Multiple solutions for a nonlinear Neumann problem involving a critical Sobolev exponent

Jan Chabrowski

Department of Mathematics, University of Queensland,
St. Lucia 4072, Queensland, Australia
`jhc@maths.uq.edu.au`

Abstract. We study the nonlinear Neumann problem (1) involving a critical Sobolev exponent and a nonlinearity of lower order. Our main results assert that for every $k \in \mathbb{N}$ problem (1) admits at least k pairs of nontrivial solutions provided a parameter μ belongs to some interval $(0, \mu^*)$.

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1 Introduction and preliminaries

In this paper we are concerned with the existence of solutions of the nonlinear Neumann problem

$$\begin{cases} -\Delta u = \mu |u|^{2^*-2}u + f(x,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega,\ u > 0 \text{ on } \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^{\mathbb{N}}$ is a bounded domain with a smooth boundary $\partial\Omega$, ν is the outward normal to the boundary, $\mu > 0$ is a parameter and $2^* = \frac{2N}{N-2}$, $N \geq 3$, is the critical Sobolev exponent.

Throughout this work we assume that the nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition and $(*)$ for every $M > 0$ $\sup\{|f(x,s)|;\ x \in \Omega,\ |s| \leq M\} < \infty$.

We impose the following conditions on f:

(f_1) There exist constants $a_1, a_2 > 0$ and $\sigma \in [0, 2)$ such that

$$\frac{1}{2} f(x,s)s - F(x,s) \geq -a_1 - a_2 |s|^{\sigma}$$

for $(x,s) \in \Omega \times \mathbb{R}$, where $F(x,s) = \int_0^s f(x,t)\,dt$.

(f_2) $\lim_{|s|\to\infty} \frac{f(x,s)}{|s|^{2^*-1}} = 0$ uniformly a. e. in $x \in \Omega$.

(f_3) There exist constants $b_1, b_2 > 0$ and $2 < q < 2^*$ such that

$$F(x,s) \leq b_1 + b_2|s|^q$$

for all $s \in \mathbb{R}$ and a. e. in $x \in \Omega$.

(f_4) There exist a constant $c_1 > 0$ and $h \in L^1(\Omega)$ and $\Omega_\circ \subset \Omega$ with $|\Omega_\circ| > 0$ such that

$$F(x,s) \geq -h(x)|s|^2 - c_1$$

for all $s \in \mathbb{R}$ and a. e. in $x \in \Omega$ and

$$\lim_{|s|\to\infty} \frac{F(x,s)}{s^2} = \infty$$

uniformly a.e. in $x \in \Omega_\circ$, where $|\Omega_\circ|$ denotes the Lebesgue measure of the set $\Omega_\circ$.

It is easy to see that $(*)$ and (f_2) yield: for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$|f(x,s)s|, |F(x,s)| \leq \epsilon|s|^{2^*} + C_\epsilon \tag{2}$$

for every $s \in \mathbb{R}$ and a. e. in $x \in \Omega$. The assumption (f_1) replaces the usual Ambrosetti-Rabinowitz type assumption in order to apply the mountain-pass principle.

Solutions of (1) are sought in the Sobolev space $H^1(\Omega)$. We recall that by $H^1(\Omega)$ we denote the usual Sobolev space equipped with the norm

$$\|u\|^2 = \int_\Omega \left(|\nabla u|^2 + u^2 \right) \, dx.$$

We recall that a C^1 functional $\phi : X \to \mathbb{R}$ on a Banach space X satisfies the Palais-Smale condition at level c ($(PS)_c$ condition for short), if each sequence $\{x_n\} \subset X$ such that (i) $\phi(x_n) \to c$ and (ii) $\phi'(x_n) \to 0$ in X^* is relatively compact in X. Finally, any sequence $\{x_n\}$ satisfying (i) and (ii) is called a Palais-Smale sequence at level c (a $(PS)_c$ for short).

Throughout this paper we denote strong convergence by "$\to$" and weak convergence by "$\rightharpoonup$". The norms in the Lebesgue spaces $L^p(\mathbb{R}^{\mathbb{N}})$ are denoted by $\| \cdot \|_p$.

The Neumann problem for the equation

$$-\Delta u + u = Q(x)|u|^{2^*-2}u \ \text{ in } \Omega, \tag{**}$$

where $\lambda > 0$ is a parameter and Q is a positive and continuous function on $\bar{\Omega}$, has an extensive literature. If $Q \equiv 1$, we refer to the papers [1, 3, 4, 6, 21], where the existence of least energy solutions and their properties have been investigated. The least-energy solutions are one-peak solutions and concentrate on $\partial\Omega$ as $\lambda \to \infty$ (see [17], [18]). These results have been extended to the case when Q is not constant in the papers [7] and [8]. The existence of multi-peak solutions has been studied in the papers [11, 22, 23]. If $Q \equiv 1$, then problem (**) obviously admits also constant solutions. If λ is small then these are the only least energy solutions [5]. It appears that the first result on the existence of multiple solutions were given in [23] and [16] for (**) with $Q \equiv 1$. The multiplicity of solutions in these two papers is expressed in terms of a relative category of $\partial\Omega$. If Ω is a ball, $Q \equiv 1$ and the right side of (**) has a nontrivial perturbation of lower order, then there exist infinitely many solutions [9, 10].

The paper is organized as follows. In Section 2 we establish the Palais-Smale condition for the variational functional corresponding to problem (1). To obtain the existence of solutions we apply some versions of the mountain-pass theorem due to E.A.B. Silva [20]. These existence results are discussed in Sections 3 and 4. In this paper we follow the approach from the paper [19] where the existence of multiple solutions for equation (1) with the Dirichlet boundary conditions has been obtained.

2   Palais-Smale condition

We look for solutions of (1) as critical points of the variational functional

$$I_\mu(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \frac{\mu}{2^*} \int_\Omega |u|^{2^*} \, dx - \int_\Omega F(x, u) \, dx.$$

It is easy to see that under our assumptions on f I_μ is a C^1 functional on $H^1(\Omega)$. To show that I_μ satisfies the Palais-Smale condition we shall use the following version of the concentration-compactness principle (see [14]). Let $u_m \rightharpoonup u$ in $H^1(\Omega)$. Then up to a subsequence there exist positive measures μ and ν on $\bar{\Omega}$ such that

$$|\nabla u_m|^2 \rightharpoonup \mu \ \text{ and } \ |u_m|^{2^*} \rightharpoonup \nu$$

weakly in the sense of measure. Moreover, there exist at most countable set J and a collection of points $\{x_j, j \in J\} \subset \bar{\Omega}$ and numbers $\nu_j > 0$, $\mu_j > 0$, $j \in J$, such that

$$\nu = |u|^{2^*} \, dx + \sum_{j \in J} \nu_j \delta_{x_j}$$

and

$$\mu \geq |\nabla u|^2 \, dx + \sum_{j \in J} \mu_j \delta_{x_j}.$$

The numbers ν_j and μ_j satisfy

$$S\nu_j^{\frac{2}{2^*}} \leq \mu_j \ \ \text{if} \ \ x_j \in \Omega, \tag{3}$$

and

$$\frac{S}{2^{\frac{2}{N}}} \nu_j^{\frac{2}{2^*}} \leq \mu_j \ \ \text{if} \ x_j \in \partial\Omega, \tag{4}$$

where S is the best Sobolev constant for the continuous embedding of $H_\circ^1(\Omega)$ into $L^{2^*}(\Omega)$. Here δ_{x_j} denotes the Dirac measure concentrated at x_j. Moreover, we have $\sum_{j \in J} \nu_j^{\frac{2}{2^*}} < \infty$.

1 Proposition. *Suppose that* (f_1) *and* (f_2) *hold. Then for every* $M > 0$ *there exists* $\mu^* > 0$ *such that* I_μ *satisfies the* $(PS)_c$ *condition for* $c < M$ *and* $0 < \mu < \mu^*$.

PROOF. Let $\{u_m\} \subset H^1(\Omega)$ be a $(PS)_c$ sequence with $c < M$. First we show that $\{u_m\}$ is bounded in $H^1(\Omega)$. For large m we have

$$c + 1 + \|u_m\| \geq I_\mu(u_m) - \frac{1}{2}\langle I_\mu'(u_m), u_m \rangle = \frac{\mu}{N} \int_\Omega |u_m|^{2^*} \, dx$$
$$+ \int_\Omega \left[\frac{1}{2} f(x, u_m)u_m - F(x, u_m) \right] \, dx.$$

It follows from (f_1) that

$$c + 1 + \|u_m\| \geq \frac{\mu}{N} \int_\Omega |u_m|^{2^*} \, dx - a_1 |\Omega| - a_2 \int_\Omega |u_m|^\sigma \, dx. \tag{5}$$

In the sequel we always denote by C a positive constant independent of m which may change from one inequality to another. Using the Young inequality we obtain

$$\int_\Omega |u_m|^\sigma \, dx \leq \kappa \int_\Omega |u_m|^{2^*} \, dx + c$$

for every $\kappa > 0$, where $C > 0$ is a constant depending on on κ and $|\Omega|$. Inserting this inequality with $\kappa = \frac{\mu}{2Na_2}$ into (5) we derive

$$\|u_m\|_{2^*}^{2^*} \leq C \left(\|u_m\| + 1 \right) \tag{6}$$

for some constant $C > 0$. To proceed further we use the equality

$$I_\mu(u_m) - \frac{1}{2^*}\langle I'_\mu(u_m), u_m\rangle = \frac{\mu}{N}\int_\Omega |\nabla u_m|^2\, dx$$

$$+ \frac{1}{2^*}\int_\Omega f(x, u_m)u_m\, dx - \int_\Omega F(x, u_m)\, dx. \quad (7)$$

Using (2) we deduce from (7) that

$$\int_\Omega |\nabla u_m|^2\, dx \le C\left(\int_\Omega |u_m|^{2^*}\, dx + \|u_m\| + 1\right).$$

This combined with (6) leads to the estimate

$$\int_\Omega |\nabla u_m|^2\, dx \le C\left(\|u_m\| + 1\right). \quad (8)$$

We now need the decomposition $H^1(\Omega) = \mathbb{R} \oplus V$, where

$$V = \{v \in H^1(\Omega); \int_\Omega v\, dx = 0\}.$$

We equip $H^1(\Omega)$ with an equivalent norm

$$\|u\|_V = \left(\int_\Omega |\nabla v|^2\, dx + t^2\right)^{\frac{1}{2}}$$

for $u = t + v$, $v \in V$ and $t \in \mathbb{R}$. Using this decomposition we can write $u_m = v_m + t_m$, $v_m \in V$, $t_m \in \mathbb{R}$. We claim that $\{t_m\}$ is bounded. Arguing by contradiction we may assume that $t_m \to \infty$. The case $t_m \to -\infty$ is similar. We put $w_m = \frac{v_m}{t_m}$. It then follows from (8) that

$$\int_\Omega |\nabla w_m|^2\, dx \le C\left[t_m^{-2} + t_m^{-1}\int_\Omega \left(|\nabla w_m|^2\, dx + 1\right)^{\frac{1}{2}}\, dx\right].$$

This yields $\int_\Omega |\nabla w_m|^2\, dx \to 0$ and hence $w_m \to 0$ in $L^p(\Omega)$ for every $2 \le p \le 2^*$. Here we have used the fact that the space V equipped with norm $\left(\int_\Omega |\nabla v|^2\, dx\right)^{\frac{1}{2}}$ is continuously embedded into $L^p(\Omega)$ for $2 \le p \le 2^*$. We now observe that

$$t_m^{-2^*}\left(I_\mu(u_m) - \frac{1}{2}\langle I'_\mu(u_m), u_m\rangle\right) = \frac{\mu}{N}\int_\Omega |w_m + 1|^{2^*}\, dx$$

$$+ t_m^{-2^*}\left[\frac{1}{2}\int_\Omega f(x, u_m)u_m\, dx - \int_\Omega F(x, u_m)\, dx\right]. \quad (9)$$

Using (2) and letting $m \to \infty$ in (9) we obtain $\frac{\mu}{N} \int_\Omega dx = 0$. This is a contradiction. Since $\{t_m\}$ is bounded we deduce from (8) that $|\nabla v_m|$ is bounded in $L^2\Omega$. Thus $\{u_m\}$ is bounded in $H^1(\Omega)$. By the concentration-compactness principle we have up to a subsequence that

$$|\nabla u_m|^2 \rightharpoonup \mu \quad \text{and} \quad |u_m|^{2^*} \rightharpoonup \nu$$

in the sense of measure. It is easy to check that the constants ν_j and μ_j from (3) and (4) satisfy $\mu_j = \mu\nu_j$. Therefore, if $\nu_j > 0$, then (a) $\nu_j > \left(\frac{S}{\mu}\right)^{\frac{N}{2}}$ and if $x_j \in \partial\Omega$, then (b) $\nu_j > \frac{1}{2}\left(\frac{S}{\mu}\right)^{\frac{N}{2}}$. Hence the set J is finite. We now consider the inequality

$$I_\mu(u_m) - \frac{1}{2}\langle I'_\mu(u_m), u_m \rangle \geq \frac{\mu}{N} \int_\Omega |u_m|^{2^*}\, dx - a_1|\Omega| - a_2|\Omega|^\alpha \|u_m\|_{2^*}^{2^*(1-\alpha)}, \quad (10)$$

which follows from (f_1), where $\alpha = \frac{2^*-\sigma}{2^*} < 1$. Put $A = a_1|\Omega| + a_2|\Omega|^\alpha$ and

$$\mu^* = \min\left(2^{-\frac{2}{N}}S, \left[\frac{S^{\frac{N}{2}}}{2\left(N(M+A)\right)^{\frac{1}{\alpha}}}\right]^{\frac{1}{\frac{N}{2}-\frac{1}{\alpha}}}\right).$$

It is easy to see that

$$1 < \frac{1}{2}\left(\frac{S}{\mu}\right)^{\frac{N}{2}} \tag{11}$$

and

$$\left(\frac{N(M+A)}{\mu}\right)^{\frac{1}{\alpha}} < \frac{1}{2}\left(\frac{S}{\mu}\right)^{\frac{N}{2}} \tag{12}$$

for $0 < \mu < \mu^*$. In the final part of the proof we show that

$$\int_\Omega d\nu < \frac{1}{2}\left(\frac{S}{\mu}\right)^{\frac{N}{2}}. \tag{13}$$

If $\int_\Omega d\nu \leq 1$, then (11) yields (13). Hence it remains to consider the case $\int_\Omega d\nu > 1$. Letting $m \to \infty$ in (10) we obtain

$$\frac{\mu}{N} \int_\Omega d\nu \leq c + a_1|\Omega| + a_2|\Omega|^\alpha \left(\int_\Omega d\nu\right)^{1-\alpha} \leq (M+A)\left(\int_\Omega d\nu\right)^{1-\alpha}.$$

In view of (12) we have

$$\int_\Omega d\nu \leq \left(\frac{N(M+A)}{\mu}\right)^{\frac{1}{\alpha}} < \frac{1}{2}\left(\frac{S}{\mu}\right)^{\frac{N}{2}}$$

and this establishes inequality (13). Using (13), (a) and (b) we see that $\nu_j = 0$ for every $j \in J$. This means that $\int_\Omega |u_m|^{2^*} dx \to \int_\Omega |u|^{2^*} dx$. It is now routine to show that up to a subsequence $u_m \to u$ in $H^1(\Omega)$.   QED

We point out here that if $\mu = 0$, then the Palais-Smale condition is not satisfied (see [19]).

3   Existence of multiple solutions

First, we recall the symmetric version of the mountain-pass theorem [20].

2 Theorem. *Let $E = V \oplus X$, where E is a real Banach space and $\dim V < \infty$. Let $I \in C^1(E, \mathbb{R})$ be an even functional satisfying $I(0) = 0$ and*

(I_1) *there exists a constant $\rho > 0$ such that*

$$I \mid \partial B_1 \cap X \geq \rho.$$

(I_2) *there exists a subspace $W \subset E$ with $\dim V < \dim W < \infty$ and there exists a constant $M > 0$ such that*

$$\max_{u \in W} I(u) < M.$$

(I_3) *I satisfies the $(PS)_c$-condition for $0 \leq c \leq M$.*

Then I has at least $\dim W - \dim V$ pairs of nontrivial critical points.

To establish the existence of multiple solutions of problem (1) we check that functional I_μ with $0 < \mu < \mu^*$ satisfies the assumptions of Theorem 2. We denote by $\{\lambda_j\}$, $j \in \mathbb{N}$, the eigenvalues of the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega. \end{cases} \tag{14}$$

Let $\{e_i\}$ be the corresponding orthonormal sequence of eigenfunctions. The first eigenvalue $\lambda_1 = 0$ and the corresponding eigenfunctions are constant. Then for each $u \in H^1(\Omega)$ we have a unique representation

$$u = \sum_{j=1}^\infty \alpha_j e_j.$$

Let e_n^*, $n \in \mathbb{N}$, be continuous linear functionals on $H^1(\Omega)$ defined by $e_n^*(u) = \alpha_n$. We define the following decomposition of the space $H^1(\Omega)$:

$$V_j = \{u \in H^1(\Omega); \ e_i^*(u) = 0, \ i > j\},$$

$$X_j = \{u \in H^1(\Omega); \ e_i^*(u) = 0, \ i \le j\},$$

so $H^1(\Omega) = V_j \oplus X_j$. Since $e_1 = \frac{1}{|\Omega|^{\frac{1}{2}}}$ on Ω and $e_1^*(u) = \int_\Omega u e_1 \, dx = \alpha_1$, we see that $\int_\Omega u \, dx = 0$ for every $u \in X_j$, $j \in \mathbb{N}$. Therefore $\|\nabla v\|_2$ is a norm equivalent to $\|\cdot\|$, on each subspaces X_j. Consequently, functions belonging to X_j satisfy the Gagliardo-Nirenberg type inequality (see [13, p. 66, inequality 2.10]). These observations allow us to formulate

3 Lemma. *Let $2 \le r < 2^*$ and $\delta > 0$ be given. Then there exists a $j \in \mathbb{N}$ such that*

$$\|u\|_r^r \le \delta \|\nabla u\|_2^r$$

for all $u \in X_j$.

For the proof we refer to [19] (see Lemma 4.1 there).

4 Lemma. *Suppose (f_3) holds. Then there exist $\bar{\mu} > 0$, $j \in \mathbb{N}$ and $\rho, \alpha > 0$ such that $I_\mu(u) \ge \alpha$ for all $u \in X_j$ with $\|u\| = \rho$ and $0 < \mu < \bar{\mu}$.*

PROOF. In the proof we shall use the equivalent norm $\|\nabla u\|_2$ on X_j. It follows from (f_3) that

$$I_\mu(u) \ge \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - C_1 - b_2 \int_\Omega |u|^q \, dx - \frac{\mu}{2^*} \int_\Omega |u|^{2^*} \, dx$$

where $C_1 = b_1 |\Omega|$. Let $\delta > 0$ and $\|\nabla u\|_2 = \rho$. We choose $\rho > 0$ so that

$$\delta b_2 \rho^{q-2} = \frac{1}{4}.$$

Since $\rho(\delta) \to \infty$ as $\delta \to 0$ we select $\delta > 0$ so that

$$\frac{\rho^2}{4} - C_1 > \frac{\rho^2}{8}.$$

With this choice of δ we apply Lemma 3 and the Sobolev inequality to obtain

$$I_\mu(u) \ge \rho^2 \left(\frac{1}{2} - b_2 \delta \rho^{q-2} \right) - C_1 - C_3 \mu \rho^{2^*} \ge \frac{\rho^2}{4} - C_1 - C_3 \mu \rho^{2^*} \ge \frac{\rho}{8} - C_3 \mu \rho^{2^*}$$

for some constant $C_3 > 0$ and for all $u \in X_j$ with $\|\nabla u\|_2 = \rho$. (Here the existence of j has been guaranteed by Lemma 3). Finally, we choose $\bar{\mu} > 0$ so that

$$I_\mu(u) \ge \frac{\rho^2}{8} - C_3 \mu \rho^{2^*} > 0$$

for $u \in X_j$ with $\|\nabla u\|_2 = \rho$ and $0 < \mu < \bar{\mu}$.   *QED*

5 Lemma. *Suppose that (f_4) holds. Then for every $m \in \mathbb{N}$ there exists a subspace $W \subset H^1(\Omega)$ (more precisely of $H^1_\circ(\Omega)$) and a constant $M_m > 0$ independent of μ such that $\dim W = m$ and $\max_{w \in W} I_0(w) < M_m$.*

PROOF. It is easy to construct a family of functions $v_1, \ldots, v_m$ in $C^\infty_\circ(\Omega)$ with supports in $B(x_1, r_1), \ldots, B(x_m, r_m)$, respectively, so that $\operatorname{supp} v_i \cap \operatorname{supp} v_j = \emptyset$ for $i \neq j$ and $|(\operatorname{supp} v_j) \cap \Omega_\circ| > 0$ for every j. We recall that $\Omega_\circ$ is a set from assumption (f_4). Let $W = \operatorname{span}\{v_1, \ldots, v_m\}$. It is clear that $\dim W = m$ and $\int_{\Omega_\circ} |v|^p \, dx > 0$ for every $v \in W - \{0\}$. We now observe that

$$\max_{u \in W - \{0\}} I_0(u) = \max_{t > 0, \|\nabla v\|_2 = 1, \, v \in W} \left\{ t^2 \left(\frac{1}{2} - \frac{1}{t^2} \int_\Omega F(x, tv) \, dx \right) \right\}.$$

To complete the proof it is sufficient to show that

$$\lim_{t \to \infty} \frac{1}{t^2} \int_\Omega F(x, tv) \, dx > \frac{1}{2} \tag{15}$$

uniformly in $v \in W$ with $\|\nabla v\|_2 = 1$. In view of (f_4) given $L > 0$ we can find $C > 0$ such that

$$F(x, s) \geq Ls^2 - C$$

for every $s \in \mathbb{R}$ and a. e. $x \in \Omega_\circ$. Hence, for $v \in W$ with $\|\nabla v\|_2 = 1$ we have

$$\int_\Omega F(x, tv) \, dx \geq Lt^2 \int_{\Omega_\circ} v^2 \, dx - C|\Omega_\circ| - t^2 \int_{\Omega - \Omega_\circ} hv^2 \, dx - c_1 |\Omega - \Omega_\circ|. \tag{16}$$

Here we have used the lower estimate for F from the assumption (f_4). Since $\dim W < \infty$, we obviously have

$$0 < r = \min_{\|\nabla v\|_2 = 1, \, v \in W} \int_{\Omega_\circ} v^2 \, dx \quad \text{and} \quad 0 < R = \max_{\|\nabla v\|_2^2 = 1, \, v \in W} \|v\|_\infty^2 < \infty.$$

Combining this with (16) and choosing $L > 0$ sufficiently large we derive (15).
$$\boxed{QED}$$

We are now in a position to formulate our first existence result.

6 Theorem. *Suppose that (f_1), (f_2), (f_3) and (f_4) hold and that f is odd in s. Then for every $k \in \mathbb{N}$ there exists $\mu_k \in (0, \infty]$ such that problem (1) has at least k nontrivial solutions for all $\mu \in (0, \mu_k)$.*

PROOF. We apply Theorem 1 with decomposition $H^1(\Omega) = V_j \oplus X_j$. By Lemma 4 there exist $j \in \mathbb{N}$ and $\tilde{\mu}$ such I_μ satisfies (I_1) with $X = X_j$ for $0 < \mu < \tilde{\mu}$. With the aid of Lemma 5 we can find a subspace $W \in H^1(\Omega)$ with $\dim W = k + j = k + \dim V_j$ such that I_μ satisfies (I_2). Finally, we select $\tilde{\mu}$ smaller if necessary so that $(PS)_c$ hold for $\mu \in (0, \tilde{\mu})$ with $c < M$, where $\max_{w \in W} I_\mu(u) < M$. The result follows from Theorem 2.
$$\boxed{QED}$$

Theorem 6 can be applied to the problem

$$\begin{cases} -\Delta u = |u|^{2^*-2}u + \lambda u + \beta|u|^{q-2}u & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{17}$$

where $\lambda \in \mathbb{R}$ and $\beta > 0$ are constants and $2 < q < 2^*$. By changing the unknown function u: $v = \beta^{\frac{1}{q-2}}u$ the above equation can be reduced to

$$-\Delta v = \mu|v|^{2^*-2}v + \lambda v + |v|^{q-2}v,$$

where $\mu = \beta^{-\frac{2^*-2}{q-2}}$. Therefore, by Theorem 6, given $k \in \mathbb{N}$ we can find $\beta_k > 0$ so that problem (17) has at least k pairs of nontrivial solutions for $\beta > \beta_k$. We point out here that problem (17) admits at most one constant solution $u = t$, where t satisfies the equation

$$|t|^{2^*-2} + \lambda + \beta|t|^{q-2} = 0.$$

4   Case of interference of nonlinearity with eigenvalues

First we consider the case where f interferes with the first eigenvalue $\lambda_1 = 0$.

7 Lemma. *Let $a(x)$ be bounded and measurable function on Ω such that $a(x) \leq 0$ with strict inequality on a set of positive measure. Then there exists $\eta > 0$ such that*

$$\int_\Omega \left(|\nabla u|^2 - a(x)u^2\right) dx \geq \eta \int_\Omega u^2 \, dx \tag{18}$$

for every $u \in H^1(\Omega)$.

PROOF. If $-a(x)$ is bounded from below by a positive constant then (18) is obvious. In a general situation we argue by contradiction. Assume that for each $m \in \mathbb{N}$ there exists $u_m \in H^1(\Omega)$ with $\|u_m\|_2 = 1$ such that

$$\int_\Omega \left(|\nabla u_m|^2 - a(x)u_m^2\right) dx \leq \frac{1}{m}.$$

Then $\{u_m\}$ is bounded in $H^1(\Omega)$. We may assume that $u_m \rightharpoonup u$ in $H^1(\Omega)$ and $u_m \to u$ in $L^2(\Omega)$. By the lower semicontinuity of norm with respect to weak convergence, we derive

$$\int_\Omega \left(|\nabla u|^2 - a(x)u^2\right) dx = 0.$$

This is a contradiction since $\int_\Omega u^2 \, dx = 1$.   QED

8 Theorem. *Suppose that (f_1), (f_2) and (f_4) hold and that*

$(\tilde{f}_3)$ $\lim_{s \to 0} \frac{2F(x,s)}{s^2} = a(x)$ *uniformly a.e. in $x \in \Omega$, where $a(x)$ satisfies assumptions of Lemma 7.*

If $f(x, s)$ is odd in s, then for every $k \in \mathbb{N}$ there exists $\mu_k > 0$ such that problem (1) has at least k pairs of nontrivial solutions for every $\mu \in (0, \mu_k)$.

PROOF. We apply Theorem 2 with $V = \{0\}$. Since assumption $(\tilde{f}_3)$ replaces (f_3) we only need to check that I_μ satisfies (I_1) of Theorem 2. It is clear that for a given $\epsilon > 0$ we can find $C_\epsilon > 0$ such that

$$F(x, s) \leq \frac{a(x) + \epsilon}{2} s^2 + C_\epsilon |s|^{2^*}$$

for every $(x, s) \in \Omega \times \mathbb{R}$. Applying Lemma 7 we have

$$I_\mu(u) \geq \frac{1 + \epsilon}{2(1 + \epsilon)} \int_\Omega \left(|\nabla u|^2 - a(x) u^2 \right) dx - \frac{\epsilon}{2} \int_\Omega u^2 \, dx$$
$$- \left(\frac{\mu}{2^*} + C_\epsilon \right) \int_\Omega |u|^{2^*} dx$$
$$\geq \frac{\eta - \epsilon(1 + \epsilon)}{2(1 + \epsilon)} \int_\Omega u^2 \, dx + \frac{\epsilon}{2(1 + \epsilon)} \int_\Omega |\nabla u|^2 \, dx$$
$$- C(\epsilon) \left(\int_\Omega \left(|\nabla u|^2 + u^2 \right) dx \right)^{\frac{2^*}{2}}.$$

We choose $\epsilon > 0$ so that $\eta - \epsilon(1 + \epsilon) > 0$. Put

$$\beta_\circ = \min \left(\frac{\eta - \epsilon(1 + \epsilon)}{2(1 + \epsilon)}, \frac{\epsilon}{2(1 + \epsilon)} \right).$$

Thus

$$I_\mu(u) \geq \beta_\circ \|u\|^2 - C(\epsilon) \|u\|^{2^*}.$$

Obviously this implies (I_1) of Theorem 2.   QED

We now consider the situation where f interferes with eigenvalues of higher order. We need the following two assumptions:

$(\tilde{f}_4)$ Let $k > 1$. There exists a constant $B \geq 0$ such that

$$F(x, s) \geq \lambda_k \frac{s^2}{2} - B$$

for all $s \in \mathbb{R}$ and a. e. in $x \in \Omega$.

$(\tilde{f}_5)$ $\lim_{s \to 0} \frac{2F(x,s)}{s^2} = a(x)$ uniformly a. e. in $x \in \Omega$, where $a(x)$ is a bounded and measurable function such that $a(x) \leq \lambda_j \leq \lambda_k$ for some $j \leq k$ and with strict inequality on a set of positive measure.

For $j > 1$ we set $V_j = \text{span}\{e_1, \ldots, e_{j-1}\}$ and $W = \text{span}\{e_1, \ldots, e_k\}$.

Lemma 9 below follows from the variational characterization of eigenfunctions.

9 Lemma. *Suppose that f satisfies $(\tilde{f}_4)$. Then there exists a constant $M_k > 0$ independent of μ such that*

$$\max_{u \in W} I_\mu(u) < M_k.$$

10 Lemma. *Suppose that $a(x)$ is a measurable and bounded function such that $a(x) \leq \lambda_j$ on Ω with a strict inequality on a set of positive measure. Then there exists $\beta > 0$ such that*

$$\int_\Omega \left(|\nabla u|^2 - a^+(x)u^2 \right) \, dx \geq \beta \int_\Omega u^2 \, dx$$

for all $u \in H^1(\Omega) \cap V_j^\perp$.

This follows from the continuation property of eigenfunctions and the fact that $\|\nabla u\|_2$ is a norm on $V_j^\perp$. The proof is similar to that of Lemma 7.

11 Theorem. *Suppose that (f_1), (f_2), $(\tilde{f}_4)$ and $(\tilde{f}_5)$ hold. If $f(x, s)$ is odd in s, then for every $k \in \mathbb{N}$ there exists $\mu_k > 0$ such that problem (1) has at least $k - j + 1$ pairs of nontrivial solutions for $\mu \in (0, \mu_k)$.*

PROOF. With the aid of Lemma 10 and repeating the argument used in the proof of Theorem 8 we show that assumption (I_1) of Theorem 2 holds. Applying this theorem and Proposition 1 we derive the existence of $k - j + 1$ pairs of nontrivial solutions. $\boxed{QED}$

Finally, we establish the existence of solutions which do not change sign. We need the following abstract result (see [20]).

12 Theorem. *Let E be a real Banach space. Suppose that $I \in C^1(E, \mathbb{R})$ satisfies $I(0) = 0$ and*

(I_1) *there exists a constant $\rho > 0$ such that $I(u) \geq 0$ for $\|u\| = \rho$.*

$(\hat{I}_2)$ *there exist $v_1 \in E$ with $\|v_1\| = 1$ and a constant M such that*

$$\sup_{t \geq 0} I(tv_1) \leq M$$

and

(I_3) *if M is a constant from $(\hat{I}_2)$, then I satisfies the $(PS)_c$ condition for $0 < c < M$.*

Then I has a nontrivial critical point.

13 Theorem. *Suppose that $f(x, 0) = 0$ on Ω and that (f_1), (f_2), $(\tilde{f}_4)$ with $\lambda_k = \lambda_1$, and $(\tilde{f}_3)$ hold. (In fact, we need only the estimate from below for F from assumption (f_4)). Then there exists $\mu_1 > 0$ such that problem (1) has a nontrivial nonnegative and nontrivial nonpositive solution for every $\mu \in (0, \mu_1)$.*

PROOF. We only show the existence of nonnegative nontrivial solution. Put $\bar{f}(x, s) = f(x, s)$ for $s \geq 0$ and $\bar{f}(x, s) = 0$ for $s < 0$. A solution will be obtained as a critical point of the functional

$$J_\mu(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \frac{\mu}{2^*} \int_\Omega (u^+)^{2^*} \, dx - \int_\Omega \bar{F}(x, u) \, dx,$$

where $\bar{F}(x, s) = \int_0^s \bar{f}(x, t) \, dt$. To check $(\hat{I}_2)$ we use $v = \frac{1}{\sqrt{|\Omega|}}$. Then for $t \geq 0$

$$J_\mu(tv) \leq -\frac{\mu}{2^*} |\Omega|^{1 - \frac{2^*}{2}} t^{2^*} + \frac{t^2}{|\Omega|} \int_\Omega h \, dx + c_1 |\Omega|.$$

It is clear that $\max_{t \geq 0} J_\mu(tv) < \infty$. To check the $(PS)_c$ condition, let $\{u_m\}$ be a $(PS)_c$ sequence. It is easy to show that $u_m^- \to 0$ in $H^1(\Omega)$. Then it suffices to apply Proposition 1 to $\{u_m^+\}$. □ QED

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