, \ldots, x_r)$, where $x_i \in GF(q^{sn})$.

2 Definition. Furthermore, the 1-dimensional $GF(q^{sn})$-subspaces may be partitioned in the following sets called 'j-(0-sets)'. A 'j-(0-set)'is the set of vectors with j of the entries equal to 0. For a specific set of j zeros among the r elements, the set of such non-zero vectors in the remaining $r-j$ non-zero entries is called a '$(j-$(0-subset))'.

Note that there are exactly $\binom{r}{r-j}(q^{sn}-1)^{r-j}$ vectors (non-zero vectors) in each j-(0-set) and exactly $(q^{sn}-1)^{r-j}$ vectors in each of the $\binom{r}{r-j}$ disjoint $j-$(0-subsets).

3 Notation. Hence, $j = 0, 1, \ldots, r-1$ and we denote the $j-$(0-sets) by Σ_j and by specifying any particular order, we index the $\binom{r}{r-j}$ $j-$(0-subsets) by $\Sigma_{j,w}$, for $w = 1, 2, \ldots, \binom{r}{r-j}$. We note that

$$\cup_{w=1}^{\binom{r}{r-j}} \Sigma_{j,w} = \Sigma_j,$$

a disjoint union.

4 Remark. Furthermore, the $(q^{rsn}-1)$ non-zero vectors are partitioned in the $j-$(0-sets) by

$$(q^{rsn}-1) = \sum\sum\sum_{j=0}^{r-1} \binom{r}{r-j} (q^{sn}-1)^{r-j} \ ,$$

and the number of 1-dimensional $GF(q^{sn})$-subspaces is

$$\frac{(q^{rsn}-1)}{(q^{sn}-1)} = \sum\sum\sum_{j=0}^{r-1} \binom{r}{r-j} (q^{sn}-1)^{r-j-1}.$$

Also, note that this is then also the number of sn-dimensional $GF(q)$-subspaces in a $r-(sn, q)$-spread.

5 Notation. Consider a vector $(x_1, x_2, \ldots, x_r)$ over $GF(q^{sn})$, we use the notation (x_1, y) for this vector. Consider a j-(0-set) Σ_j and let x_{j_1} denote the first non-zero entry. Then all of the other entries are of the form $x_{j_1}m$, for $m \in GF(q^{sn})$. For example, the elements of an element of a 0-(0-set) may be presented in the form $(x_1, x_1m_1, \ldots, x_1m_{r-1})$, for x_1 non-zero and m_i also non-zero in $GF(q^{sn})$. That is, $y = (x_1m_1, \ldots, x_1m_{r-1})$. More importantly, if we vary x_1 over $GF(q^{sn})$, then

$$y = (x_1m_1, \ldots, x_1m_{r-1}),$$

is a 1-dimensional $GF(q^{sn})$-subspace. However, we now consider this subspace as an sn-dimensional $GF(q)$-subspace.

In this notation, a Desarguesian 1-spread leads to an affine translation plane by defining 'lines' to be translates of the 1-dimensional $GF(q^{sn})$-subspaces.

6 Definition. In general, for $r > 2$, a Desarguesian r-spread leads to a 'Desarguesian translation Sperner space' (simply the associated affine space) by the same definition on lines. Every 1-dimensional $GF(q^{sn})$-space may be considered an sn-space over $GF(q)$. When this occurs we have what we shall call an '$r-(sn$-spread)' (or also a '$r-(sn,q)$-spread').

More generally,

7 Definition. A partition of an rsn-dimensional vector space over $GF(q)$ by mutually disjoint sn-dimensional subspaces shall be called an '$r-(sn,q)$-spread'. In the literature, this is often called an 'sn-spread' or projectively on the associated projective space as an '$sn-1$-spread'.

If $r = 2$, any '$2-(sn,q)$-spread' is equivalent to a translation plane of order q^{2n}, with kernel containing $GF(q)$.

8 Definition. Let Σ be a r-(sn-spread). We define the 'collineation group' of Σ to be the subgroup of $\Gamma L(rsn,q)$ that permutes the spread elements (henceforth called 'components').

For a Desarguesian r-spread Σ, the subgroup with elements

$$(x_1, x_2, \ldots, x_r) \longmapsto (dx_1, dx_2, \ldots, d_r x_r)$$

for all d nonzero in $GF(q^{sn})$ is called the 'sn-kernel' subgroup of Σ. The group fixes each Desarguesian component and acts transitively on its points. The group K_{sn}^* union the zero mapping is isomorphic to $GF(q^{qn})$. K_{sn}^* has a subgroup K_s^*, where d above is restricted to $GF(q^s)^*$, and K_s^* union the zero mapping is isomorphic to $GF(q^s)$. K_s^* is called the 's-kernel' subgroup'.

9 Definition. More generally, also for a Desarguesian spread Π, we note that the group $G^{(sn)^r}$ of order $(q^{sn}-1)^r$ with elements

$$(x_1, x_2, \ldots, x_r) \longmapsto (d_1x_1, d_2x_2, \ldots, d_r x_r); d_i \in GF(q^{sn})^*, i = 1, 2, \ldots, r$$

also acts as a collineation group of Π. We call $G^{(sn)^r}$, the 'generalized kernel group'.

Let Σ be a Desarguesian r-spread with vectors $(x_1, x_2, \ldots, x_r)$. Consider any set Σ_j, and suppress the set of j zeros and write vectors in the form $(x_1^*, x_2^*, \ldots, x_{r-j}^*)$, in the order of non-zero elements within $(x_1, \ldots, x_r)$ Assume that $j \leq r-1$.

10 Definition. We consider such vectors of the following form

$$\left(x_1^*, x_1^{*q^{\lambda_1}} m_1, \ldots, x_1^{*q^{\lambda_{r-j}}} m_{r-j}\right).$$

If x_1^* varies over $GF(q^{sn})$, and consider Σ as a rsn-vector space over $GF(q)$, we then have a sn-vector subspace over $GF(q)$ that we call

$$y = \left(x_1^{*q^{\lambda_1}} m_1, \ldots, x_1^{*q^{\lambda_{r-j-1}}} m_{r-j-1}\right),$$

where λ_i are integers between 0 and $sn-1$.

11 Definition. We are interested in the set of Desarguesian sn-subspaces

$$y = (x_1^* w_1, \ldots, x_1^* w_{r-j-1})$$

(using the same notation) that can intersect

$$y = \left(x_1^{*q^{\lambda_1}} m_1, \ldots, x_1^{*q^{\lambda_{r-j-1}}} m_{r-j-1}\right).$$

We note that we have a non-zero intersection if and only if

$$x_1^{*q^{\lambda_i - 1}} = \frac{w_i}{m_i}, \text{ for all } i = 1, 2, \ldots, r-j-1.$$

This set of non-zero intersections of Σ shall be called an 'extended André set of type $(\lambda_1, \lambda_2, \ldots, \lambda_{r-j-1})$'. The set of all subspaces

$$y = \left(x_1^{*q^{\lambda_1}} n_1, \ldots, x_1^{*q^{\lambda_{r-j-1}}} n_{r-j-1}\right),$$

such that

$$x_1^{*q^{\lambda_i - 1}} = \frac{w_i}{n_i}, \text{ for all } i = 1, 2, \ldots, r-j-1,$$

has a solution is called an 'extended André replacement'.

2.1 Examples

To get a feel for where this idea arose, we consider when $r-j-1=1$. So, we have vectors of the basic form (x_1^*, x_2^*) and sn-subspaces $y = x_1^{*q^{\lambda_1}} m_1$ that are covered by sets of Desarguesian sn-subspaces $y = x_1^* w_1$, such that $x_1^{*q^{\lambda_1 - 1}} = \frac{w_1}{m_1}$. In this setting,

$$\left\{ y = x_1^* w_1; w_1^{\frac{(q^{sn}-1)}{(q^{(\lambda_1, sn)}-1)}} = m_1^{\frac{(q^{sn}-1)}{\left(q^{(\lambda_1, sn)}-1\right)}} = \delta \right\},$$

is called an 'André set A_δ', where $\delta \in GF(q^{(\lambda_1, sn)})$. This set has replacements sets

$$A_\delta^\rho = \left\{ y = x^{q^{(\lambda_1, sn)\rho}} w_1; w_1^{\frac{(q^{sn}-1)}{\left(q^{(\lambda_1, sn)}-1\right)}} = \delta \right\},$$

where $1 \leq \rho \leq sn/(\lambda_1, sn) - 1$.

12 Example. For some examples, let $r-j=3$, and consider the sn-space

$$y=\left(x_1^{*q}m_1,x_1^{*q}m_2\right),$$

this space generates the corresponding André net

$$\{y=(x_1^{*}w_1,x_1^{*}w_2)\}$$

as follows: The intersections are

$$x_1^{*q-1}=\frac{w_1}{m_1}=\frac{w_2}{m_2}.$$

Let $\frac{w_1}{m_1}=\frac{w_2}{m_2}=\tau$, so that $\tau^{\frac{(q^{sn}-1)}{(q-1)}}=1$. Then, $w_1=m_1\tau$, and $w_2=m_2\tau$, and hence we have

$$\left\{y=(x_1^{*}m_1\tau,x_1^{*}m_2\tau);\tau^{\frac{(q^{sn}-1)}{(q-1)}}=1\right\}.$$

Now consider the kernel group K_{sn}, which fixes element of the André set and maps

$$y=(x_1^{*q}m_1,x_1^{*q}m_2)\longmapsto y=(x_1^{*q}m_1d^{1-q},x_1^{*q}m_2d^{1-q}).$$

Since K_{sn} is transitive on each component of the André set, it follows that we have a replacement set

$$\left\{y=(x_1^{*q}m_1d^{1-q},x_1^{*q}m_2d^{1-q});d\in GF(q^{sn})^*\right\}.$$

Note that in this case, each component of the replacement set intersects each component of the André set in a 1-dimensional $GF(q)$-subspace.

13 Example. Now consider again $r-j=3$ and the set

$$y=\left(x_1^{*q}m_1,x_1^{*q^2}m_2\right).$$

We would then obtain a typical Desarguesian intersection of the form

$$y=\left(xm_1\tau,m_2\tau^{q+1}\right);\tau^{\frac{(q^{sn}-1)}{(q-1)}}=1.$$

The kernel group would then map

$$y=\left(x_1^{*q}m_1,x_1^{*q^2}m_2\right)\longmapsto y=\left(x_1^{*q}m_1d^{1-q},x_1^{*q^2}m_2d^{1-q^2}\right);d\in GF(q^{sn})^*,$$

and we would have an André set

$$\left\{y=\left(x_1m_1\tau,x_1m_2\tau^{q+1}\right);\tau^{\frac{(q^{sn}-1)}{(q-1)}}=1\right\}$$

with replacement set

$$\left\{y=\left(x_1^{*q}m_1d^{1-q},x_1^{*q^2}m_2d^{1-q^2}\right);d\in GF(q^{sn})^*\right\}.$$

Note that $d^{1-q}=\tau$ implies that $d^{1-q^2}=\tau^{q+1}$, so we obtain a cover.

We now show that any sn-subspace of the type

$$y=\left(x_1^{*q^{\lambda_1}}n_1,\ldots,x_1^{*q^{\lambda_{r-j-1}}}n_{r-j-1}\right),$$

generates an extended André set and an extended André replacement set.

3 The main theorem on extended André replacements

14 Theorem. *Let Σ be a Desarguesian r-spread of order q^{sn}. Let $\Sigma_{j,w}$ be any the j-(0-subset) for $j=0,1,2,\ldots,r-1$.*

Choose any sn-dimensional subspace

$$y=\left(x_1^{*q^{\lambda_1}}n_1,\ldots,x_1^{*q^{\lambda_{r-j-1}}}n_{r-j-1}\right);$$

where $n_i\in GF(q^{sn})^*$, $i=1,2,\ldots,r-j-1$. Let $d=(\lambda_1,\lambda_2,\ldots,\lambda_{r-j-1})$, where $0\leq\lambda_i\leq sn-1$.

(1) Then

$$A_{(n_1,\ldots,n_{r-j-1})}=\left\{y=(x_1^*w_1,\ldots,x_1^*w_{r-j-1});\text{ there is an }x_1^*\text{ such that}\right.$$
$$\left.x_1^{*q^{\lambda_i}-1}=\frac{w_i}{n_i},\text{ for all }i=1,2,\ldots,r-j-1\right\}$$

is a set of $\frac{(q^{sn}-1)}{(q^d-1)}$ sn-subspaces, which is covered by the set of $\frac{(q^{sn}-1)}{(q^d-1)}$

$$A^{(\lambda_1,\ldots,\lambda_{r-j-1})}_{(n_1,\ldots,n_{r-j-1})}=\left\{y=(x_1^{*q^{\lambda_1}}n_1d^{1-q^{\lambda_1}},\ldots,x_1^{*q^{\lambda_{r-j-1}}}n_{r-j-1}d^{1-q^{\lambda_{r-j-1}}});\right.$$
$$\left.d\in GF(q^{sn})^*\right\}.$$

(2) Let $C_{\frac{(q^{sn}-1)}{(q-1)}}$ denote the cyclic subgroup of $GF(q^{sn})^*$ of order $\frac{(q^{sn}-1)}{(q-1)}$. Then, for each

$$y=(x_1^*w_1,\ldots,x_1^*w_{r-j-1}),$$

there exists an element τ in $C_{\frac{(q^{sn}-1)}{(q-1)}}$ such that

$$w_i=n_i\tau^{\frac{(q^{\lambda_i}-1)}{(q-1)}}.$$

(3) The
$$\frac{(q^{sn}-1)}{(q^{(\lambda_1,\dots,\lambda_{r-j-1},sn)}-1)}$$
components of
$$\left\{y=\left(x_1^{*q^{\lambda_1}}n_1d^{1-q^{\lambda_1}},\dots,x_1^{*q^{\lambda_{r-j-1}}}n_{r-j-1}d^{1-q^{\lambda_{r-j-1}}}\right);d\in GF(q^{sn})^*\right\}$$
are in
$$\frac{\left(\frac{(q^{sn}-1)}{\left(q^{(\lambda_1,\dots,\lambda_{r-j-1})}-1\right)}\right)}{\left(\frac{(q^{s}-1)}{\left(q^{(\lambda_1,\lambda_2,\dots,\lambda_{r-j-1},s)}-1\right)}\right)}$$
orbits of length
$$\frac{(q^{s}-1)}{\left(q^{(\lambda_1,\lambda_2,\dots,\lambda_{r-j-1},s)}-1\right)}$$
under the s-kernel homology group K_s.

Proof. There exists an integer i_0 such that $\lambda_{i_0}=d\rho_{i_0}$, for $(\rho_{i_0},sn/d)=1$. For fixed n_{i_0} non-zero in $GF(q^{sn})^*$, consider the set of all elements w_{i_0} such that $w_{i_0}/n_{i_o}=\tau_0$ for some τ_0 such that $\tau^{\frac{(q^{sn}-1)}{(q^{d\rho_{i_0}}-1)}}=1$, clearly a set of $\frac{(q^{sn}-1)}{(q^d-1)}$ elements in $GF(q^{sn})^*$.

There exists an element $x_1^{*(q-1)}=\tau$ so that
$$x_1^{*q^{\lambda_i}-1}=\frac{w_i}{n_i}=\tau^{\frac{(q^{\lambda_i}-1)}{(q-1)}}.$$
This proves part (2).

Now to show that the indicated set
$$\left\{y=\left(x_1^{*q^{\lambda_1}}n_1d^{1-q^{\lambda_1}},\dots,x_1^{*q^{\lambda_{r-j-1}}}n_{r-j-1}d^{1-q^{\lambda_{r-j-1}}}\right);d\in GF(q^{sn})^*\right\}$$
covers the set
$$\left\{y=\left(x_1^*n_1\tau^{\frac{(q^{\lambda_1}-1)}{(q-1)}},\dots,x_1^*n_{r-j-1}\tau^{\frac{(q^{\lambda_{r-j-1}}-1)}{(q-1)}}\right);\tau\in C_{\frac{(q^{sn}-1)}{(q-1)}}\right\}.$$
First note that the following sets are equal:
$$\left\{x_1^{*q^{\lambda_i}}n_id^{1-q^{\lambda_i}};x_i^*,d\in GF(q^{sn})^*\right\}$$
$$=\left\{x_1^*n_i\tau^{(q^{\lambda_i}-1)/(q-1)};x_i\in GF(q^{sn}),\tau\in C_{\frac{(q^{sn}-1)}{(q-1)}}\right\}.$$

Assume for a fixed x_1^* that

$$x_1^{*(q-1)}d^{1-q}=\tau.$$

Then

$$(x_1^*d^{-1})^{(q^{\lambda_i}-1)}=\tau^{(q^{\lambda_i}-1)/(q-1)},$$

which is true if and only if

$$x_1^{*q^{\lambda_i}}n_id^{1-q^{\lambda_i}}=x_1^*n_i\tau^{\frac{(q^{\lambda_i}-1)}{(q-1)}}.$$

Therefore, as

$$\left\{x_1^{*q}\tau;\tau\in C_{\frac{(q^{sn}-1)}{(q-1)}}\right\}$$

covers

$$\left\{x_1^*d^{1-q};d\in GF(q^{sn})^*\right\},$$

it follows that

$$\left\{y=(x_1^*n_1\tau^{(q^{\lambda_1}-1)/(q-1)},\ldots,x_1^*n_{r-j-1}\tau^{\frac{(q^{\lambda_{r-j-1}}-1)}{(q-1)}});\tau\in C_{\frac{(q^{sn}-1)}{(q-1)}}\right\}.$$

is covered by

$$\left\{y=(x_1^{*q^{\lambda_1}}n_1d^{1-q^{\lambda_1}},\ldots,x_1^{*q^{\lambda_{r-j-1}}}n_{r-j-1}d^{1-q^{\lambda_{r-j-1}}});d\in GF(q^{sn})^*\right\}.$$

Now to determine the total number of components in the extended André net $(\lambda_1,\lambda_2,\ \ldots,\lambda_{r-j-1})$. Thus, the question is, when is

$$\tau^{\frac{(q^{\lambda_i}-1)}{(q-1)}}=1$$

for all $i=1,2,\ldots,r-j-1$, where τ is an arbitrary element of $C_{\frac{(q^{sn}-1)}{(q-1)}}$. It is then clear that τ must have order

$$\frac{\left(q^{(\lambda_1,\ldots,\lambda_{r-j-1})}-1\right)}{(q-1)}.$$

In other words, there are exactly

$$\frac{(q^{sn}-1)}{\left(q^{(\lambda_1,\ldots,\lambda_{r-j-1})}-1\right)}$$

components of the extended André set $A_{(\lambda_1,\ldots,\lambda_{r-j-1})}$.

Now consider an orbit in the replacement set under the s-kernel homology group K_s.

$$y = (x_1^{*q^{\lambda_1}} n_1, \ldots, x_1^{*q^{\lambda_{r-j-1}}} n_{r-j-1})$$

maps to

$$y = (x_1^{*q^{\lambda_1}} {n_1}^{1-q^{\lambda_1}} d^{1-q^{\lambda_1}}, \ldots, x_1^{*q^{\lambda_{r-j-1}}} n_{r-j-1} d^{1-q^{\lambda_{r-j-1}}})$$

for $d \in GF(q^s)^*$. This orbit clearly has the cardinality indicated. $\boxed{QED}$

15 Definition. The André planes of order q^{sn} and kernel containing $GF(q)$ are defined as follows. We let

$$A_\delta = \left\{ y = xm; m^{\frac{(q^{sn}-1)}{(q-1)}} = \delta \right\}, \delta \in GF(q),$$

called an 'André partial spread' of degree $\frac{(q^{sn}-1)}{(q-1)}$ and order q^{sn}. This partial spread is replaceable by any partial spread

$$A_\delta^{q^\lambda} = \left\{ y = x^{q^\lambda} m; m^{\frac{(q^{sn}-1)}{(q-1)}} = \delta \right\}, \delta \in GF(q), 0 \le \lambda \le sn-1,$$

called an 'André replacement'. Hence, there are exactly $sn-1$ non-trivial replacements and, of course, if $\lambda = 0$, the partial spread has not been replaced. There are exactly $q-1$ André nets each admitting sn replacements. An 'André plane' is defined as any translation plane obtained with spread consisting of $q-1$ André replacement partial spreads together with $x = 0, y = 0$.

Therefore, there are $(sn)^{q-1} - 1$ distinct André spreads obtained from a given Desarguesian affine plane.

16 Remark. If we take $y = x^{q^\lambda} m_0$ for a fixed element of $GF(q^{sn})^*$, Let N_{λ,m_0} define the set of components of the associated Desarguesian affine plane which non-trivially intersect this subspace. Then the set of images under the kernel homology group of order $(q^{sn}-1)$ is

$$\left\{ y = x^{q^\lambda} m_0 d^{1-q^\lambda} \right\},$$

and we see that we obtain a net of degree

$$\frac{(q^{sn}-1)}{\left(q^{(\lambda,sn)}-1\right)}.$$

The partial $sn-$spread

$$A = \left\{ y = \left(x_1^* n_1 \tau^{\frac{(q^{\lambda_1}-1)}{(q-1)}}, \ldots, x_1^* n_{r-j-1} \tau^{\frac{(q^{\lambda_{r-j-1}}-1)}{(q-1)}} \right); \tau \text{ has order} \right.$$
$$\left. \text{dividing } \frac{(q^{sn}-1)}{(q-1)}. \right\}$$

is a set of

$$\frac{(q^{sn}-1)}{(q^{(\lambda_1,\lambda_2,\ldots,\lambda_{r-j-1},sn)}-1)}$$

sn-subspaces, which is covered by the set of

$$\frac{(q^{sn}-1)}{(q^{(\lambda_1,\lambda_2,\ldots,\lambda_{r-j-1},sn)}-1)}$$

sn-subspaces

$$A_{(n_1,\ldots,n_{r-j-1})}^{(\lambda_1,\ldots,\lambda_{r-j-1})} = \left\{ y = (x_1^{*q^{\lambda_1}} n_1 d^{1-q^{\lambda_1}}, \ldots, x_1^{*q^{\lambda_{r-j-1}}} n_{r-j-1} d^{1-q^{\lambda_{r-j-1}}}); \right.$$
$$\left. d \in GF(q^{sn})^* \right\}.$$

17 Definition. We shall call $A_{(n_1,\ldots,n_{r-j-1})}$ an 'extended André partial spread' of degree $(q^{sn}-1)/(q^{(\lambda_1,\lambda_2,\ldots,\lambda_{r-j-1},sn)}-1)$' and order q^{sn}. So, we note that $A_{(n_1,\ldots,n_{r-j-1})}^{(\lambda_1,\ldots,\lambda_{r-j-1})}$ is a replacement partial spread of the same degree and order, called an 'extended André replacement'.

3.1 Extended André replacements

Note that is more problematic to define all of the André replacements for a given extended André partial spread. Furthermore, by purposely not trying to make such a definition will free us to consider more general situations. However, if we are looking for extended André partial spreads of the same degree, we take other exponent sets $\{\rho_1, \rho_2, \ldots, \rho_{r-j-1}\}$ so that

$$(\rho_1, \rho_2, \ldots, \rho_{r-j-1}, sn) = (\lambda_1, \lambda_2, \ldots, \lambda_{r-j-1}, sn),$$

there are are exactly

$$(q^{(\lambda_1,\lambda_2,\ldots,\lambda_{r-j-1},sn)}-1)(q^{sn}-1)^{r-j-2},$$

possible extended André partial spreads of degree

$$\frac{(q^{sn}-1)}{(q^{(\lambda_1,\lambda_2,\dots,\lambda_{r-j-1},sn)}-1)}.$$

Therefore, if we let $\rho_i = \lambda_i t_i$, for $0 \leq t_i \leq sn/(\lambda_i, sn) - 1$, then let

$$(\lambda_1, \lambda_2, \dots, \lambda_{r-j-1}, sn) = s^*,$$

$$\begin{aligned}(\rho_1, \rho_2, \dots, \rho_{r-j-1}, sn) &= (\lambda_1 t_1, \lambda_2 t_2, \dots, \lambda_{r-j-1} t_{r-j-1}, sn) \\ &= \left(\frac{\lambda_1 t_1}{s^*}, \frac{\lambda_2 t_2}{s^*}, \dots, \frac{\lambda_{r-j-1} t_{r-j-1}}{s^*}, \frac{sn}{s^*}\right).\end{aligned}$$

18 Remark. Now assume that all $\lambda_i = 1$, for $i = 1, 2, \dots, r-j-1$. Then the extended André partial spread

$$A_{(n_1,\dots,n),}$$

has degree $\frac{(q^{sn}-1)}{(q-1)}$ and we have a replacement partial spread with components

$$A^{(1,\dots,1)}_{(n_1,\dots,n_{r-j-1})} = \left\{ y = \left(x_1^q n_1 d^{1-q}, x_2^q n_2 d^{1-q}, \dots, x_{r-j-1}^q n_{r-j-1} d^{1-q} \right) \right\}.$$

As noted,

$$\begin{aligned}&A_{(n_1 \tau^{(q^{\lambda_1}-1)/(q-1)}, \dots, n_{r-j-1} \tau^{(q^{\lambda_{r-j-1}}-1)/(q-1)})} \\ &\quad = \left\{ y = \left(x_1^* n_1 \tau^{\frac{(q^{\lambda_1}-1)}{(q-1)}}, \dots, x_1^* n_{r-j-1} \tau^{\frac{(q^{\lambda_{r-j-1}}-1)}{(q-1)}} \right); \right. \\ &\qquad\qquad \left. \tau \text{ has order dividing } \frac{(q^{sn}-1)}{(q-1)}. \right\}\end{aligned}$$

is a set of

$$\frac{(q^{sn}-1)}{(q^{(\lambda_1,\lambda_2,\dots,\lambda_{r-j-1},sn)}-1)}$$

sn-subspaces, which is covered by the set of

$$\frac{(q^{sn}-1)}{(q^{(\lambda_1,\lambda_2,\dots,\lambda_{r-j-1},sn)}-1)}$$

$sn-$subspaces

$$\begin{aligned}&A^{(\lambda_1,\dots,\lambda_{r-j-1})}_{(n_1,\dots,n_{r-j-1})} \\ &\quad = \left\{ y = (x_1^{*q^{\lambda_1}} n_1 d^{1-q^{\lambda_1}}, \dots, x_1^{*q^{\lambda_{r-j-1}}} n_{r-j-1} d^{1-q^{\lambda_{r-j-1}}}); d \in GF(q^{sn})^* \right\}.\end{aligned}$$

19 Remark. Now consider

$$\left\{(n_1 d^{1-q}, n_2 d^{1-q}, \ldots, n_{r-j-1} d^{1-q}); d \in GF(q^{sn})^*\right\}.$$

If $\tau^{\frac{(q^{\lambda_k}-1)}{(q-1)}} = \tau^{\frac{(q^{\lambda_i}-1)}{(q-1)}}$, for all k, i, so for example $\lambda_i = \lambda_z$, for all $i = 1, 2, \ldots, r-j-1$ then we would obtain that $\frac{(q^{sn}-1)}{(q^{(\lambda_1, sn)}-1)}$ components in this extended André partial spread and if we vary over the cosets

$$\frac{(q^{sn}-1)}{(q-1)} / \frac{(q^{sn}-1)}{(q^{(\lambda_1, sn)}-1)} = \frac{(q^{(\lambda_1, sn)}-1)}{(q-1)}$$

of the cyclic group of order

$$\frac{(q^{sn}-1)}{(q^{(\lambda_1, sn)}-1)}$$

with respect to the group of order

$$\frac{(q^{sn}-1)}{(q-1)}.$$

Therefore, we would have

$$(n_i d^{1-q})^{\frac{(q^{sn}-1)}{(q-1)}} = n_i^{\frac{(q^{sn}-1)}{(q-1)}} = \delta_i\,.$$

Hence, we might then call this net

$$A_{\delta_1, \delta_2, \ldots, \delta_{r-j-1}},$$

which is then covered by

$$A(S)^{(\lambda_1, \ldots, \lambda_{r-j-1})}_{\delta_1, \ldots, \delta_{r-j-1}} = \left\{ \begin{array}{l} y = (x_1^{q^{\lambda_1}} n_1^*, x_1^{q^{\lambda_1}} n_2^*, \ldots, x_1^{q^{\lambda_1}} n_{r-j-1}^*); \\ n_i^{*\frac{(q^{sn}-1)}{(q-1)}} = \delta_i, i = 1, 2, \ldots, r-j-1. \end{array} \right\}$$

So that there are $sn-1$ non-trivial extended André replacements for this particular extended André partial spread.

In this setting, there are

$$(q^n-1)^{r-j-2}(q-1),$$

mutually disjoint extended André partial spreads and each admit $sn-1$ non-trivial extended André replacements.

20 Remark. To consider the analogous situation, for André planes, we note that we can make such replacements for all j-(0-subset)'s subsets, for each j such that $r - j \geq 2$.

Hence, we define 'extended André r-(sn)-spreads' to be any of the sn replacements for each of the $\binom{r}{r-j}$ j-(0-subsets), for each of the $j \geq r-2$ j-(0-sets). This then constructs a set of

$$\sum\sum\sum_{j=0}^{r-2} \binom{r}{r-j} (sn)^{(q^n-1)^{r-j-2}(q-1)} - 1$$

distinct non-trivial $r-(sn)-\mathit{spreads}$, not equal to the original Desarguesian $r-(sn)-$spread.

4 Multiple extended André replacement

An André plane is obtained by making replacements all of the same degree, say $(q^{sn}-1)/(q-1)$. However, if we partition a given André partial spread in $(q^{s^*}-1)/(q-1)$ André partial spreads of degree $(q^{sn}-1)/(q^{s^*}-1)$, where s^* divides sn, in this setting a typical subspace of a replacement has the form $y = x^{h^i}m$, where $h = q^{s^*}$. Normally, one would take $0 \leq i \leq sn/s^* - 1$. Any of these André partial spreads may be further subdivided and André replacements of various different degrees may be considered. Any translation plane constructed by making André replacements of various degrees is not (always) an André plane but is a 'generalized André plane', since the components of the constructed translation plane have the general form $y = x^{q^{\lambda(m)}}m$, where m is in $GF(q^{sn})$ and λ is a function from $GF(q^{sn})^*$ to the set of integers $0, 1, \ldots, sn-1$.

21 Definition. We say that the generalized André plane constructed using the method above is constructed by 'multiple André replacement'.

We now consider an analogous construction procedure using a different approach. First we note:

22 Lemma.

$$A_{(n_1,\ldots,n_{r-j-1})} = \left\{ y = \left(x_1^* n_1 \tau^{\frac{(q^{\lambda_1}-1)}{(q-1)}}, x_1^* n_2 \tau^{\frac{(q^{\lambda_2}-1)}{(q-1)}}, \ldots, x_1^* n_{r-j-1} \tau^{\frac{(q^{\lambda_{r-j-1}}-1)}{(q-1)}} \right); \right.$$
$$\left. \tau \textit{ has order dividing } \frac{(q^{sn}-1)}{(q-1)} \right\},$$

and

$$A_{(n_1^*,\dots,n_{r-j-1}^*)} = \left\{ y = \left(x_1^* n_1^* \tau^{\frac{(q^{\lambda_1^*}-1)}{(q-1)}}, x_1^* n_2^* \tau^{\frac{(q^{\lambda_2^*}-1)}{(q-1)}}, \dots, x_1^* n_{r-j-1}^* \tau^{\frac{(q^{\lambda_{r-j-1}^*}-1)}{(q-1)}} \right); \tau \text{ has order dividing } \frac{(q^{sn}-1)}{(q-1)} \right\},$$

share a component if and only if there exist elements τ_1 and τ_1^* of order dividing $\frac{(q^{sn}-1)}{(q-1)}$ such that

$$n_i \tau_1^{\frac{(q^{\lambda_i}-1)}{(q-1)}} = n_i^* \tau_1^{*(q^{\lambda_i^*}-1)/(q-1)}; i = 1, 2, \dots, r-j-1.$$

Proof. Suppose in the first listed set we have τ_1 in place of τ and in the second listed set have τ_1^* in place of τ, such that the corresponding components

$$y = \left(x_1^* n_1 \tau_1^{\frac{(q^{\lambda_1}-1)}{(q-1)}}, x_1^* n_2 \tau_1^{\frac{(q^{\lambda_2}-1)}{(q-1)}}, \dots, x_1^* n_{r-j-1} \tau_1^{\frac{(q^{\lambda_{r-j-1}}-1)}{(q-1)}} \right) = y = \left(x_1^* n_1^* \tau_1^{*\frac{(q^{\lambda_1^*}-1)}{(q-1)}}, x_1^* n_2^* \tau_1^{*\frac{(q^{\lambda_2^*}-1)}{(q-1)}}, \dots, x_1^* n_{r-j-1}^* \tau_1^{*\frac{(q^{\lambda_{r-j-1}^*}-1)}{(q-1)}} \right)$$

are equal. Then we must have

$$n_i \tau_1^{\frac{(q^{\lambda_i}-1)}{(q-1)}} = n_i^* \tau_1^{*(q^{\lambda_i^*}-1)/(q-1)}; i = 1, 2, \dots, r-j-1.$$

Since the extended André partial spread is an orbit, we may assume that $\tau_1 = 1$. So, the only way that this could occur is if $(n_1, \dots, n_{r-j-1})$ and $(n_1^*, n_2^*, \dots, n_{r-j-1}^*)$ are related by the set of equations above. This completes the proof of the lemma. $\boxed{QED}$

4.1 Algorithm for constructing $r-(sn,q)$-spreads

The approach that we have taken to construct $r-(sn)$-spreads will be as follows:

I. Choose a j-(0-set) then choose any of the $\binom{r}{r-j}$ $j-$(0-subsets).

II.

(a) Within this subset choose an ordered set E_1 of exponents of q,

$$(\lambda_1, \lambda_2, \dots, \lambda_{r-j-1}), \text{ where } 0 \leq \lambda_i \leq sn-1, \text{ for each } i = 1, 2, \dots, r-j-1.$$

(b) Choose an ordered set C_1 of coefficients $(n_1, n_2, \ldots, n_{r-j-1})$.

(c) From (a) and (b), form the corresponding sn-dimensional $GF(q)$-subspace:

$$y = (x_1^{*q^{\lambda_1}} n_1, \ldots, x_1^{*q^{\lambda_{r-j-1}}} n_{r-j-1}).$$

III. Determine the minimal extended André partial spread non-trivially intersecting the given subspace. This will be

$$A_{(n_1,\ldots,n_{r-j-1})} = \left\{ y = \left(x_1^* n_1 \tau^{\frac{(q^{\lambda_1}-1)}{(q-1)}}, x_1^* n_2 \tau^{\frac{(q^{\lambda_2}-1)}{(q-1)}}, \ldots, x_1^* n_{r-j-1} \tau^{\frac{(q^{\lambda_{r-j-1}}-1)}{(q-1)}} \right); \tau \text{ has order dividing } \frac{(q^{sn}-1)}{(q-1)} \right\},$$

which has $\frac{(q^{sn}-1)}{\left(q^{(\lambda_1,\lambda_2,\ldots,\lambda_{r-j-1},sn)}-1\right)}$ components

IV. Apply the kernel group of order $(q^{sn}-1)$ to

$$y = \left(x_1^{*q^{\lambda_1}} n_1, \ldots, x_1^{*q^{\lambda_{r-j-1}}} n_{r-j-1} \right).$$

This constructs the following replacement partial spread.

$$A_{(n_1,\ldots,n_{r-j-1})}^{(\lambda_1,\ldots,\lambda_{r-j-1})} = \left\{ y = \left(x_1^{*q^{\lambda_1}} n_1 d^{1-q^{\lambda_1}}, \ldots, x_1^{*q^{\lambda_{r-j-1}}} n_{r-j-1} d^{1-q^{\lambda_{r-j-1}}} \right); d \in GF(q^{sn})^* \right\}.$$

V. There are

$$(q^{sn}-1)^{r-j-1} - \frac{(q^{sn}-1)}{\left(q^{(\lambda_1,\lambda_2,\ldots,\lambda_{r-j-1},sn)}-1\right)}$$

sn-dimensional $GF(q)$-subspaces remaining in Σ_{j,w_1}, and return to II and choose another ordered set E_2 of exponents $(\lambda_1^*, \lambda_2^*, \ldots, \lambda_{r-j-1}^*)$ and another ordered set C_2 of coefficients $(n_1^*, n_2^*, \ldots, n_{r-j-1}^*)$ such that the following set of equations is not valid, for any τ_1, τ_1^* of order dividing $\frac{(q^{sn}-1)}{(q-1)}$:

$$n_i \tau_1^{\frac{(q^{\lambda_i}-1)}{(q-1)}} = n_i^* \tau_1^{*\frac{(q^{\lambda_i^*}-1)}{(q-1)}}; i = 1, 2, \ldots, r-j-1.$$

This makes the corresponding replacement partial spreads and the corresponding generated extended André partial spread mutually disjoint on $sn - GF(q)$-subspaces.

Repeat II, III, IV. There are

$$(q^{sn}-1)^{r-j-1}-\frac{(q^{sn}-1)}{\left(q^{(\lambda_1,\lambda_2,\ldots,\lambda_{r-j-1},sn)}-1\right)}-\frac{(q^{sn}-1)}{\left(q^{(\lambda_1^*,\lambda_2^*,\ldots,\lambda_{r-j-1}^*,sn)}-1\right)}$$

remaining $sn-$dimensional $GF(q)$-subspaces remaining.

VI. If this number at least $\frac{(q^{sn}-1)}{(q-1)}$, go back to II and repeat II, III, IV. Continue as long as possible.

VII. Do this for each j and for each of the $\binom{r}{r-j}$ j-(0-subsets).

Note that when the process terminates, we add to our sn-spreads whatever left over Desarguesian 1-spaces over $GF(q^{sn})$ remain.

Let D_{sn} denote the set of divisors of sn (including 1 and sn). When a replacement set of $\frac{(q^{sn}-1)}{(q^{d^*}-1)}$ sn-spaces is obtained for $d^*\in D_{sn}$, let k_{d^*} denote the number of different and mutually disjoint replacement sets of $\frac{(q^{sn}-1)}{(q^{d^*}-1)}$ sn-spaces (k_{d^*} could be 0). Then we merely require that

$$\sum_{d^*\in D_{sn}}\left(\frac{q^{sn}-1}{q^{d^*}-1}\right)k_{d^*}=(q^{sn}-1)^{r-j-1}.$$

For example, if we take $k_{d^*}=(q^{d^*}-1)k_{d^*}^*$, then we would require that

$$\sum_{d^*\in D_{sn}}k_{d^*}^*=(q^{sn}-1)^{r-j-2}.$$

23 Conclusion. The process above constructs $r-(sn,q)$-spreads by finding replacement sets of extended André sets of $(\frac{q^{sn}-1}{q^{d^*}-1})$ sn-dimensional subspaces for $d^*\in D_{sn}$ that we term 'generalized extended André $r-(sn,q)$-spreads obtained by multiple extended André replacement'.

5 Variations

The algorithm of the previous section, produces a vast number of new $r-(sn,q)$-spreads. In this section, we give a few other constructions, using a variation of the same theme.

5.1 Algorithm for constructing partitions of j-(0-sets)

In this setting, we partition the sets using the action of the subgroup K_s of order q^s-1 of the kernel subgroup K_{sn} of order $q^{sn}-1$

I. Choose a j-(0-set) then choose any of the $\binom{r}{r-j}$ j-(0-subsets).

II_{s^*}: Choose any divisor of s, say s^*.

(a) Within this subset choose an ordered set E_1 of exponents of q,

$$(\lambda_1, \lambda_2, \ldots, \lambda_{r-j-1}), \text{ such that } \gcd(\lambda_1, \lambda_2, \ldots, \lambda_{r-j-1}, s) = s^*,$$

$$\text{where } 0 \leq \lambda_i \leq sn - 1, \text{ for each } i = 1, 2, \ldots, r - j - 1.$$

(b) Choose an ordered set C_1 of coefficients $(n_1, n_2, \ldots, n_{r-j-1})$.

(c) From (a) and (b), form the corresponding sn-dimensional $GF(q)$-subspace:

From (a) and (b), form the corresponding sn-dimensional $GF(q)$-subspace:

$$y = \left(x_1^{*q^{\lambda_1}} n_1, \ldots, x_1^{*q^{\lambda_{r-j-1}}} n_{r-j-1}\right).$$

III. Determine the minimal extended André partial spread non-trivially intersecting the given subspace. This will be

$$A_{(n_1,\ldots,n_{r-j-1})} = \left\{ y = \left(x_1^* n_1 \tau^{\frac{(q^{\lambda_1}-1)}{(q-1)}}, x_1^* n_2 \tau^{\frac{(q^{\lambda_2}-1)}{(q-1)}}, \ldots, x_1^* n_{r-j-1} \tau^{\frac{(q^{\lambda_{r-j-1}}-1)}{(q-1)}} \right); \right.$$
$$\left. \tau \text{ has order dividing } \frac{(q^{sn}-1)}{(q-1)} \right\},$$

which has

$$\frac{(q^{sn}-1)}{(q^{(\lambda_1,\lambda_2,\ldots,\lambda_{r-j-1},sn)}-1)} = \frac{(q^{sn}-1)}{(q^{(\lambda_1,\lambda_2,\ldots,\lambda_{r-j-1},sn)}-1)}$$

components

IV. Apply the kernel group of order $(q^{sn} - 1)$ to

$$y = \left(x_1^{*q^{\lambda_1}} n_1, \ldots, x_1^{*q^{\lambda_{r-j-1}}} n_{r-j-1}\right).$$

This constructs the following replacement partial spread.

$$A_{(n_1,\ldots,n_{r-j-1})}^{(\lambda_1,\ldots,\lambda_{r-j-1})} = \left\{ y = \left(x_1^{*q^{\lambda_1}} n_1 d^{1-q^{\lambda_1}}, \ldots, x_1^{*q^{\lambda_{r-j-1}}} n_{r-j-1} d^{1-q^{\lambda_{r-j-1}}} \right); d \in GF(q^{sn})^* \right\}.$$

V_s: Now determine the orbit lengths under the kernel subgroup of order $(q^s - 1)$, which will turn out to be

$$\frac{(q^s-1)}{(q^{s^*}-1)}.$$

Noting that

$$s^* = \gcd(\lambda_1, \lambda_2, \ldots, \lambda_{r-j-1}, s),$$

we have then partitioned the original

$$\frac{(q^{sn}-1)}{\left(q^{(\lambda_1,\lambda_2,\ldots,\lambda_{r-j-1},sn)}-1\right)}$$

sn-subspaces into

$$\frac{\left(\frac{q^{sn}-1}{q^{(\lambda_1,\ldots,\lambda_{s-j-1},sn)}-1}\right)}{\left(\frac{q^{s}-1}{q^{(\lambda_1,\lambda_2,\ldots,\lambda_{r-j-1},s)}-1}\right)}.$$

In this case, we may either take the same set of exponents or compatible sets so that the number of components is the same and partition the original set of

$$\frac{(q^{sn}-1)}{(q^{(\lambda_1,\lambda_2,\ldots,\lambda_{r-j-1},sn)}-1)}.$$

Hence, if we take

$$\frac{\left(\frac{q^{sn}-1}{q^{(\lambda_1,\ldots,\lambda_{s-j-1},sn)}-1}\right)}{\left(\frac{q^{s}-1}{q^{(\lambda_1,\lambda_2,\ldots,\lambda_{r-j-1},s)}-1}\right)} z_{s^*}$$

orbits of length $(q^s-1)/(q^{s^*}-1)$, we end up with

$$\left(\frac{q^{sn}-1}{q^{(\lambda_1,\ldots,\lambda_{s-j-1},sn)}-1}\right) z_{s^*}$$

total sn-dimensional $GF(q)$-subspaces.

$$\sum_{d^*\in D_{sn}} \left(\frac{q^{sn}-1}{q^{d^*}-1}\right) k_{d^*} = (q^{sn}-1)^{r-j-1}.$$

For example, if let D_s denote the set of divisors of s. If we take $z_{s^*} = (q^{s^*}-1)z^*_{s^*}$, then we would require that

$$\sum_{s^*\in D_{sn}} z^*_{d^*} = (q^{sn}-1)^{r-j-2}.$$

The distinction is that now we have constructed a set of replacement partial spreads that are orbits under a subgroup of order q^s-1 of the kernel homology group of order $q^{sn}-1$.

If we repeat as in the previous section, we have a specific instance of the previous algorithm, this one constructing orbits of various lengths under a subgroup of the kernel group.

6 Large groups

We note that the generalized kernel group $G_{sn,r}$ of order $(q^{sn}-1)^r$ acts on the Desarguesian $r-(sn,q)$-spread and the kernel group of order $(q^{sn}-1)$ also acts on each extended André replacement partial spread, as this is the way that the replacements are determined. Indeed, the generalized kernel group of order $(q^{sn}-1)^r$ is transitive on 1-dimensional $GF(q^{sn})$-subspaces.

Furthermore, the group $G_{sn,r}$ fixes each j-(0-subset) and acts transitively on each set of (non-zero) vectors. Furthermore, if we take a given extended André replacement set

$$\begin{aligned} &A^{(\lambda_1,\dots,\lambda_{r-j-1})}_{(n_1,\dots,n_{r-j-1})} \\ &= \left\{ y = \left(x_1^{*q^{\lambda_1}} n_1 d^{1-q^{\lambda_1}}, \dots, x_1^{*q^{\lambda_{r-j-1}}} n_{r-j-1} d^{1-q^{\lambda_{r-j-1}}} \right); d \in GF(q^{sn})^* \right\}, \end{aligned}$$

the kernel group K_{sn} is transitive on the subspaces and the group R_{sn} of order $q^{sn}-1$

$$\left\langle (x_1,x_2,\dots,x_{r-j-1}) \longmapsto \left(x_1^* m_0, x_1^* m_0^{q^{\lambda_1}}, \dots, x_1^* m_0^{q^{\lambda_{r-j-1}}} \right); m_0 \in GF(q^{sn}) \right\rangle,$$

fixes each sn-dimensional subspace of $A^{(\lambda_1,\dots,\lambda_{r-j-1})}_{(n_1,\dots,n_{r-j-1})}$. Note that the remaining entries that are 0 are omitted. In order that this group act on the $r-(sn,q)$-spread, choose exactly one j and exactly one $j-(0\text{-subset})$. Then the resulting generalized extended André sn-spread will admit a group of order $(q^{sn}-1)^r$ of which there is a group of order $(q^{sn}-1)^2$ that acts transitively on the components of the replaced partial spread and there is a subgroup of order $(q^{sn}-1)^j$ that acts fixes each vector of the $j-(0\text{-subset (set)})$ (just take the j 0-entries to have arbitrary coefficients in $GF(q^{sn})^*$ and take the other coefficients to be 1). Note that the remaining sn-dimensional $GF(q)$ subspaces are actually 1-dimensional $GF(q^{sn})$-subspaces and since $G_{sn,r}$ just maps 1-dimensional $GF(q^{sn})$-subspaces to 1-dimensional $GF(q^{sn})$-subspaces, therefore there is an Abelian group of order $(q^{sn}-1)^{j+2}$, acting on such an sn-spread.

24 Theorem. *Choose any subspace that generates*

$$\begin{aligned} &A^{(\lambda_1,\dots,\lambda_{r-j-1})}_{(n_1,\dots,n_{r-j-1})} \\ &= \left\{ y = \left(x_1^{*q^{\lambda_1}} n_1 d^{1-q^{\lambda_1}}, \dots, x_1^{*q^{\lambda_{r-j-1}}} n_{r-j-1} d^{1-q^{\lambda_{r-j-1}}} \right); d \in GF(q^{sn})^* \right\}, \end{aligned}$$

that in turn generates

$$A_{(n_1,\dots,n_{r-j-1})} = \left\{ y = \left(x_1^* n_1 \tau^{\frac{(q^{\lambda_1}-1)}{(q-1)}}, x_1^* n_2 \tau^{\frac{(q^{\lambda_2}-1)}{(q-1)}}, \dots, x_1^* n_{r-j-1} \tau^{\frac{(q^{\lambda_{r-j-1}}-1)}{(q-1)}} \right); \tau \text{ has order dividing } \frac{(q^{sn}-1)}{(q-1)} \right\}.$$

Now in the same j-(0-subset), for each ordered set of coefficients, form the corresponding extended André set as above. Now partition the associated j-(0-subset) by constructing extended André sets using possibly different ordered sets of coefficients $(n_1, n_2, \dots, n_{r-j-1})$. There are exactly

$$(q^{sn}-1)^{r-j-2}(q^{(\lambda_1,\lambda_2,\dots,\lambda_{r-j-1},sn)}-1)$$

possible extended André sets. Now for each extended André choose either $(\lambda_1, \dots, \lambda_{r-j-1})$ or $(0,0,0,\dots,0)$. Then construct the $r-(sn,q)$-spread obtained by replacing the various $A_{(n_1,\dots,n_{r-j-1})}$ by $A^{(\lambda_1,\dots,\lambda_{r-j-1})}_{(n_1,\dots,n_{r-j-1})}$, or by $A_{(n_1,\dots,n_{r-j-1})}$, where the remaining sn-subspaces are the remaining uncovered 1-dimensional $GF(q^{sn})$-subspaces.

(1) Then any such extended André spread admits an Abelian group of order $(q^{sn}-1)^{j+2}$, which is the direct product of $j+2$ cyclic groups of order $(q^{sn}-1)$, and $r-j \geq 2$.

(2) Let $N_{(\lambda_1,\dots,\lambda_{r-j-1})}$ denote the number of different ordered sets of exponents

$$(\lambda_1^*, \lambda_2^*, \dots, \lambda_{r-j-1}^*),$$

such that

$$\gcd(\lambda_1^*, \lambda_2^*, \dots, \lambda_{r-j-1}^*, sn) = \gcd(\lambda_1, \lambda_2, \dots, \lambda_{r-j-1}, sn).$$

There are then

$$\binom{r}{r-j} \left(2^{(q^{sn}-1)^{r-j-2}(q^{(\lambda_1,\lambda_2,\dots,\lambda_{r-j-1},sn)}-1)} - 1 \right) N_{(\lambda_1,\dots,\lambda_{r-j-1})}$$

proper sn-spreads that admit Abelian groups of order $(q^{sn}-1)^{j+2}$.

6.1 The Ebert-Mellinger $r-(rn,q)$-spreads

Recently, Ebert and Mellinger [3], construct new $r-(rn,q)$-spreads admitting Abelian groups of order $(q^{rn}-1)^2$ that may be constructed with the methods of the previous theorem. The construction in Ebert and Mellinger begins with the construction of new subgeometry partitions in $PG(rn-1, q^r)$ by subgeometries isomorphic to $PG(rn-1, q)$ and $PG(n-1, q^r)$. They describe a 'lifting' procedure that constructs new classes of $r-(rn, q)$-spreads (in our notation). As their method is completely different than ours, so we will describe the spreads using Theorem 24, take $r = s$ and $j = 0$ and $(\lambda_1, \lambda_2, \ldots, \lambda_{r-1}) = (q, q^2, \ldots, q^{r-1})$. Then, for any set of coefficients $(n_1, n_2, \ldots, n_{r-j-1})$, for $n_i \in GF(q^{rn})^*$, there is a set of

$$2^{(q^{rn}-1)^{r-2}(q-1)} - 1$$

$r-(rn, q)$-spreads admitting an Abelian group of order $(q^{rn}-1)^2$, which is a direct product of cyclic groups of orders $(q^{rn}-1)$.

These $r-(rn, q)$-spreads are the ones due to Ebert and Mellinger by their lifting methods.

Now when $r = 2$, the corresponding $2-(2n, q)$-spread corresponds to a translation plane of order q^{2n}. Ebert and Mellinger point out that due to the group action, this translation plane is a generalized André plane. However, from Theorem 24, the plane is necessarily an André plane of order q^{2n}.

25 Theorem. *The $r-(rn, q)$-spreads of Ebert and Mellinger are extended André spreads. When $r = 2$, the spreads correspond to André planes of order q^{2n}.*

26 Remark. In our constructions of generalized extended André spreads, we have found replacements (the extended André replacements) of extended André partial spreads of various sizes using the kernel homology group of order $q^{sn}-1$. Hence, all of our new spreads necessarily admit the kernel group of order $q^{sn}-1$. It is an open question whether is might be possible to find replacements of the extended André partial spreads that do not admit this kernel group. Any such sn-spreads would necessary be non-isomorphic to any of the spreads we construct in this article.

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