Note di Matematica
Note Mat. 1(2008), 1-13
ISSN 1123-2536, e-ISSN 1590-0932
DOI 10.1285/i15900932v28n1p1
http://siba2.unile.it/notemat

Variations on a theme by James Stirling

Diego Dominici
Department of Mathematics, State University of New York at New Paltz
dominicd@newpaltz.edu

Received: 19/07/2006; accepted: 23/11/2006.

Abstract. We review the history and various approaches to the derivation of Stirling's series. We use a different procedure, based on the asymptotic analysis of the difference equation $\Gamma(z+1)=z\Gamma(z)$. The method reproduces Stirling's series very easily and can be applied to analyze more complicated difference equations.

Keywords: Stirling's series, gamma function, difference equations, asymptotic analysis

MSC 2000 classification: primary 33B15, secondary 41A60

Introduction

The most widely known and used result in asymptotics is probably *Stirling's formula*,

$$n!\sim\sqrt{2\pi n}n^{n}e^{-n},\quad n\to\infty \tag{1}$$

named after James Stirling (May 1692 – 5 Dec. 1770). The formula provides an extremely accurate approximation of the factorial numbers for large values of n. The asymptotic formula (1), for which Stirling is best known, appeared as Example 2 to Proposition 28 in his most important work, *Methodus Differentialis*, published in 1730 [41]. In it, he asserted that $\log(n!)$ is approximated by "three or four terms" of the series

$$\left(n+\frac{1}{2}\right)\log\left(n+\frac{1}{2}\right)-a\left(n+\frac{1}{2}\right)+\frac{1}{2}\log(2\pi)\\ -\frac{a}{24\left(n+\frac{1}{2}\right)}+\frac{7a}{2880\left(n+\frac{1}{2}\right)^{3}}-\cdots, \tag{2}$$

where log means the base-10 logarithm and $a=[\ln(10)]^{-1}$ (see [40]).

In 1730 Stirling wrote to Abraham De Moivre (26 May 1667 – 27 Nov. 1754) pointing out some errors that he had made in a table of logarithms of factorials in his book and also telling him about (2). After seeing Stirling's results, De Moivre derived the formula

$$\ln[(n-1)!]\sim\left(n-\frac{1}{2}\right)\ln(n)-n+\frac{1}{2}\ln(2\pi)+\sum_{k\geq1}\frac{B_{2k}}{2k\left(2k-1\right)n^{2k-1}}, \tag{3}$$

which he published in his *Miscellaneis Analyticis Supplementum* a few months later. Equation (3) is called *Stirling's series* and the numbers B_k are called the *Bernoulli numbers*, and are defined by

$$B_0 = 1, \quad \sum_{j=0}^{k} \binom{k+1}{j} B_j = 0, \quad k \geq 1. \tag{4}$$

Clearly Stirling and De Moivre regularly corresponded around this time, for in September 1730 Stirling related the new results of De Moivre in a letter to Gabriel Cramer.

In 1729 Leonhard Euler (15 April 1707 – 18 Sept. 1783) proposed a generalization of the factorial function from natural numbers to positive real numbers [11]. It is called the *gamma function,* $\Gamma(z)$, which he defined as

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{z(z+1)\cdots(z+n)}, \tag{5}$$

and it is related to the factorial numbers by

$$\Gamma(n+1) = n!, \quad n = 0, 1, 2, \ldots$$

From Euler's definition (5), we immediately obtain the fundamental relation

$$\Gamma(z+1) = z\Gamma(z) \tag{6}$$

and the value $\Gamma(1) = 1$. In fact, the gamma function is completely characterized by the *Bohr-Mollerup theorem* [8]:

1 Theorem. *The gamma function is the only function* $\Gamma : (0,\infty) \to (0,\infty)$ *which satisfies*

(i) $\Gamma(1) = 1$

(ii) $\Gamma(x+1) = x\Gamma(x)$

(iii) $\ln[\Gamma(x)]$ *is convex for all* $x \in (0,\infty)$.

Proof. See [4] and [30]. QED

Another complex-analytic characterization is due to Wielandt [43]:

2 Theorem. *The gamma function is the only holomorphic function in the right half plane* $\mathbb{A}$ *satisfying*

(i) $\Gamma(1) = 1$

(ii) $\Gamma(z+1) = z\Gamma(z)$ *for all* $z \in \mathbb{A}$

(iii) $\Gamma(z)$ *is bounded in the strip* $1 \leq \mathrm{Re}(z) < 2$

PROOF. See [35]. QED

In terms of $\Gamma(z)$, we can re-write (1) as

$$\ln[\Gamma(z)] \sim \mathcal{P}(z), \quad z \to \infty \tag{7}$$

with

$$\mathcal{P}(z) = z\ln(z) - z - \frac{1}{2}\ln(z) + \frac{1}{2}\ln(2\pi) \tag{8}$$

and (3) in the form

$$\ln[\Gamma(z)] \sim \mathcal{P}(z) + R_N(z), \quad z \to \infty \tag{9}$$

where $R_0(z) = 0$ and

$$R_N(z) = \sum_{k=1}^{N} \frac{B_{2k}}{2k(2k-1)z^{2k-1}}, \quad N \geq 1. \tag{10}$$

Estimations of the remainder $\ln[\Gamma(z)] - \mathcal{P}(z) - R_N(z)$ were computed in [38].

1 Previous results

Over the years, there have been **many** different approaches to the derivation of (7) and (9), including:

(i) Aissen [2] studied the sequence $V_n = \frac{n^n e^{-n}}{n!}$. Using his lemma

3 Lemma. *If*

$$\frac{y_{n+1}}{y_n} = 1 + \frac{\alpha}{n} + O\left(n^{-2}\right)$$

and $y_n \neq 0$ *for all* n*, then* $y_n \sim Cn^{\alpha}, \quad n \to \infty$ *for some non-zero constant* C*.*

he showed that $n! \sim C\sqrt{n}n^n e^{-n}$.

(ii) Bender & Orszag [5], Bleistein & Handelsman [6], Diaconis & Freedman [13], Dingle [14], Olver [33] and Wong [45] applied Laplace's method to the *Euler integral of the second kind*

$$\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}dt, \quad \mathrm{Re}(z) > 0. \tag{11}$$

(iii) Bender & Orszag [5] and Temme [39] used *Hankel's contour integral* [1]

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{(0+)} t^{-z} e^t dt$$

and the method of steepest descent.

(iv) Bleistein & Handelsman [6], Lebedev [25], Sasvári [37] and Temme [39] used *Binet's first formula*

$$\ln\left[\Gamma(z)\right] = \mathcal{P}(z) + \int_0^\infty \frac{1}{t}\left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) e^{-tz} dt, \quad \mathrm{Re}(z) > 0$$

and

$$\frac{1}{t}\left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) = \sum_{k\geq 1} \frac{B_{2k}}{(2k)!} t^{2k-2}, \quad |t| < 2\pi.$$

(v) Blyth & Pathak [7] and Khan [23] used probabilistic arguments, applying the Central Limit Theorem and the limit theorem for moment generating functions to Gamma and Poison random variables.

(vi) Coleman [9] defined

$$c_n = \left(n + \frac{1}{2}\right)\ln(n) - n + 1 - \ln\left(n!\right),$$

and showed that $c_n \to 1 - \frac{1}{2}\ln(2\pi)$ as $n \to \infty$. A similar result was proved by Aissen [2], using the concavity of $\ln(x)$.

(vii) Dingle [14] used *Weierstrass' infinite product*

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}\right],$$

(where γ is *Euler's constant*) and Mellin transforms.

(viii) Feller [17], [18] proved the identity

$$\ln\left(n!\right) - \frac{1}{2}\ln(n) = I(n) - I\left(\frac{1}{2}\right) + \sum_{k=1}^{n-1}\left(a_k - b_k\right) + a_n, \qquad (12)$$

where

$$I(n) = \int_0^n \ln(t)dt, \quad a_k = \int_{k-\frac{1}{2}}^{k} \ln\left(\frac{k}{t}\right) dt, \quad b_k = \int_k^{k+\frac{1}{2}} \ln\left(\frac{t}{k}\right) dt$$

and showed that

$$\sum_{k=1}^{\infty} (a_k - b_k) - I\left(\frac{1}{2}\right) = \frac{1}{2}\ln(2\pi).$$

There is a mistake in his Equation (2.4), where he states that

$$\ln(n!) - \frac{1}{2}\ln(n) + I(n) - I\left(\frac{1}{2}\right) = \sum_{k=1}^{n-1} (a_k - b_k) + a_n,$$

instead of (12).

(ix) Hayman [19] used the exponential generating function

$$e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k$$

and his method for admissible functions.

(x) Hummel [20], established the inequalities

$$\frac{11}{12} < r_n + \frac{1}{2}\ln(2\pi) < 1, \quad n = 2, 3, \ldots,$$

where

$$r_n = \ln\left(\frac{n!e^n}{\sqrt{2\pi n}n^n}\right).$$

Impens [21], [22], showed that for $x > 0$

$$R_{2n}(x) < \ln[\Gamma(x)] - \mathcal{P}(x) < R_{2m+1}(x), \quad n, m \geq 0,$$

where $R_n(x)$ was defined in (10). Maria [26] showed that

$$\left[12n + \frac{3}{2(2n+1)}\right]^{-1} < r_n, \quad n = 1, 2, \ldots.$$

Mermin [28] proved the identity

$$e^{r_n} = \prod_{k=n}^{\infty} e^{-1}\left(1 + \frac{1}{k}\right)^{k+\frac{1}{2}},$$

which he used to show that $r_n \sim R_3(n)$. Michel [29], proved the inequality

$$\left| e^{r_n} - 1 - \frac{1}{12n} - \frac{1}{288n^2} \right| \leq \frac{1}{360n^3} + \frac{1}{108n^4}, \quad n = 3, 4 \ldots .$$

Nanjundiah [32], showed that

$$R_2(n) < r_n < R_1(n) = 1, 2, \ldots .$$

Robbins [36], established the double inequality

$$\frac{1}{12n+1} < r_n < \frac{1}{12n}, \quad n = 1, 2, \ldots .$$

(xi) Marsaglia & Marsaglia [27] derived from (11) the asymptotic expansion

$$n! \sim n^{n+1} e^{-n} \sum_{k=1}^{\infty} b_k \left(\frac{2}{n}\right)^{\frac{k}{2}} \Gamma\left(\frac{k}{2}\right) k,$$

where the generating function $G(z) = \sum_{k \geq 0} b_k z^k$ is defined by

$$G(z) \exp\left[1 - G(z)\right] = \exp\left(-\frac{1}{2} z^2\right), \quad G'(0) = 1.$$

(xii) Namias [31] introduced the function $F(n) = \frac{\Gamma(n)}{\mathcal{P}(n)}$, with $\mathcal{P}(n)$ defined in (8). From *Legendre's duplication formula*

$$\Gamma(2n) = \frac{2^{2n-1}}{\sqrt{\pi}} \Gamma(n) \Gamma\left(n + \frac{1}{2}\right), \tag{13}$$

he derived a functional equation for $F(n)$

$$\frac{F(2n)}{F(n) F\left(n - \frac{1}{2}\right)} = \sqrt{e} \left(1 - \frac{1}{2n}\right)^n,$$

from which he obtained (9). He also considered the triplication case, using *Gauss' multiplication formula*

$$\Gamma(mz) = \left(\sqrt{2\pi}\right)^{1-m} m^{mz - \frac{1}{2}} \prod_{k=0}^{m-1} \Gamma\left(z + \frac{k}{m}\right), \quad m = 2, 3, \ldots$$

with $m = 3$. His results where extended by Deeba & Rodriguez in [12].

(xiii) Olver [33] used Euler's definition (5)

$$\ln\left[\Gamma(z)\right] = -\ln(z) + \lim_{n\to\infty} z\ln(n) + \sum_{k=1}^{n}\left[\ln(k) - \ln(z+k)\right]$$

and the Euler-Maclaurin formula. A similar analysis was done by Knopp [24] and Wilf [44].

(xiv) Patin [34] used (11) and the Lebesgue Dominated Convergence Theorem.

(xv) Whittaker & Watson [42] used *Binet's second formula*

$$\ln\left[\Gamma(z)\right] = \mathcal{P}(z) + 2\int_0^\infty \frac{\arctan\left(\frac{t}{z}\right)}{e^{2\pi t}-1}dt, \quad \mathrm{Re}(z) > 0$$

and

$$\arctan(x) = x\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}x^{2k}, \quad |x| \le 1.$$

Thus, there have been a huge variety of approaches to Stirling's result, ranging from elementary to heavy-machinery methods. In an effort to join such illustrious company, we present still another direction for deriving (9).

Our starting point shall be the difference equation (6). A parallel approach was considered in [3]. For a different analysis of (6) using the method of controlling factors, see [5]. Extensions and other applications of the method used can be found in [10], [15] and [16].

2 Asymptotic analysis

2.1 Stirling's formula

We begin with a derivation of (7), to better illustrate how the method works. We assume that

$$\ln\left[\Gamma(z)\right] \sim f(z) + g(z), \quad z \to \infty \tag{14}$$

with

$$g = o(f), \quad z \to \infty. \tag{15}$$

Using (14) in (6), we have

$$f(z+1) - f(z) + g(z+1) - g(z) \sim \ln(z). \tag{16}$$

Expanding $f(z+1)$ and $g(z+1)$ in a Taylor series, we obtain

$$f'(z) + \frac{1}{2}f''(z) + g'(z) \sim \ln(z). \tag{17}$$

From (15) and (17) we get the system

$$f'(z) = \ln(z), \quad \frac{1}{2}f''(z) + g'(z) = 0$$

and thus,

$$f(z) = z\ln(z) - z, \quad g(z) = -\frac{1}{2}\ln(z) + C. \tag{18}$$

To find the constant C in (18), we replace $\Gamma(z) \sim e^{f(z)+g(z)}$ in (13) and obtain

$$e^{f(2z)+g(2z)} \sim \frac{2^{2z-1}}{\sqrt{\pi}} e^{f(z)+g(z)} e^{f\left(z+\frac{1}{2}\right)+g\left(z+\frac{1}{2}\right)},$$

or

$$e^{C-\frac{1}{2}} \left(1 + \frac{1}{2z}\right)^z \sim \sqrt{2\pi}, \quad z \to \infty,$$

from which we conclude that $C = \ln\left(\sqrt{2\pi}\right)$.

Hence, we have shown that

$$\ln\left[\Gamma(z)\right] \sim z\ln(z) - z - \frac{1}{2}\ln(z) + \frac{1}{2}\ln\left(2\pi\right), \quad z \to \infty.$$

2.2 Stirling's series

To extend the result of the previous section, we now assume that

$$\ln\left[\Gamma(z)\right] \sim \sum_{k=0}^{N} f_k(z), \quad z \to \infty \tag{19}$$

with

$$f_{k+1} = o\left(f_k\right), \quad z \to \infty, \quad k = 0, 1, \ldots, N-1. \tag{20}$$

Using (19) in (6) we have

$$\sum_{k=0}^{N} f_k(z+1) - f_k(z) \sim \ln\left(z\right), \quad z \to \infty. \tag{21}$$

Replacing the Taylor series

$$f_k(z+1) = \sum_{j\geq 0} \frac{1}{j!} \frac{d^j}{dz^j} f_k(z)$$

in (21), we have

$$\sum_{k=0}^{N}\sum_{j\geq 1}\frac{1}{j!}\frac{d^j}{dz^j}f_k(z)\sim \ln(z)\,,\quad z\to\infty. \tag{22}$$

From (20), we obtain the system of ODEs

$$\frac{d}{dz}f_0=\ln(z)$$

and

$$\sum_{j=0}^{k-1}\frac{1}{(k+1-j)!}\frac{d^{k+1-j}}{dz^{k+1-j}}f_j(z)+\frac{d}{dz}f_k=0,\quad k\geq 1,$$

which imply

$$f_0(z)=z\ln(z)-z \tag{23}$$

and

$$f_k(z)=-\sum_{j=1}^{k}\frac{1}{(j+1)!}\frac{d^j}{dz^j}f_{k-j}(z),\quad k\geq 1, \tag{24}$$

where we have omitted (for the time being) any constant of integration. To find the functions $f_k(z)$, we set

$$f_k(z)=a_k\frac{d^k}{dz^k}f_0(z) \tag{25}$$

in (24) and get $a_0=1$ and

$$a_k\frac{d^k}{dz^k}f_0(z)=-\sum_{j=1}^{k}\frac{1}{(j+1)!}\frac{d^j}{dz^j}\frac{d^{k-j}}{dz^{k-j}}f_0(z),\quad k\geq 1,$$

which gives

$$a_0=1,\quad a_k=-\sum_{j=1}^{k}\frac{1}{(j+1)!}a_{k-j},\quad k\geq 1,$$

or

$$a_0=1,\quad \sum_{j=0}^{k}\frac{1}{(k+1-j)!}a_j=0,\quad k\geq 1. \tag{26}$$

Multiplying both sides by $(k+1)!$, we can write (26) as

$$a_0=1,\quad \sum_{j=0}^{k}\frac{(k+1)!}{(k+1-j)!j!}j!a_j=0,\quad k\geq 1,$$

or

$$a_0 = 1, \quad \sum_{j=0}^{k} \binom{k+1}{j} j! a_j = 0, \quad k \geq 1. \tag{27}$$

Comparing (4) and (27) we conclude that

$$a_k = \frac{B_k}{k!}, \quad k \geq 0. \tag{28}$$

Thus, from (23), (25) and (28) we have

$$f_k(z) = \frac{B_k}{k!} \frac{d^k}{dz^k} \left[z \ln(z) - z \right], \quad k \geq 0,$$

from which we obtain

$$f_1(z) = -\frac{1}{2} \ln(z) \tag{29}$$

and

$$f_k(z) = \frac{B_k}{k!} (-1)^k \frac{(k-1)!}{z^{k-1}} = \frac{(-1)^k B_k}{k\,(k+1)\, z^{k-1}}, \quad k \geq 2.$$

Since $B_{2k+1} = 0$ for all $k \geq 1$, we need to consider even values of k only,

$$f_{2k}(z) = \frac{B_{2k}}{2k\,(2k+1)\, z^{2k-1}}, \quad k \geq 1. \tag{30}$$

So far, we haven't included any constant of integration in our calculations. We could add a constant to one of the functions $f_k(z)$, let's say to $f_1(z)$, and proceed as in Section 2.1 to find it. Doing this, we would obtain from (23), (29) and (30) that

$$\sum_{k=0}^{N} f_k(z) = \mathcal{P}(z) + R_N(z)$$

where $\mathcal{P}(z)$, $R_N(z)$ were defined in (8) and (10) respectively.

Another possibility, would be to assume no previous knowledge of $\Gamma(z)$, except for the difference equation $\Gamma(z+1) = z\Gamma(z)$ and the value at 1, $\Gamma(1) = 1$. In doing so, (19) would imply the initial conditions $f_k(1) = 0$, for all $k \geq 0$. Hence, we would have

$$f_0(z) = z \ln(z) - z + 1, \quad f_1(z) = -\frac{1}{2} \ln(z)$$
$$f_{2k}(z) = \frac{B_{2k}}{2k\,(2k+1)\, z^{2k-1}} - \frac{B_{2k}}{2k\,(2k+1)}, \quad k \geq 1$$

and therefore

$$\ln\left[\Gamma(z)\right] \sim z\ln(z) - z - \frac{1}{2}\ln(z) + C_N + R_N(z), \tag{31}$$

with

$$C_N = 1 - \sum_{k=1}^{N} \frac{B_{2k}}{2k\,(2k+1)}. \tag{32}$$

Computing the first few C_N, we would get

$$C_1 \simeq .91667,\ C_2 \simeq .91944,\ C_3 \simeq .91865,\ C_4 \simeq .91925,\ C_5 \simeq .91840,$$
$$C_6 \simeq .92032,\ C_7 \simeq .91391,\ C_8 \simeq .94346,\ C_9 \simeq .76382,\ C_{10} \simeq 2.1562,$$

and increasingly greater numbers (in absolute value). We would conclude that, before the sum starts diverging, the C_N seem to be approaching a value close to .918. Given our previous discussion of Stirling's formula, it is not surprising to find that $\frac{1}{2}\ln(2\pi) \simeq .91894$.

3 Conclusion

We have presented the history and previous approaches to the proof of Stirling's series (9). We have used a different procedure, based on the asymptotic analysis of the difference equation (6). The method reproduces (9) very easily and can be extended to use in more complicated difference equations.

Bender and Orszag observed in [5, Page 227] that

> without further information the constant $\frac{1}{2}\ln(2\pi)$ cannot be determined. The difference equation that we have solved is linear and homogeneous, so any arbitrary multiple of a solution is still a solution.

While agreeing with them completely, we have shown that by imposing the additional condition $\Gamma(1) = 1$ one can find an approximation to the value of $\frac{1}{2}\ln(2\pi)$, without any other assumptions. Thus, local behavior at $z = 1$ and asymptotic behavior as $z \to \infty$ can be combined fruitfully.

References

[1] M. Abramowitz, I. A. Stegun, editors: Handbook of mathematical functions with formulas, graphs, and mathematical tables, Dover Publications Inc., New York 1992.

[2] M. I. Aissen: *Some remarks on Stirling's formula*, Amer. Math. Monthly, **61**, 687–691, 1954.

[3] G. E. Andrews, R. Askey, R. Roy: Special functions, Cambridge University Press, Cambridge 1999.

[4] E. Artin: The gamma function, Holt, Rinehart and Winston, New York 1964.

[5] C. M. Bender, S. A. Orszag: Advanced mathematical methods for scientists and engineers, McGraw-Hill Book Co., New York 1978.

[6] N. Bleistein, R. A. Handelsman: Asymptotic expansions of integrals, Dover Publications Inc., New York, second edition, 1986.

[7] C. R. Blyth, P. K. Pathak: *Notes: A Note on Easy Proofs of Stirling's Theorem*, Amer. Math. Monthly, **93**, n. 5, 376–379, 1986.

[8] H. Bohr, J. Mollerup: Laerebog i Matematisk Analyse: Afsnit III, Funktioner af flere reelle Variable. Jul. Gjellerups Forlag, Copenhagen 1922.

[9] A. J. Coleman: *Classroom Notes: A Simple Proof of Stirling's Formula*, Amer. Math. Monthly, **58**, n. 5, 334–336, 1951.

[10] O. Costin, R. Costin: *Rigorous WKB for finite-order linear recurrence relations with smooth coefficients*, SIAM J. Math. Anal., **27**, n. 1, 110–134, 1996.

[11] P. J. Davis: *Leonhard Euler's integral: A historical profile of the gamma function*, Amer. Math. Monthly, **66**, 849–869, 1959.

[12] E. Y. Deeba, D. M. Rodriguez: *Stirling's series and Bernoulli numbers*, Amer. Math. Monthly, **98**, n. 5, 423–426, 1991.

[13] P. Diaconis, D. Freedman: *An elementary proof of Stirling's formula*, Amer. Math. Monthly, **93**, n. 2, 123–125, 1986.

[14] R. B. Dingle: Asymptotic expansions: their derivation and interpretation, Academic Press, New York 1973.

[15] R. B. Dingle, G. J. Morgan: WKB *methods for difference equations. I, II*, Appl. Sci. Res., **18**, 221–237; 238–245, 1967/1968.

[16] D. Dominici: *Asymptotic analysis of the Hermite polynomials from their differential-difference equation*, J. Difference Equ. Appl., **13**, n. 12, 1115–1128, 2007.

[17] W. Feller: *A direct proof of Stirling's formula*, Amer. Math. Monthly, **74**, 1223–1225, 1967.

[18] W. Feller: *Correction to: "A direct proof of Stirling's formula"*, Amer. Math. Monthly, **75**, 518, 1968.

[19] W. K. Hayman: *A generalisation of Stirling's formula*, J. Reine Angew. Math., **196**, 67–95, 1956.

[20] P. M. Hummel: *Questions, Discussions, and Notes: A Note on Stirling's Formula*, Amer. Math. Monthly, **47**, n. 2, 97–99, 1940.

[21] C. Impens: *Stirling's formula for $n!$ made easy*, Real Anal. Exchange, (26th Summer Symposium Conference, suppl.), 67–71, 2002.

[22] C. Impens: *Stirling's series made easy*, Amer. Math. Monthly, **110**, n. 8, 730–735, 2003.

[23] R. A. Khan: *A probabilistic proof of Stirling's formula*, Amer. Math. Monthly, **81**, 366–369, 1974.

[24] K. Knopp: Theory and Application of Infinite Series, Dover Publications Inc., New York 1990.

[25] N. N. Lebedev: Special functions and their applications, Dover Publications Inc., New York 1972.

[26] A. J. Maria: *A remark on Stirling's formula*, Amer. Math. Monthly, **72**, 1096–1098, 1965.

[27] G. Marsaglia, J. C. W. Marsaglia: *A new derivation of Stirling's approximation to n!*, Amer. Math. Monthly, **97**, n. 9, 826–829, 1990.

[28] N. D. Mermin: *Stirling's formula!*, Amer. J. Phys., **52**, n. 4, 362–365, 1984.

[29] R. Michel: *On Stirling's formula*, Amer. Math. Monthly, **109**, n. 4, 388–390, 2002.

[30] M. E. Muldoon: *Some monotonicity properties and characterizations of the gamma function*, Aequationes Math., **18**, n. 1–2, 54–63, 1978.

[31] V. Namias: *A simple derivation of Stirling's asymptotic series*, Amer. Math. Monthly, **93**, n. 1, 25–29, 1986.

[32] T. S. Nanjundiah: *Note on Stirling's formula*, Amer. Math. Monthly, **66**, 701–703, 1959.

[33] F. W. J. Olver: Asymptotics and special functions, AKP Classics. A K Peters Ltd., Wellesley, MA 1997.

[34] J. M. Patin: *A very short proof of Stirling's formula*, Amer. Math. Monthly, **96**, n. 1, 41–42, 1989.

[35] R. Remmert: *Wielandt's theorem about the* Γ*-function*, Amer. Math. Monthly, **103**, n. 3, 214–220, 1996.

[36] H. Robbins: *A remark on Stirling's formula*, Amer. Math. Monthly, **62**, 26–29, 1955.

[37] Z. Sasvári: *An elementary proof of Binet's formula for the gamma function*, Amer. Math. Monthly, **106**, n. 2, 156–158, 1999.

[38] R. Spira: *Calculation of the gamma function by Stirling's formula*, Math. Comp., **25**, 317–322, 1971.

[39] N. M. Temme: Special functions, A Wiley-Interscience Publication. John Wiley & Sons Inc., New York 1996.

[40] I. Tweddle: *Approximating n!: historical origins and error analysis*, Amer. J. Phys., **52**, n. 6, 487–488, 1984.

[41] I. Tweddle: James Stirling's *Methodus differentialis*, Sources and Studies in the History of Mathematics and Physical Sciences. Springer-Verlag London Ltd., London 2003. An annotated translation of Stirling's text.

[42] E. T. Whittaker, G. N. Watson: A course of modern analysis, Cambridge Mathematical Library. Cambridge University Press, Cambridge 1996.

[43] H. Wielandt: Mathematische Werke/Mathematical works. Vol. 2, Walter de Gruyter & Co., Berlin 1996.

[44] H. S. Wilf: Mathematics for the physical sciences, Dover Publications Inc., New York 1978.

[45] R. Wong: Asymptotic approximations of integrals, volume 34 of Classics in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA 2001.