The Hall plane of order 9-revisited

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Abstract. It is shown that the Hall plane of order 9 admits a collineation group isomorphic to $SL(2,3)$ generated by Baer 3-collineations, whose Baer axes are disjoint as subspaces, where the union of whose components completely cover the components of the Hall plane. This group acts doubly transitive on a set of four points on the line at infinity of the plane.

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1   Introduction

The Hall planes of order q^2 are now well known to be exactly those translation planes that may be derived from an affine Desarguesian plane of order q^2 by the replacement (i.e. derivation) of a single regulus net. When q is not 3, the full collineation group of the Hall plane of order q^2 has component orbits of lengths $(q+1)$ and (q^2-q) and admits $SL(2,q)$ acting doubly transitively on the orbit of length $q+1$. However, when q is 3, the situation is truly remarkable. It is known by Hall [8], that the group is transitive on the components and that if G is the full translation complement and T is the translation group that GT is doubly transitive on the affine points. Furthermore, by André [1], the set of ten infinite points may be partitioned into five sets of two each, which are permuted by GT. Indeed, André [1] shows that the group action on this set of 5 is S_5 and

that the full collineation group has order $2^5 \cdot 5!$. It would not seem that like there is much more to be said about the collineation groups of the Hall planes. This is where the work of Foulser [4] on groups generated by Baer p-collineations in translation planes of order $p^{2r} = q^2$, p a prime, becomes very much of interest.

When $p > 3$, Foulser [4] shows that the group generated by Baer p-collineations in the translation complement in the non-solvable case is $SL(2, p^t)$ and the Baer axes (subspaces pointwise fixed by the Baer p-groups) are mutually disjoint as vector subspaces and 'line up', in the sense that all such Baer subplanes are in the same net of degree $q + 1$. When $p = 3$ and one makes the assumption that the Baer subplanes pointwise fixed by Baer 3-groups as mutually disjoint as vector subspaces, then, assuming that there are at least two Baer subplanes, the group generated is either $SL(2, 3)$ or $SL(2, 5)$. Furthermore, it is possible to prove, as Foulser [4] does, if $SL(2, 3^t)$, for $t > 1$, is generated by Baer 3-groups then the Baer axes are disjoint and line up in the same net of degree $q + 1$. When the group is $SL(2, 5)$, two of the authors (Jha and Johnson [9]) show if the group is generated by Baer 3-collineations then indeed the 10 Baer subplanes are mutually disjoint as vector subspaces and line up on a net of degree $q + 1$.

This leaves the question of the group $SL(2, 3)$ as generated by Baer 3-collineations in translation planes of order 3^{2r} and whether the four Baer subplanes pointwise fixed by the Baer 3-collineations are mutually disjoint and/or whether all lie in the same net of degree $3^r + 1$. Actually, Foulser [5] shows that there are translation planes of order 81 that admit overlapping Baer 3-axes (i.e. the axes are not disjoint) and whose Baer groups generate $SL(2, 3)$, and, of course, the Baer axes cannot lie in the same net of degree 10. Indeed such a phenomenon exists in the Hall planes as well.

But, here we are interested in asking if there are translation planes of order 3^{2r} admitting $SL(2, 3)$ generated by Baer 3-elements, where the Baer axes are now disjoint but still do not lie in the same net of degree $3^r + 1$. Actually, this problem arises in the analysis of the following problem:

1 Problem. Completely determine those projective translation planes π^+ of order q^2, extending the affine translation planes π that admit a collineation group G that acts doubly transitive on a set Γ of $q + 1$ points.

Since we are interested in $SL(2, p^t)$ groups, if we assume that the group G is $SL(2, q)$ or $PSL(2, q)$ then the problem is solved by the work of Foulser and Johnson.

2 Theorem (Foulser and Johnson [6, 7]). *Let π denote an affine translation plane of order q^2. Assume that π admits a collineation group G in the translation complement isomorphic to $SL(2, q)$.*

Then π is one of the following planes:

(1) *Desarguesian,*

(2) *Hall,*

(3) *Hering,*

(4) *Ott-Schaeffer*

(5) *one of three planes of Walker of order* 25 *or*

(6) *the Dempwolff plane of order* 16.

Going back to the problem listed above, the question is in the projective plane π^+, which of these types of translation planes admit a group isomorphic to $SL(2, q)$ acting doubly transitive on a set of $q + 1$ points? The Desarguesian, Hering and Ott-Schaeffer planes do admit such permutation group actions and we in this note show that the Hall planes of order larger than 9, the Walker planes of order 25 (which are not Hering) and the Dempwolff plane of order 16 'never' admit such groups.

This leaves the Hall plane of order 9. There is a natural group $SL(2, 3)$ generated by Baer 3-collineations and inherited from the associated Desarguesian plane constructing the Hall plane. However, this group has infinite point orbit lengths of 1 or 6. It would appear that the Hall plane of order 9 does not admit a group isomorphic to $SL(2, 3)$ that acts doubly transitive on a set of four points. However, in this note, we show that in fact, it does! Furthermore, the group is generated by Baer 3-collineations whose Baer axes are mutually disjoint as vector spaces but 'do not' fall into the same net of degree $4 + 1$.

Actually, we prove the following theorem.

3 Theorem. *The Hall plane π of order* 9 *admits exactly* 10 *sets* Γ_i, $i = 1, 2, \ldots, 10$, *of four infinite points each and* 10 *collineation groups* G_i *isomorphic to* $SL(2, 3)$, *where* G_i *acts doubly transitively on* Γ_i. *Each group is generated by Baer* 3-*collineations, where the Baer axes are disjoint as vector spaces and the union of whose components is the complete spread of* π.

So, we see that there is an associated problem which is of interest.

4 Problem. Determine the projective translation planes of order q^2 admitting two distinct sets Γ_1 and Γ_2 of $q + 1$ points each such that there exist collineation groups G_1 and G_2 such that G_i acts doubly transitive on Γ_i, for $i = 1, 2$.

We solve part of this problem as follows:

5 Theorem. *Let π^+ be a projective translation planes of order q^2 admitting two distinct sets Γ_1 and Γ_2 of $q + 1$ points each such that there exist collineation*

groups G_1 and G_2 such that G_i acts doubly transitive on Γ_i, for $i = 1, 2$ and is isomorphic to $SL(2, q)$. Then π^+ is either Desarguesian or Hall of order 9.

2   Hering and Ott-Schaeffer cases

In general, if a group G acts doubly transitively on a set Γ of $q+1$ points on a projective plane of order q^2, Γ could be on a line, a $(q+1)$-arc, or a $2-(v, k, 1)$-design. However, it turns out that the latter case can never occur (see e.g. [3]), when considering a non-Desarguesian projective translation plane. So, here we consider only when Γ is contained in a line or is possible a $(q+1)$-arc. There are more general arguments given in [3], showing that affine cases do not occur. But, the arguments can be given more directly for the individual planes and so these are included here for the various classes of planes Hering, Ott-Schaeffer, Hall, Walker of order 25 and Dempwolff of order 16.

Assume the plane is Hering or Ott-Schaeffer. There is a group H_0 isomorphic to $SL(2, q)$, fixing an affine point 0, with three infinite point orbits, one of length $q+1$ and two of length $(q^2 - q)/2$. The full collineation group of either plane is $N_{\Gamma L(4,q)}(H_0)T$, where T is the translation group of order q^4 (see e.g. Lüneburg [11], p. 262, 50.3). If Γ is affine and G contains a normal subgroup W isomorphic to $SL(2, q)$ acting doubly transitively on Γ then W contains no translations. Assume that Γ is contained in an affine line L, so that W leaves L invariant and is a subgroup of $N_{\Gamma L(4,q)}(H_0)T$. The subgroup H_{0_L} of H_0 that fixes L fixes exactly q points on L and these points are within a 1-dimensional $GF(q)$-subspace, and is transitive on the remaining q 1-dimensional $GF(q)$-subspaces. Assume that Γ contains the zero vector 0. This means that one point of Γ may be considered the zero vector 0 and so W_0 permutes $q+1$ 1-dimensional $GF(q)$-subspaces and the unique Sylow S_0 p-subgroup of order q of W_0 must fix one X and since W is a subgroup of $GL(4, q)T$, it follows that W_0 is a subgroup of $GL(4, q)$. Therefore, S_0 must fix X pointwise. There is a Sylow p-subgroup S^{H_0} of H_0 that fixes X pointwise. Assume that S_0 is not a subgroup of S^{H_0}. Since S^{H_0} fixes X pointwise, then S_0 normalizes S^{H_0}, as it fixes X. Then in $S^{H_0}S_0$ there is a collineation group that fixes L pointwise, a contradiction (see e.g. Lüneburg [11], p. 261, 50.1). Hence, W_0 fixes one 1-dimensional $GF(q)$-subspace of L pointwise and is transitive on the remaining q 1-dimensional $GF(q)$-subspaces on L. Therefore, Γ consists of 0 and one point from each of q 1-dimensional $GF(q)$-subspaces. The normalizer $N_{H_0}(S_0)$ of S_0 in H_0 has order $q(q-1)$, fixes X and fixes a second 1-dimensional subspace X_1 incident with L and is transitive on the remaining $q-1$ one dimensional $GF(q)$-subspaces on L. Note that G is a subgroup of $GL(2, q)$ as it fixes L, fixes 0 and is in $GL(4, q)T$ and fixes a 1-dimensional $GF(q)$-subspace X. Hence, the order of G

must divide $q(q-1)^2$. Let Z^* denote the kernel homology group of order $q-1$. Then, $Z^* \cap W$ and $Z^* \cap H_0$ are groups of order dividing 2. Thinking of L as a Desarguesian spread, it follows that $N_{H_0}(S_0)$ is normal in G and $N_{H_0}(S_0)Z^*$ has index 1 or 2 in G. But, this also says that $W_0 S_0 \leq H_0 S_0 Z^*$. Since H_0 fixes exactly two 1-dimensional $GF(q)$-subspaces X and X_1 on L, it follows that we may assume that $W_0 \leq H_0^t Z^*$, where t is in S_0. Since Z^* fixes all 1-dimensional $GF(q)$-subspaces, it is then clear that we may assume that W_0 and H_0 fix X and X_1 so that $W_0 \leq H_0 Z^*$. However, W_0 must act on Γ and fix X_1, and $X_1 \cap \Gamma$ is a unique point. Therefore, W_0 fixes X_1 pointwise. So, W_0 is forced into being a Baer group of order $q-1$, which is a contradiction to the known action of the collineation groups of the Hering and Ott-Schaeffer planes.

Now assume that Γ is a $(q+1)$-arc. Again, we may assume that 0 is incident with Γ. Let S_0 be a Sylow p-subgroup of W_0. There are then exactly q components $0P$, where $P \in \Gamma - \{0\}$, permuted by a group W_0 of order $q(q-1)$. Note that S_0 is normal in W_0, must fix a component L and fixes a 1-dimensional $GF(q)$-subspace pointwise on L. The previous argument applies to show that S_0 may be considered a subgroup of H_0 and W_0 is a subgroup of $N_{H_0}(S_0)Z^*$. Therefore, as a group S_0 has a unique fixed component L. Hence, L is not $0P$ for any point $P \in \Gamma - \{0\}$. So, S_0 is regular on this set of q components. There is a cyclic subgroup C_0 of W_0 that fixes L and acts on the remaining q-components and hence fixes one a component M. But, M contains a unique point of $\Gamma - \{0\}$, which then must be fixed by C_0. Since we know that C_0 is in $GL(4,q)$, it follows that C_0 fixes a 1-dimensional $GF(q)$-subspace on M pointwise. But, since $W_0 \leq N_{H_0}(S_0)Z^*$, it follows that C_0 also fixes two 1-dimensional subspaces. Hence, C_0 is a Baer collineation group of order $q-1$, a contradiction as noted previously

Hence, if the plane is Hering or Ott-Schaeffer then Γ can only be a set of infinite points of π.

3 The Hall planes of order > 9

Assume that Γ is affine and let 0 be a point of Γ. Assume that Γ is a $q+1$-arc. Form the components $0P$, for $P \in \Gamma - \{0\}$ and let W be a subgroup isomorphic to $SL(2,q)$ that acts doubly transitive on Γ. Let H_0 be the subgroup in the translation complement that fixes 0, which is isomorphic to $SL(2,q)$ and generated by Baer p-collineations, for $q = p^r$. All Baer axes are on the same net N of degree $q+1$, and H_0 is transitive on the set of components outside of N. Consider W_0, the subgroup of W of order $q(q-1)$ and let S_0 denote the Sylow p-subgroup of W_0. S_0 will fix a component L and fix a 1-dimensional $GF(q)$-subspace pointwise on L. Since S_0 is regular on the set of components

$0P$, $P \in \Gamma - \{0\}$, it follows that L is not one of these components.

Therefore, let C_0 be a cyclic subgroup of W_0 and note that C_0 must fix one of the q components, say M. It follows that C_0 will fix a 1-dimensional $GF(q)$-subspace pointwise on W_0. Since C_0 has order $q - 1$, each element of prime power order τ will be a Baer collineation. Since S_0 has order q, will have fix at least one of these components L. But, H_0 then will also fix L (or a group isomorphic to H_0, if the order of the Hall planes is 9). The fixed point space of S_0 on L is then also fixed pointwise by a Baer group B_0 of H_0. Clearly, both S_0 and B_0 are regular on the remaining 1-dimensional $GF(q)$-subspaces of L. Hence, S_0 and B_0 are quartic groups on L, are therefore identical on L and so have identical point orbits. If S_0 is not B_0 and s_0 is not in B_0 then there is an element b_0 of B_0 such that $s_0 b_0$ fixes L pointwise, a contradiction by the known action of the Hall planes. Therefore, S_0 is in H_0, so S_0 is Baer of order q. Hence, the set of q components $0P$, for $P \in \Gamma - \{0\}$, is outside the net N. So, each element τ of prime power of C_0 will fix one of these q-components and is Baer with Baer subplane π_0 outside of the net N. Assume that $q - 1$ is not 2^a, for some a. If $q > 3$, the full collineation group of the Hall plane leaves N invariant. In this case, then there is an element τ_0 of odd prime power order that fixes at least two components of N. This element τ_0 is Baer and then acts on the associated Desarguesian affine plane obtained by the derivation of N still as a Baer collineation of odd order. However, this is a contradiction to the structure theory for Desarguesian affine groups. Therefore, assume that $q - 1 = 2^a$. Therefore, a generator τ of order 2^a will fix a component of the plane and since $(q - 1, q + 1) = 2$, then for $q > 3$, τ^2 will fix a component of N. Hence, again we have a Baer collineation of the associated Desarguesian affine plane of order $(q - 1)/2$ so $(q - 1)/2 = 2$, implying that $q = 5$. If $q = 5$, and there is a Baer collineation in C_0 on Σ but this means that C_0 contains an affine homology acting on the Desarguesian subplane of order 5 fixed pointwise by S_0 in the Hall plane. So, this means that C_0, since Abelian must fix two components of N, which implies that there actually is a Baer group of order 4 in Σ, a contradiction.

So, if q is not 3 then Γ cannot be a $q + 1$-arc.

Assume that Γ is a set of infinite points. Then we may assume that W fixes an affine point by analysis by the authors in the paper [2]. Again, if $q > 3$, there is an invariant net N of $q + 1$ components, which has a subgroup of $GL(2, q)$ that fixes N componentwise, and this is the full subgroup of $GL(2, q^2)$ of the associated Desarguesian affine plane constructing the Hall plane that acts on the Hall plane. It now follows that no element of Γ can lie on the subline at infinity of N. Since W fixes 0, it follows that W is a subgroup of $GL(2, q)Z_2$, where Z_2 is a Baer involution, a contradiction.

Therefore, if q is not 3 then Γ cannot be contained on the infinite line.

Now assume that Γ is contained in an affine line L. Similar arguments as given above will show that W_0 contains a Baer group $B_0 = S_0$ of H_0 and if $W_0 = S_0 C_0$, then C_0 must fix a non-zero point and hence fix a 1-dimensional subspace on L pointwise. It now is easy to see that if $q > 3$, then C_0 must be a central collineation group of order $q - 1$ of the associated Desarguesian affine plane Σ constructing the Hall plane. Hence, $W_0 \leq GL(2,q)$ acting on Σ. Moreover, since $W \leq GL(2,q)TZ_2$, where T is the translation group of order q^4 and Z_2 is a Baer involution, it follows that $W = \langle S_0, S_0^g \rangle$, for some element g of $GL(2,q)TZ_2$. This means that S_0^g is a Baer collineation fixing a line of Σ pointwise. Clearly, S_0 and S_0^g must have a common fixed point since they fix pointwise line of Σ that are in distinct parallel classes. Hence, we may then assume that that common point is 0, without loss of generality, which implies that W is a subgroup of $GL(2,q)$, and hence W is H_0. But, then the group of order $q - 1$ must be Baer in H_0, a contradiction, since H_0 is $SL(2,q)$.

If $q > 3$ then Γ cannot be contained in an affine line.

Summary: The Hall planes of order q^2, for q not 3 cannot admit a collineation group isomorphic to $SL(2,q)$ acting doubly transitive on $q + 1$ points on the projective extension.

4   The Walker planes of order 25

There are three Walker planes of order 25, of which one is the Hering plane of order 25. In the other two planes, there is always a component fixed L, under the full translation complement. Hence, under a group W isomorphic to $SL(2,q)$, this means that W fixes an infinite point. Furthermore, all elements of order 5 are quartic (fix exactly $q = 5$ affine points). This means that Γ must lie on a component L or be a 6-arc. If this case, the stabilizer of a point 0 of Γ contains a group S_0 of order 5, which is then a quartic group and fixes exactly $q = 5$ points of L. Let S_1 be another subgroup of order 5 of W which then fixes exactly 5 points on a line M in the same parallel class (α) as L. Suppose $M = L$. If the fixed point spaces share a point then W fixes an affine point 0, since otherwise, there would $6 \cdot 5$ affine points on L. If M is not L, there there are exactly 6 lines of (α) permuted by W. But there are exactly 25 affine lines on (α), implying that W permutes the remaining 19. But, then S_0 would fix a second line of (α), a contradiction to the known action of S_0. Hence, we may assume that W fixes an affine point. If Γ lies on a component, this means that the points of Γ lie one each in each of the 6 $GF(5)$-subspaces relative to 0. That normalizer of a group of order 5 then fixes 0 and fixes a second point of Γ, implying that the element

fixes a second and hence third 1-dimensional $GF(5)$-subspace, both pointwise. This means that there is a group of order 4 that fixes L pointwise, but then there is a normal subgroup of $SL(2,5)$ that fixes L pointwise, a contradiction. Clearly Γ cannot be on the infinite line so assume that Γ is a 6-arc, so in this case, we do not assume that Γ contains the point 0 fixed by W. But then there is a point of Γ fixed by S_0 which is outside of L, forcing S_0 to be a Baer group, a contradiction.

Hence, the Walker planes of order 25 **do not admit a collineation group isomorphic to** $SL(2,4)$ **acting doubly transitive on** 6 **points**.

5 The Dempwolff plane of order 16

This leaves us to consider the Dempwolff plane of order 16. From Johnson [10], we have the following results. The Dempwolff plane of order 16 has $\Gamma L(2,4)$ as the full translation complement and (1) has orbits lengths on the line at infinity of size $1, 1, 15$, (2) The groups of order 4 of $SL(2,4)$ fix Baer subplanes pointwise that share 0 but otherwise are disjoint, (3) the elements of order 3 in the center of $GL(2,4)$ are affine homologies. Let W be a group isomorphic to $SL(2,4)$ and acting doubly transitively on Γ. If Γ is on the infinite line then we may assume that W fixes an affine point 0. However, the group action of the translation complement provides such an action.

Hence, assume that Γ is affine. First assume that Γ lies on an affine line L, so clearly L is one of the component orbits of length 1. Let 0 be an element of Γ. Our group is in $SL(2,4)T$, where T is the translation group leaving L invariant. Now there is a subgroup B fixing 0 of order 4 which then is in $SL(2,4)$ and hence is Baer. L becomes a $GF(4)$-subspace and there is a spread induced on L by the fixed point spaces of Baer groups of $SL(2,4)$. Since B is transitive on this set of 4 remaining subspaces, again it follows that the points of Γ fall one each in 5 $GF(4)$-subspaces on L (one point is 0). The normalizer of order 3 of B must then fix another point, fix the corresponding $GF(4)$-subspace pointwise and fix another $GF(4)$-subspace pointwise, implying we have a homology group of order 3 in $SL(2,4)$, a contradiction.

Finally assume that Γ is a $16+1$-arc and let 0 be an element of Γ, so there is a group W_0 of order $4 \cdot 3$ fixing 0. All 2-groups that fix an affine point are Baer groups (noting that any such Baer axis defines a derivable net by Johnson [10]).

6 The Hall plane of order 9

The 3-elements in the Hall plane of order 9 are Baer. It is known that the full collineation group G has order $2^5 \cdot 5! \cdot 9^4$ (see e.g. Lüneburg [11] p. 36, 8.3).and G

acts on a set of five pairs of infinite points $\{(\infty),(0)\}^G$ exactly as S_5, so there is a normal subgroup Z of order 2^5 that fixes each of these five pairs. Every group of order 3 must fix two of these pairs and when the 3-element fixes a pair, it must fix the pair pointwise. Hence, every group B_1 of order 3 fixes exactly two pairs and has two orbits of length 3 on the remaining pairs. Suppose that we have a group G that acts doubly transitive on a set Γ of four points on the line at infinity. It follows that if we have a group acting 2-transitively on 4 points on the line at infinity then the four points are all from different pairs. This means that the group G must fix the 5th pair, implying that all of the 3-elements will fix the same two infinite points. If two Baer 3-groups have the same set of infinite fixed points then there is a group $SL(2,3)$ fixing four points and acting transitively on the remaining 6 points, but no subgroup is doubly transitive on a set of four infinite points. Assuming that G fixes an affine point, no two Baer groups of order 3 can fix pointwise the same set of four infinite points. Assume that the order of $\overline{G} > 12$. Then the group stabilizer of two pairs is non trivial. Our group $\overline{G}$ is a subgroup of $A\Gamma L(1,4)$ acting on Γ. There are exactly four Baer groups of order 3 in $\overline{G}$. The normalizer of a Baer group of order 3 in the full group has order 12 and the 2-group is generated by the kernel involution and the Frobenius automorphism collineation of the associated Desarguesian affine plane. An involution in the normalizer of a Baer 3-group must permute the two orbits of length 3 and by our assumptions must fix both of these. This means that this involution is the kernel involution which would not act in $\overline{G}$. Hence, $\overline{G}$ has order 12 and is generated by two Baer 3-elements and so is $AGL(1,4)$. If the Baer axes are not disjoint, it follows from Foulser p. 115, [5] that the group generated would contain a quaternion homology group of order 8 (the group action also works in the order 9 case), which has an infinite orbit of length 8. So, the Baer axes are disjoint and share exactly two components. There is such a group action, as is now demonstrated.

We create a convenient coordinate system for the Hall plane of order 9 as follows. Let

$$x = 0, y = x \begin{bmatrix} u & -t \\ t & u \end{bmatrix}; u, t \in GF(3)$$

be the spread for a Desarguesian affine plane Σ of order 9. Derive the net with partial spread R, where

$$R = \{x = 0, y = x\alpha; \alpha \in GF(3)\}.$$

Apply the 'Albert Switch' to determine the spread for the Hall plane. That is, coordinates for the Desarguesian plane (x_1, x_2, y_1, y_2) are switched by (x_1, y_1, x_2, y_2) to determine the spread for the Hall plane. Let $x_1^* = x_1, x_2^* = y_1$, so that

$y = x \begin{bmatrix} u & -t \\ t & u \end{bmatrix}$, for t non-zero as $(x_1, x_2, x_1 u + x_2 t, -x_1 t, +x_2 u)$ becomes

$$y = x \begin{bmatrix} -ut^{-1} & -(t + t^{-1} u^2) \\ t^{-1} & ut^{-1} \end{bmatrix}.$$

Now $t^{-1} = t$ and $u^2 = 0$ or 1. So the spread for the Hall plane π is

$$x = 0, \ y = x\alpha; \ \alpha \in GF(3), \ y = x \begin{bmatrix} -ut & -(t + tu^2) \\ t & ut \end{bmatrix}.$$

We know that the Hall plane is the regular nearfield plane of order 9, so it follows that

$$g : (x, y) \mapsto (x, y) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

is an affine homology of π. We also know that

$$B = \left\langle \tau = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\rangle,$$

is a collineation group (Baer 3-group) of π, inherited from an affine elation of Σ as is B^t, where

$$B^t = \left\langle \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \right\rangle.$$

If we conjugate B^t by g, we then obtain a Baer group $B_1 = B^{tg}$ of π, where an easy calculation shows that B_1 is as follows:

$$B_1 = \left\langle \rho = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \right\rangle.$$

We now take the group $\langle B, B_1 \rangle$. Now B fixes $\{(0, x_2, 0, y_2)\} = \pi_1$ pointwise and B_1 fixes $\{(x_1, 0, y_1, y_1)\} = \pi_2$ pointwise, for all x_i, y_i, $i = 1, 2$ in $GF(3)$. It follows directly that τ maps π_2 onto $\pi_3 = \{(x_1, x_1, y_1, -y_1), \text{ and } \tau^2 \text{ maps } \pi_2 \text{ onto } \pi_3 = \{(x_1, -x_1, y_1, 0)\}$. Moreover, ρ of B_1 maps π_1 onto π_3 and ρ^2 maps π_1 onto π_4. Hence, we observe the following: $\{\pi_1, \pi_2, \pi_3, \pi_4\}$ is a set of

mutually disjoint Baer subplanes (as subspaces) in an orbit under the group $\langle B, B_1 \rangle$. Since the groups are Baer and the Baer subspaces are mutually disjoint, it follows essentially from the ideas behind the Hering-Ostrom theorem that $\langle B, B_1 \rangle \simeq SL(2,3)$ and furthermore that the center of the group is -1 on both of the fixed axes $x = 0, y = 0$. This means that the central involution is the kernel involution of π. Hence, the group restricted to the line at infinity is $PSL(2,3)$ and acts on a set of 8 infinite points. The elementary Abelian normal subgroup A has two orbits of length 4, which are necessarily permuted by every element $\overline{\tau}$ of order 3. Hence, $\overline{\tau}$ fixes both of the A-orbits of length 4 and necessarily fixes exactly one point in each orbit (since the Baer group also fixes $x = 0, y = 0$ and has exactly four fixed infinite points). Therefore, we have two sets Γ_1 and Γ_2 that admit collineation groups acting doubly transitive on them.

Since there are five possible ways to choose the fixed axes $x = 0, y = 0$, we observe the following theorem.

6 Theorem. *The Hall plane of order 9 admits exactly 10 sets Γ_i, $i = 1, 2, \ldots, 10$, of four infinite points each and 10 collineation groups G_i isomorphic to $SL(2,3)$, where G_i acts doubly transitively on Γ_i. Each group is generated by Baer 3-collineations, where the Baer axes are disjoint as vector spaces and the union of whose components is the complete spread of π.*

We also note that the affine cases Γ, when Γ is contained in an affine line or is a 4-arc cannot hold as follows.

Let K denote the kernel of the action on Γ. If Γ is contained in an affine line, then K fixes four points on an affine line L. But, we know from André [1] that the stabilizer of two points in the full collineation group has order 6. But, a Baer 3-element will fix but three affine points. There is at most an involutory homology fixing L pointwise. But, there is an elementary Abelian group A of order 4 induced on Γ and hence induced on L. L becomes a Desarguesian spread of order 3 admitting an elementary Abelian group of order 4. But, A must fix a point there are $9 - 5 = 5$ remaining points of $L - \Gamma$. But, there is no elementary Abelian group of order 4 in the translation complement of a Desarguesian plane of order 3.

So, assume that Γ is a 4-arc. The kernel, if non-trivial, can only be a Baer 3-group. Still there is an elementary Abelian group A of order 4 acting on Γ and further acting on the remaining $9^2 - 4 = 77$ points, implying again that A fixes an affine point 0. So, A does not contain the kernel homology, since A is transitive on a set of 4 components incident with 0. If K is trivial then all involutions of A are forced to be Baer involutions, which cannot occur. So, K is a Baer 3-group. But, then G has order divisible by 9, a contradiction.

Hence, we have

7 Theorem. *Let π be a translation plane of order q^2 and let π^+ denote the*

projective extension. Assume that π^+ admits a collineation group G inducing a two-transitive group isomorphic to $SL(2,q)$ or $PSL(2,q)$ on a set Γ of $q+1$ points.

Then π is one of the following types of planes:

(1) Desarguesian,

(2) Hering,

(3) Ott-Schaeffer, or

(4) the Hall plane of order 9.

(5) Furthermore, if the plane π^+ is not Desarguesian then Γ is a set of infinite points of π.

8 Theorem. *Let π^+ be a projective translation plane of order q^2 admitting two distinct sets Γ_1 and Γ_2 of $q+1$ points each such that there exist collineation groups G_1 and G_2 such that G_i acts doubly transitive on Γ_i, for $i = 1, 2$ and is isomorphic to $SL(2,q)$. Then π^+ is either Desarguesian or Hall of order 9.*

PROOF. By our previous result, we need only be concerned with the Hering and Ott-Schaeffer planes. Recalling that the orbit lengths of components is $q+1, (q^2-q)/2, (q^2-q)/2$, it follows that the only set of $q+1$ points possible must be the set of infinite points in the orbit of length $q+1$ since the generated group can only be $SL(2,q)$. Hence, the only remaining cases are the Desarguesian and Hall of order 9.   QED

9 Remark. The above theorem is actually must more general and can be proved without the assumption that the groups G_i are isomorphic to $SL(2,q)$. This is done by the authors in [3].

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