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Groups in which every subgroup is almost pronormal

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Abstract. A subgroup of a group is called almost normal if it has only finitely many conjugates, or equivalently if its normalizer has finite index in the group. A famous theorem by B.H. Neumann states that all subgroups of a group G are almost normal if and only if the centre $Z(G)$ has finite index in G. Here the structure of groups in which every subgroup is pronormal in a subgroup of finite index is investigated.

Keywords: pronormal subgroup, almost normal subgroup

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1 Introduction

A subgroup X of a group G is said to be *pronormal* if for every element g of G the subgroups X and X^g are conjugate in $\langle X, X^g \rangle$. The concept of a pronormal subgroup arises naturally in many questions; it was introduced by P. Hall, and the first general results about pronormality appeared in a paper by J.S. Rose [12]. Obvious examples of pronormal subgroups are normal subgroups and maximal subgroups of arbitrary groups; moreover, Sylow subgroups of finite groups and Hall subgroups of finite soluble groups are always pronormal. Recently, several authors have investigated the behaviour of pronormal subgroups, and in particular infinite groups which are rich in pronormal subgroups have been studied (for more details, see the survey article [5]). It is well known that pronormal subgroups are strictly related to the structure of T-groups (recall that a group G is called a *T-group* if normality in G is a transitive relation, i.e.

if every subnormal subgroup of G is normal, and G is a $\bar{T}$*-group* if all its subgroups are T-groups); in fact, since a subgroup of a group is normal if and only if it is ascendant and pronormal, groups in which all subgroups are pronormal must have the property $\bar{T}$. Moreover, it has been proved by T.A. Peng [8] that a finite group is a soluble T-group if and only if all its subgroups are pronormal, and this result has recently been extended to some classes of infinite groups (see [4]).

A subgroup X of a group G is called *almost normal* if the conjugacy class of X in G is finite, or equivalently if the normalizer $N_G(X)$ has finite index in G. A famous theorem of B.H. Neumann [7] states that all subgroups of a group G are almost normal if and only if the centre $Z(G)$ of G has finite index. Recently, similar conditions have been considered for other relevant types of subgroups, like for instance permutable and modular subgroups (see [2]). The aim of this paper is to study the situation obtained by replacing in the above context normality by pronormality. Thus we shall say that a subgroup X of a group G is *almost pronormal* if there exists a subgroup K of G containing X such that X is pronormal in K and the index $|G : K|$ is finite. Among other results, we shall prove that any finitely generated soluble-by-finite group in which all cyclic subgroups are almost pronormal is finite over its centre; it follows that torsion-free soluble-by-finite groups with such property are abelian. On the other hand, an example will be given in the last section in order to show that soluble groups with all subgroups almost pronormal need not contain a subgroup of finite index with the property T. Finally, we mention that groups in which every subgroup contains a pronormal subgroup of finite index have recently been studied (see [3]).

Most of our notation is standard and can for instance be found in [10].

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2 Solubility

It is well known that a subgroup of a group is normal if and only if it is ascendant and pronormal; it follows that every ascendant subgroup which is almost pronormal has finitely many conjugates.

1 Lemma. *Let G be a group in which every subgroup is almost pronormal, and let H be a locally nilpotent ascendant subgroup of G. Then every subgroup of H is almost normal in G, and in particular H is central-by-finite.*

Proof. Let X be any subgroup of H. As pronormal subgroups of locally nilpotent groups are normal, the normalizer $N_H(X)$ has finite index in H and hence X is subnormal in H. Thus X is ascendant in G and so also almost normal. In particular, all subgroups of H are almost normal and $H/Z(H)$ is finite by Neumann's theorem. $\boxed{QED}$

Our next lemma shows in particular that every finite-by-abelian group with all subgroups almost pronormal is central-by-finite.

2 Lemma. *Let G be a group in which every subgroup is almost pronormal. If G contains a finite normal subgroup N such that G/N is locally nilpotent, then the factor group $G/Z(G)$ is finite.*

Proof. The group G/N is central-by-finite by Lemma 1, so that G has finite commutator subgroup by Schur's theorem and hence each subgroup of G has finite index in its normal closure. Let X be any subgroup of G and let K be a subgroup of finite index of G such that X is pronormal in K. Then $K = X^K N_K(X)$ and so the index $|K : N_K(X)|$ is finite. It follows that the normalizer $N_G(X)$ has finite index in G. Therefore all subgroups of G are almost normal and hence $G/Z(G)$ is finite. $\boxed{QED}$

Of course, there exist infinite simple groups in which every subgroup is pronormal, like for instance Tarski groups (i.e. infinite simple groups whose proper non-trivial subgroups have prime order). On the other hand, we can now prove that generalized soluble groups with all subgroups almost pronormal are metabelian-by-finite. Recall that the *FC-centre* of a group G is the subgroup consisting of all elements of G having only finitely many conjugates, and a group G is called an *FC-group* if it coincides with the FC-centre; in particular, a group has the property FC if and only if all its cyclic subgroups are almost normal. It is well known that if G is any FC-group, then the factor group $G/Z(G)$ is periodic and residually finite (for this and other properties of groups with finite conjugacy classes we refer to the monograph [13]).

3 Theorem. *Let G be a group in which every subgroup is almost pronormal. If G has an ascending series whose factors either are finite or locally nilpotent, then G contains a metabelian subgroup of finite index.*

Proof. As every ascendant subgroup of G is almost normal, it follows from Dietzmann's lemma that G has an ascending normal series

$$\{1\} = G_0 < G_1 < \cdots < G_\alpha < G_{\alpha+1} < \cdots < G_\tau = G$$

whose factors either are finite or locally nilpotent. In order to prove that G contains a hyperabelian subgroup of finite index, it is clearly enough to show that G contains an abelian non-trivial normal subgroup, provided that G is infinite. Assume that this is false, so that for each positive integer n the subgroup G_n is finite by Lemma 2, and replacing G by its infinite subgroup

$$G_\omega = \bigcup_{n\in\mathbb{N}} G_n,$$

it can be assumed without loss of generality that G is a periodic FC-group. In particular, G is residually finite since $Z(G) = \{1\}$. Thus there exists a collection $(E_n)_{n\in\mathbb{N}}$ of finite non-trivial normal subgroups of G such that

$$\langle\, E_n \mid n \in \mathbb{N}\,\rangle = \mathrm{Dr}_{n\in\mathbb{N}}\, E_n;$$

of course, we may choose each E_n to be a minimal normal subgroup of G, so that

$$E = \mathrm{Dr}_{i\in I}\, S_i$$

is the direct product of an infinite collection of finite simple non-abelian groups. Every S_i contains a subgroup X_i which is not pronormal (see [8]), and the direct product

$$X = \mathrm{Dr}_{i\in I}\, X_i$$

must be pronormal in a suitable subgroup K of G with finite index. Since also the core K_G of K has finite index in G, the subgroup $K_G \cap E$ is the direct product of all but finitely many S_i (see [10] Part 1, p.179), and in particular there exists $j \in I$ such that S_j is contained in K. As

$$X = X_j \times \big(\mathrm{Dr}_{i\neq j}\, X_i\big)$$

is a pronormal subgroup of

$$S_j \times \big(\mathrm{Dr}_{i\neq j}\, X_i\big) \leq K,$$

it follows that X_j is pronormal in S_j, a contradiction. Therefore G is hyperabelian-by-finite, and hence we may suppose that G is even hyperabelian.

Let H be the Hirsch-Plotkin radical of G. By Lemma 1 the commutator subgroup H' of H is finite, so that in particular H is nilpotent; thus H/H' is the Hirsch-Plotkin radical of G/H' and $C_G(H/H') = H$. Moreover, it follows from Lemma 1 that all subgroups of H are almost normal in G. Let T be the subgroup consisting of all elements of finite order of H. Then $G/C_G(T/H')$ is abelian-by-finite and $G/C_G(H/T)$ is finite (see [1], Theorem 2.4 and Lemma 2.5), so that G/H is metabelian-by-finite and G is soluble. Since all subnormal subgroups of G are almost normal, the group G is metabelian-by-finite (see [1], Corollary 3.4).
$\boxed{QED}$

3 Groups with almost pronormal cyclic subgroups

It follows from the definition that every finitely generated FC-group is finite over its centre, and this property will be generalized in this section to the case of finitely generated soluble-by-finite groups in which all cyclic subgroups are almost pronormal. Note also that finitely generated soluble groups with all cyclic subgroups pronormal have the property $\bar{T}$, and hence they either are finite or abelian (see [9]).

4 Lemma. *Let G be a finitely generated soluble group in which all cyclic subgroups are almost pronormal. Then G is polycyclic.*

PROOF. Let A be the smallest non-trivial term of the derived series of G. Since every cyclic subgroup of G/A is almost pronormal, by induction on the derived length of G it can be assumed that G/A is polycyclic. Thus A is the normal closure of a finite subset of G. As each cyclic subgroup of A is almost normal in G, it follows that A is finitely generated and G is polycyclic. $\boxed{QED}$

5 Lemma. *Let G be a finitely generated soluble group in which all cyclic subgroups are almost pronormal. Then every subnormal subgroup of G is almost normal.*

PROOF. Let X be a subnormal subgroup of G, and suppose first that X is abelian. As G is polycyclic by Lemma 4, the subgroup X is finitely generated; moreover, each cyclic subgroup of X has finitely many conjugates in G and hence X is almost normal in G. Assume now that X has derived length $k > 1$. By induction on k it can be assumed that the commutator subgroup X' of X is almost normal in G, so that its normalizer $N_G(X')$ has finite index in G. Clearly, all cyclic subgroups of the group $N_G(X')/X'$ are almost pronormal, and hence it follows from the previous case that X is almost normal in $N_G(X')$, and so also in G. $\boxed{QED}$

We can now prove the main result of this section.

6 Theorem. *Let G be a finitely generated soluble-by-finite group whose cyclic subgroups are almost pronormal. Then $G/Z(G)$ is finite.*

PROOF. Let R be the largest soluble normal subgroup of G. Since all subnormal subgroup of R are almost normal by Lemma 5, it follows that R is abelian-by-finite (see [1], Corollary 3.5), and hence G contains a torsion-free abelian normal subgroup A of finite index. Let $\{x_1A, \dots, x_tA\}$ be a finite set of generators of G/A such that the order of each x_iA is a power of a prime number p_i. Moreover, for any $i = 1, \dots, t$, let H_i be a subgroup of finite index of G such that $x_i \in H_i$ and the cyclic subgroup $\langle x_i \rangle$ is pronormal in H_i. Put $B_i = A \cap H_i$, so that $\langle x_i \rangle$ is pronormal in $K_i = B_i\langle x_i \rangle$ and $B_i \cap \langle x_i \rangle = A \cap \langle x_i \rangle$. Consider the subgroup $C_i/B_i \cap \langle x_i \rangle$ consisting of all elements of finite order of $B_i/B_i \cap \langle x_i \rangle$; then C_i is a cyclic normal subgroup of K_i and B_i/C_i is torsion-free. If $\bar{K}_i = K_i/C_i$, for each positive integer n we have that $\bar{K}_i/\bar{B}_i^{p_i^n}$ is a finite p_i-group, so that its pronormal subgroup $\langle \bar{x}_i \rangle \bar{B}_i^{p_i^n}/\bar{B}_i^{p_i^n}$ is normal, and hence

$\langle \bar{x}_i \rangle \bar{B}_i^{p_i^n}$ is a normal subgroup of $\bar{K}_i$. It follows that also

$$\langle \bar{x}_i \rangle = \bigcap_{n>0} \langle \bar{x}_i \rangle \bar{B}_i^{p_i^n}$$

is a normal subgroup of $\bar{K}_i$. Thus $[\bar{B}_i, \langle \bar{x}_i \rangle] = \{1\}$, so that $[B_i, \langle x_i \rangle] \leq C_i$ and hence B_i/C_i is contained in the centre of K_i/C_i. On the other hand, $B_i \cap \langle x_i \rangle$ lies in $Z(K_i)$, so that $C_i \leq Z(K_i)$ and B_i is a subgroup of $Z_2(K_i)$. In particular, $K_i/Z_2(K_i)$ is finite, and hence $\gamma_3(K_i)$ is likewise finite. But

$$\gamma_3(K_i) \leq K_i' \leq B_i,$$

so that $\gamma_3(K_i) = \{1\}$ and K_i is nilpotent. Thus all subgroups of K_i are almost normal by Lemma 5, and $K_i/Z(K_i)$ is finite by Neumann's theorem; as the index $|G : K_i|$ is finite, it follows that also the centralizer $C_G(x_i)$ has finite index in G. Therefore

$$A \cap \left(\bigcap_{i=1}^{t} C_G(x_i) \right)$$

is a subgroup of finite index of G, which is obviously contained in $Z(G)$, and hence $G/Z(G)$ is finite. $\boxed{QED}$

The commutator subgroup of any FC-group is periodic, and the following consequence of Theorem 6 extends this property to groups in which all cyclic subgroups are almost pronormal.

7 Corollary. *Let G be a locally (soluble-by-finite) group in which every cyclic subgroup is almost pronormal. Then the commutator subgroup G' of G is periodic.*

PROOF. Let E be any finitely generated subgroup of G. Then $E/Z(E)$ is finite by Theorem 6, and in particular E' is finite by Schur's theorem. Therefore G' is periodic. $\boxed{QED}$

8 Corollary. *Let G be a torsion-free locally (soluble-by-finite) in which every cyclic subgroup is almost pronormal. Then G is abelian.*

4 Groups with almost pronormal subgroups

Because of Neumann's theorem, the class of groups in which every subgroup is almost normal is strictly smaller than that of FC-groups. A similar consideration for almost pronormality can be done, as there exists a soluble group G such that each cyclic subgroup of G is even pronormal but G contains subgroups which are not almost pronormal. In fact, L.G. Kovacs, B.H. Neumann and H. de Vries [6] constructed an uncountable metabelian group G of exponent 6 in which every subgroup has the property T, so that in particular all cyclic subgroups of G are pronormal; on the other hand, an easy cardinality argument shows that the Sylow 2-subgroups of G cannot be almost pronormal.

If the almost pronormality assumption is imposed to all subgroups instead that just to cyclic subgroups, it is possible to prove that results like those obtained in Section 3 hold within a larger universe. Let $\mathfrak{L}_0$ be the class of linear groups over integral domains, and let $\mathfrak{L}$ be the smallest group class containing $\mathfrak{L}_0$ which is locally and extension closed. Thus $\mathfrak{L}$ is a quite large class and in particular every group having an ascending series with locally (soluble-by-finite) factors belongs to $\mathfrak{L}$.

Recall that a group is said to be *HNN-free* if it contains no subgroups which are proper HNN-extensions; groups with such property have recently been considered in [11]. It is easy to show that a group G is HNN-free if and only if $X^g = X$ whenever X is a subgroup and g is an element of G such that $X^g \leq X$. Thus our next result shows in particular that groups with all subgroups almost pronormal are HNN-free.

9 Lemma. *Let G be a group and let X be an almost pronormal subgroup of G. If $X^g \leq X$ for some element g of G, then $X^g = X$.*

Proof. Let K be a subgroup of finite index of G such that X is pronormal in K. Then $g^n \in K$ for some positive integer n, and so X and X^{g^n} are conjugate in $\langle X, X^{g^n} \rangle = X$. It follows that $X^{g^n} = X$ and hence $X^g = X$. $\boxed{QED}$

10 Corollary. *Let G be an $\mathfrak{L}$-group in which every subgroup is almost pronormal. Then every finitely generated subgroup of G is central-by-finite. In particular, G' is periodic and G locally satisfies the maximal condition on subgroups.*

Proof. Let E be any finitely generated subgroup of G. As G is HNN-free by Lemma 9, the subgroup E is polycyclic-by-finite (see [11], Theorem A) and hence it follows from Theorem 6 that $E/Z(E)$ is finite. $\boxed{QED}$

11 Corollary. *Let G be a torsion-free $\mathfrak{L}$-group in which every subgroup is almost pronormal. Then G is abelian.*

All previous results suggest that (generalized) soluble groups with all subgroups almost pronormal behave similarly to groups with the property FC. However, in contrast to the theory of FC-groups, we will produce here an example of a metabelian non-periodic group G with trivial centre such that all subgroups of G are almost pronormal. The same example also shows that for groups in which all subgroups are almost pronormal a result corresponding to Neumann's theorem does not hold; in fact, in the above mentioned group G every subgroup of finite index contains a subgroup which is not pronormal.

For our purposes, we need an easy lemma on abnormal subgroups. Recall that a subgroup X of a group G is said to be *abnormal* if each element g of G belongs to $\langle X, X^g \rangle$. Clearly, abnormal subgroups are pronormal and self-normalizing; moreover, the normalizer of any pronormal subgroup is abnormal and any subgroup containing an abnormal subgroup is likewise abnormal.

12 Lemma. *Let G be a group and let A be a periodic abelian normal subgroup of G whose non-trivial primary components have prime exponent. If X is an abelian subgroup of G containing an element x such that every subgroup of A is fixed by x and $C_A(x) = \{1\}$, then X is abnormal in XA.*

Proof. Let g be any element of XA, and write $g = ya$ with $y \in X$ and $a \in A$. In order to prove that g belongs to $\langle X, X^g \rangle$, it can obviously be assumed that $a \neq 1$, so that $a = a_1 \dots a_t$ where each a_i has prime order p_i and $p_i \neq p_j$ if $i \neq j$. Then

$$xax^{-1} = (xa_1x^{-1}) \cdots (xa_tx^{-1}) = a_1^{r_1} \cdots a_t^{r_t},$$

where $r_1, \dots, r_t$ are positive integers and $a_i^{r_i} \neq a_i$ for each i since $C_A(x) = \{1\}$. It follows that

$$g^{-1}xg = a^{-1}xa = a_1^{r_1-1} \cdots a_t^{r_t-1}x$$

and hence the product $a_1^{r_1-1} \cdots a_t^{r_t-1}$ is an element of $\langle X, X^g \rangle$. On the other hand $\langle a_i^{r_i-1} \rangle = \langle a_i \rangle$ for all i, so that

$$\langle a \rangle = \langle a_1^{r_1-1} \cdots a_t^{r_t-1} \rangle$$

and $g = ya$ belongs to $\langle X, X^g \rangle$. Therefore X is abnormal in XA. $\boxed{QED}$

13 Example. There exists a metabelian non-periodic group G with trivial centre such that all subgroups of G are almost pronormal, but every subgroup of finite index of G contains some non-pronormal subgroup.

Proof. Let $(p_n)_{n \in \mathbb{N}}$ and $(q_n)_{n \in \mathbb{N}}$ be two sequences of distinct prime numbers such that q_n divides $p_n - 1$ for all n; moreover, for each positive integer n consider a group $\langle a_n \rangle$ of order p_n and let α_n be an automorphism of $\langle a_n \rangle$ of order q_n. Define now an automorphism g of the group

$$A = \mathrm{Dr}_{n \in \mathbb{N}} \langle a_n \rangle$$

by putting $a_n^g = a_n^{\alpha_n}$ for every n, and let $G = \langle g \rangle \ltimes A$ be the semidirect product of A by $\langle g \rangle$. Clearly, $Z(G) = \{1\}$ and all subgroups of A are normal in G. Let X be any non-normal subgroup of G, so that $X = \langle x, U \rangle$, where U is a subgroup of A and $x = ag^k$ with $a \in A$ and $k \neq 0$. As

$$A = C_A(x) \times [A, x],$$

it follows from Lemma 12 that the abelian subgroup $\langle\, x, C_U(x)\,\rangle$ is abnormal in $\langle\, x, C_U(x), [A, x]\,\rangle$ and hence also X is abnormal in $\langle\, x, C_U(x), [A, x]\,\rangle$. On the other hand, $C_A(x) = C_A(g^k)$ is finite, so that $[A, x]$ has finite index in A and $\langle\, x, C_U(x), [A, x]\,\rangle$ has finite index in G. Therefore all subgroup of G are almost pronormal. Finally, every subgroup of finite index H of G neither is periodic nor abelian, so that in particular H does not have the property $\bar{T}$ and hence it contains a cyclic non-pronormal subgroup. $\boxed{QED}$

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