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Games on classes of spaces

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Abstract. Using the construction of Containing Spaces given in [1] we define some kind of games considered on topological classes of spaces.

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Agreement

In the present paper we denote by τ a fixed infinite cardinal. The first infinite cardinal larger than τ is denoted by τ^+. By ω we denote the first infinite cardinal. An ordinal α is identified with the set of all ordinals less than α. A cardinal is identified with the corresponding initial ordinal. The cardinality of a set X is denoted by $|X|$. By $\mathcal{F}$ we denote the set of all finite subsets (including the empty set) of τ.

An arbitrary considered space is assumed to be a T_0-space of weight less than or equal to τ.

Let $\mathbf{S}$ be an indexed collection of spaces. In [1] a poset $\mathrm{C}(\mathbf{S})$, whose order is denoted by $\prec^{\mathbf{S}}$, is constructed. This poset is directed and each subset A of $\mathrm{C}(\mathbf{S})$ of cardinality $\leq \tau$ has supremum denoted by $\sup(A)$. (Therefore $\mathrm{C}(\mathbf{S})$ is directed and τ-complete) Also, for each element $c \in \mathrm{C}(\mathbf{S})$ a topological space, denoted by $\mathrm{T}(c)$ and called *a Containing Space*, is constructed. The purpose of the introduction of these notions is the construction of some special spectrum in the "theory of Containing Spaces".

We shall use the above notions to introduce games on classes of spaces. We

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start given some necessary notions and notations of [1] in order to explain briefly the constructions of $C(\mathbf{S})$ and $\mathrm{T}(c)$.

1 The constructions of the poset $\mathrm{C}(\mathbf{S})$ and spaces $\mathrm{T}(c)$, $c \in \mathrm{C}(\mathbf{S})$

An indexed collection

$$\mathbf{M} \equiv \{\, \{\, U_\delta^X : \delta \in \tau \,\} : X \in \mathbf{S} \,\}, \tag{1}$$

where $\{\, U_\delta^X : \delta \in \tau \,\}$ is an indexed base for X, is called a *co-mark* of $\mathbf{S}$.

A co-mark (1) of $\mathbf{S}$ is called a *co-extension* of a co-mark

$$\mathbf{M}^+ \equiv \{\, \{\, V_\delta^X : \delta \in \tau \,\} : X \in \mathbf{S} \,\}$$

of $\mathbf{S}$ if there exists an one-to-one mapping θ of τ into itself such that $V_\delta^X = U_{\theta(\delta)}^X$, for every $X \in \mathbf{S}$ and $\delta \in \tau$.

Consider the co-mark (1) of $\mathbf{S}$. We define a family

$$\mathrm{R}_{\mathbf{M}} \equiv \{\, \sim_{\mathbf{M}}^s : s \in \mathcal{F} \,\}$$

of equivalence relations on $\mathbf{S}$ as follows: two elements X and Y of $\mathbf{S}$ are $\sim_{\mathbf{M}}^s$-*equivalent* if and only if there exists an isomorphism i of the algebra of subsets of X generated by the finite set $\{\, U_\delta^X : \delta \in s \,\}$ of X onto the algebra of subsets of Y generated by the finite set $\{\, U_\delta^Y : \delta \in s \,\}$ of Y such that $i(U_\delta^X) = U_\delta^Y$, $\delta \in s$.

An indexed family $\mathrm{R} \equiv \{\, \sim^s : s \in \mathcal{F} \,\}$ of equivalence relations on $\mathbf{S}$ is said to be *admissible* if the following conditions are satisfied:

(a) $\sim^t \subset \sim^s$, if $s \subset t \in \mathcal{F}$,

(b) the number of $\sim^s$-equivalence classes is finite for every $s \in \mathcal{F}$, and

(c) $\sim^s = \mathbf{S} \times \mathbf{S}$ if $s = \emptyset$.

The set of all $\sim^s$-equivalence classes for all $s \in \mathcal{F}$, is denoted by $\mathrm{C}(\mathrm{R})$. By $\mathrm{C}^\diamondsuit(\mathrm{R})$ we denote the ring of subsets of $\mathbf{S}$ generated by $\mathrm{C}(\mathrm{R})$.

Note that the family $\mathrm{R}_{\mathbf{M}}$ is admissible.

Let

$$\mathrm{R}_0 \equiv \{\, \sim_0^s : s \in \mathcal{F} \,\}$$

and

$$\mathrm{R}_1 \equiv \{\, \sim_1^s : s \in \mathcal{F} \,\}$$

be two indexed families of equivalence relations on $\mathbf{S}$. It is said that R_1 is a *final refinement* of R_0 if for every $s \in \mathcal{F}$ there exists an element $t \in \mathcal{F}$ such that $\sim_1^t \subset \sim_0^s$.

A family R of equivalence relations on $\mathbf{S}$ is called $\mathbf{M}$-*admissible* if R is final refinement of $\mathrm{R}_{\mathbf{M}}$.

Let $\mathrm{R} \equiv \{\sim^s : s \in \mathcal{F}\}$ be an $\mathbf{M}$-admissible family of equivalence relations on $\mathbf{S}$. On the set of all pairs (x, X), where $x \in X \in \mathbf{S}$, we consider an equivalence relation, denoted by $\sim_{\mathrm{R}}^{\mathbf{M}}$, as follows: $(x, X) \sim_{\mathrm{R}}^{\mathbf{M}} (y, Y)$ if and only if $X \sim^s Y$ for every $s \in \mathcal{F}$, and either $x \in U_\delta^X$ and $y \in U_\delta^Y$ or $x \notin U_\delta^X$ and $y \notin U_\delta^Y$ for every $\delta \in \tau$. The set of all equivalence classes of the relation $\sim_{\mathrm{R}}^{\mathbf{M}}$ is denoted by $\mathrm{T} \equiv \mathrm{T}(\mathbf{M}, \mathrm{R})$. For every $s \in \mathcal{F}$, an equivalence class $\mathbf{H}$ of $\sim^s$, and $\delta \in \tau$ we denote by $U_\delta^{\mathrm{T}}(\mathbf{H})$ (respectively, by $\mathrm{T}(\mathbf{M}, \mathrm{R}, \mathbf{H})$) the set of all elements $\mathbf{a}$ of $\mathrm{T}(\mathbf{M}, \mathrm{R})$ such that there exists a pair (x, X) of $\mathbf{a}$ for which $x \in U_\delta^X$ and $X \in \mathbf{H}$ (respectively, for which $X \in \mathbf{H}$). The set of all $U_\delta^{\mathrm{T}}(\mathbf{H})$ is denoted by B^{T}. This set is a base for a topology on the set T and the corresponding space is a T_0-space of weight $\leq \tau$.

For every element X of $\mathbf{S}$ there exists a natural embedding i_{T}^X of X into the space T defined as follows: for every $x \in X$, $i_{\mathrm{T}}^X(x) = \mathbf{a}$, where $\mathbf{a}$ is the element of T containing the pair (x, X). Thus, we have constructed Containing Spaces $\mathrm{T}(\mathbf{M}, \mathrm{R})$ for $\mathbf{S}$ of weight $\leq \tau$.

A class $I\!\!P$ of spaces is said to be *saturated* if for every indexed collection $\mathbf{S}$ of elements of $I\!\!P$ there exists a co-mark $\mathbf{M}^+$ of $\mathbf{S}$ satisfying the following condition: for every co-extension $\mathbf{M}$ of $\mathbf{M}^+$ there exists an $\mathbf{M}$-admissible family R^+ of equivalence relations on $\mathbf{S}$ such that $\mathrm{T}(\mathbf{M}, \mathrm{R}, \mathbf{H}) \in I\!\!P$ for every admissible family R, which is final refinement of R^+, and $\mathbf{H} \in \mathrm{C}^{\diamondsuit}(\mathrm{R})$. The above co-mark $\mathbf{M}^+$ is called an *initial co-mark* of $\mathbf{S}$ corresponding to the class $I\!\!P$ and the family R^+ is called an *initial family* of $\mathbf{S}$ corresponding to the co-mark $\mathbf{M}$ and the class $I\!\!P$.

Now, we give the construction of $\mathrm{C}(\mathbf{S})$. Any indexed collection

$$\{U^X : X \in \mathbf{S}\},$$

where U^X is an open subset of $X \in \mathbf{S}$, is called an $\mathbf{S}$-*open* set. Let $\mathbf{M} = \{\{U_\varepsilon^X : \varepsilon \in \tau\} : X \in \mathbf{S}\}$ be a co-mark of $\mathbf{S}$. For every $\varepsilon \in \tau$ the $\mathbf{S}$-open set $\{U_\varepsilon^X : X \in \mathbf{S}\}$ is called the ε-*component* (or a *component*) of $\mathbf{M}$.

On the set of all co-marks of $\mathbf{S}$ we define a partial order, denoted by $\prec^{cm}$, as follows. For two co-marks $\mathbf{M}_0$ and $\mathbf{M}_1$ we write $\mathbf{M}_0 \prec^{cm} \mathbf{M}_1$ if for every $\delta \in \tau$ there exists an element ε of τ such that the δ-component of $\mathbf{M}_0$ coincides with the ε-component of $\mathbf{M}_1$. We note that if $\mathbf{M}_1$ is a co-extension of $\mathbf{M}_0$ of $\mathbf{S}$, then $\mathbf{M}_0 \prec^{cm} \mathbf{M}_1$.

By $\mathrm{P}(\mathbf{S})$ we denote the set of all pairs $(\mathbf{M}, \mathrm{R})$, where $\mathbf{M}$ is a co-mark of $\mathbf{S}$ and R is an $\mathbf{M}$-admissible family of equivalence relations on $\mathbf{S}$. On the set $\mathrm{P}(\mathbf{S})$

we define a (partial) preorder, denoted by $\prec^{cm}_{af}$, and an equivalence relation on $\mathbf{S}$, denoted by $\sim^{cm}_{af}$, as follows. For two elements $(\mathbf{M}_0, \mathrm{R}_0)$ and $(\mathbf{M}_1, \mathrm{R}_1)$ of $\mathrm{P}(\mathbf{S})$ we write:

(1) $(\mathbf{M}_0, \mathrm{R}_0) \prec^{cm}_{af} (\mathbf{M}_1, \mathrm{R}_1)$ if $\mathbf{M}_0 \prec^{cm} \mathbf{M}_1$ and R_1 is final refinement of R_0.

(2) $(\mathbf{M}_0, \mathrm{R}_0) \sim^{cm}_{af} (\mathbf{M}_1, \mathrm{R}_1)$ if

$$(\mathbf{M}_0, \mathrm{R}_0) \prec^{cm}_{af} (\mathbf{M}_1, \mathrm{R}_1) \text{ and } (\mathbf{M}_1, \mathrm{R}_1) \prec^{cm}_{af} (\mathbf{M}_0, \mathrm{R}_0).$$

By $\mathrm{C}(\mathbf{S})$ we denote the set of all $\sim^{cm}_{af}$-equivalence classes. On the set $\mathrm{C}(\mathbf{S})$ we define a partial order, denoted by $\prec^{\mathbf{S}}$, as follows: for two elements $c_0 \equiv \{(\mathbf{M}_0, \mathrm{R}_0)\}$ and $c_1 \equiv \{(\mathbf{M}, \mathrm{R}_1)\}$ of $\mathrm{C}(\mathbf{S})$ we write $c_0 \prec^{\mathbf{S}} c_1$ if and only if $(\mathbf{M}_0, \mathrm{R}_0) \prec^{cm}_{af} (\mathbf{M}_1, \mathrm{R}_1)$. The partial ordered set $\mathrm{C}(\mathbf{S})$ is directed and each subset A of $\mathrm{C}(\mathbf{S})$ of cardinality $\leq \tau$ has supremum denoted by $\sup(A)$.

If $(\mathbf{M}_0, \mathrm{R}_0), (\mathbf{M}_1, \mathrm{R}_1) \in c \in \mathrm{C}(\mathbf{S})$, then $\mathrm{T}(\mathbf{M}_0, \mathrm{R}_0) = \mathrm{T}(\mathbf{M}_1, \mathrm{R}_1)$. By $\mathrm{T}(c)$ we denote the space $\mathrm{T}(\mathbf{M}, \mathrm{R})$, where $(\mathbf{M}, \mathrm{R}) \in c$.

For a class $I\!\!P$ of spaces we set

$$\mathrm{C}(I\!\!P) = \{ c \in \mathrm{C}(\mathbf{S}) : \mathrm{T}(c) \in I\!\!P \}.$$

A non-empty class $I\!\!P$ of spaces is said to be *second-type saturated* if for every indexed collection $\mathbf{S}$ of elements of $I\!\!P$ the set $\mathrm{C}(I\!\!P)$ contains a cofinal τ-closed subset of $\mathrm{C}(\mathbf{S})$ (that is the set $\mathrm{C}(I\!\!P)$ contains a cofinal subset A such that for every chain A' of A with $|A'| \leq \tau$, we have $\sup(A') \in A$ whenever the $\sup(A')$ exists).

In what follows, we denote by $I\!\!P(\mathbf{S})$ the class of all spaces homeomorphic to an element of $\mathbf{S}$.

2 The game $G_\nu(\mathbf{S})$

Let ν be an infinite cardinal, $\nu \leq \tau$. Players I and II take turns in playing elements of $\mathrm{C}(\mathbf{S})$ as follows:

Player I:	c^1_0		c^1_1	$\cdots$	c^1_δ		$\cdots$
							$\cdots$
Player II:		c^2_0		c^2_1 $\cdots$		c^2_δ	$\cdots$

Note that at all limit stages Player I goes first. So, we have two (transfinite) sequences $(c^1_\delta : \delta \in \nu)$ and $(c^2_\delta : \delta \in \nu)$ of elements of $\mathrm{C}(\mathbf{S})$. We say that Player II *wins* this run of the game provided that T(c) belongs to $I\!\!P(\mathbf{S})$, where c is the supremum of the set

$$\{ c^1_\delta : \delta \in \nu \} \cup \{ c^2_\delta : \delta \in \nu \}.$$

Thus, Player I wins if $\mathrm{T(c)} \notin I\!\!P(\mathbf{S})$. This game is denoted by $G_\nu(\mathbf{S})$.

3 The game $G^{cm}(\mathbf{S})$

This game is defined as follows: Players I and II take turns in playing $\mathbf{S}$-open sets $U_\delta^{\mathbf{S}} \equiv \{ U_\delta^X : X \in \mathbf{S} \}$, $\delta \in \tau$, as follows:

Player I:	$U_0^{\mathbf{S}}$		$U_2^{\mathbf{S}}$		$\cdots$	$U_\delta^{\mathbf{S}}$		$\cdots$
								$\cdots$
Player II:		$U_1^{\mathbf{S}}$		$U_3^{\mathbf{S}}$	$\cdots$		$U_{\delta+1}^{\mathbf{S}}$	$\cdots$

Note that at all limit stages Player I goes first.

We say that player II *wins* this run of the game if the indexed collection

$$\mathbf{M} \equiv \{ \{ U_\delta^X : \delta \in \tau \} : X \in \mathbf{S} \}$$

is a co-mark of $\mathbf{S}$ and $\mathrm{T}(\mathbf{M}, \mathrm{R}_{\mathbf{M}}) \in I\!\!P(\mathbf{S})$. Thus, player I wins if either $\mathbf{M}$ is not a co-mark of $\mathbf{S}$ or $\mathbf{M}$ is a co-mark of $\mathbf{S}$ and $\mathrm{T}(\mathbf{M}, \mathrm{R}_{\mathbf{M}}) \notin I\!\!P(\mathbf{S})$.

1 Proposition. *Let $\mathbf{S}$ be an indexed collection of spaces such that $I\!\!P(\mathbf{S})$ is a second type saturated class of spaces. Then, Player II has a winning strategy in the game $G_\nu(\mathbf{S})$.*

Proof. Since $I\!\!P(\mathbf{S})$ is a second type saturated class the set $\mathrm{C}(I\!\!P(\mathbf{S}))$ contains a cofinal τ-closed subset of $\mathrm{C}(\mathbf{S})$. Denote this subset by A and let $\{ a_\delta : \delta \in \mu \}$ be an indication of A, where μ is the cardinal of A.

We define a strategy for the Player II choosing the element c_δ^2, $\delta \in \nu$, as follows. Let

$$C_\delta = \{ c_\eta^1 : \eta \in \delta \} \cup \{ c_\eta^2 : \eta \in \delta \} \cup \{ c_\delta^1 \}.$$

Obviously $|C_\delta| \leq \nu$. If $\sup(C_\delta) \in A$, then we set $c_\delta^2 = \sup(C_\delta)$. If $\sup(C_\delta) \notin A$, then we denote by ε the minimal of all ordinals η such that $\sup(C_\delta) \prec^{\mathbf{S}} a_\eta$. In this case, we set $c_\delta^2 = a_\varepsilon$. Thus, $c_\delta^2 \in A$ for every $\delta \in \nu$.

Since A is τ-closed and $\nu \leq \tau$, by the choose of elements of c_δ^2 we have

$$c \equiv \sup(\{ c_\delta^1 : \delta \in \nu \} \cup \{ c_\delta^2 : \delta \in \nu \}) = \sup(\{ c_\delta^2 : \delta \in \nu \}) \in A,$$

which means that $T(c) \in I\!\!P(\mathbf{S})$. Thus, the defined strategy for the Player II is winning. QED

2 Proposition. *Let $\mathbf{S}$ be an indexed collection of spaces such that $I\!\!P(\mathbf{S})$ is a second type saturated class of spaces. Then, Player II has winning strategy in the game $G^{cm}(I\!\!P(\mathbf{S}))$.*

Proof. Since $I\!\!P(\mathbf{S})$ is a second type saturated class there exists a set A which is a cofinal τ-closed subset of $\mathrm{C}(\mathbf{S})$ such that $T(c) \in I\!\!P(\mathbf{S})$ for every $c \in A$.

For every $\mathbf{S}$-open set $U^{\mathbf{S}}$ we denote by $c(U^{\mathbf{S}})$ an element of A and by $(\mathbf{M}(\mathrm{U}^{\mathbf{S}}), \mathrm{R}(\mathrm{U}^{\mathbf{S}}))$ an element of $c(U^{\mathbf{S}})$ such that $U^{\mathbf{S}}$ is the ε-component of $\mathbf{M}(U^{\mathbf{S}})$ for some $\varepsilon \in \tau$. In what follows, we shall identify the $\mathbf{S}$-open set $U^{\mathbf{S}}$ with the sequence $\overline{s} \equiv (U_0^{\mathbf{S}})$ consisting of one element.

By induction, for every (transfinite) sequence $\overline{s} \equiv (U_0^{\mathbf{S}}, U_1^{\mathbf{S}}, \ldots, U_\delta^{\mathbf{S}})$ of $\mathbf{S}$-open sets of the ordinal-type $\delta + 1$, where $\delta \in \tau \setminus \{0\}$, we denote by $c(\overline{s})$ an element of A and by $(\mathbf{M}(\overline{\mathrm{s}}), \mathrm{R}(\overline{\mathrm{s}}))$ an element of $c(\overline{s})$ such that: (a) for every $\delta' \in \delta$ we have

$$(\mathbf{M}(\overline{\mathrm{s}}'), \mathrm{R}(\overline{\mathrm{s}}')) \prec_{af}^{cm} (\mathbf{M}(\overline{\mathrm{s}}), \mathrm{R}_{\mathbf{M}(\overline{\mathrm{s}})}) \prec_{af}^{cm} (\mathbf{M}(\overline{\mathrm{s}}), \mathrm{R}(\overline{\mathrm{s}})),$$

where $\overline{s}' \equiv (U_0^{\mathbf{S}}, U_1^{\mathbf{S}}, \ldots, U_{\delta'}^{\mathbf{S}})$, (therefore, $c(\overline{s}') \prec^{\mathbf{S}} c(\overline{s})$), and (b) $U_\delta^{\mathbf{S}}$ is the ε-component of $\mathbf{M}(\overline{s})$ for some $\varepsilon \in \tau$. Note that the existence of a co-mark $\mathbf{M}(\overline{\mathrm{s}})$ satisfying relation

$$(\mathbf{M}(\overline{\mathrm{s}}'), \mathrm{R}(\overline{\mathrm{s}}')) \prec_{af}^{cm} (\mathbf{M}(\overline{\mathrm{s}}), \mathrm{R}_{\mathbf{M}(\overline{\mathrm{s}})})$$

follows by Lemma 8.2.6 of [1].

We denote by ψ a one-to-one map of $\tau \times \tau$ into τ satisfying the following conditions: (a) the subset $\psi(\tau \times \tau)$ coincides with the set of all odd ordinal of τ (the limits ordinal are considered to be even) and (b) for every odd ordinal δ of τ, $\psi^{-1}(\delta)$ is an element of the set $\tau \times \{\delta'\}$ for some $\delta' \in \delta$. The existence of such a map is clear.

Now, we define a strategy for the Player II. This means that for each odd ordinal $\delta = \delta_0 + 1$ the (transfinite) sequence $\overline{s}_{\delta_0} \equiv (U_0^{\mathbf{S}}, \ldots, U_{\delta_0}^{\mathbf{S}})$ must uniquely determine the $\mathbf{S}$-open set $U_\delta^{\mathbf{S}}$. For this purpose, this set is chosen by induction as follows. If $\psi^{-1}(\delta) = (\varepsilon, \eta)$, then the sequence $\overline{s}_\eta \equiv (U_0^{\mathbf{S}}, \ldots, U_\eta^{\mathbf{S}})$ is already defined and, therefore, is defined the pair $(\mathbf{M}(\overline{\mathrm{s}}_\eta), \mathrm{R}(\overline{\mathrm{s}}_\eta))$. Let

$$\mathbf{M}(\overline{\mathrm{s}}_\eta) = \{\{U_\kappa^{\eta,X} : \kappa \in \tau\} : X \in \mathbf{S}\}.$$

Then, we set $U_\delta^{\mathbf{S}} = \{U_\varepsilon^{\eta,X} : X \in \mathbf{S}\}$.

Now, we prove that the defined strategy for the Player II is winning. That is, if $U_\delta^{\mathbf{S}} = \{U_\delta^X : X \in \mathbf{S}\}$, $\delta \in \tau$, then the indexed collection

$$\mathbf{M} \equiv \{\{U_\delta^X : \delta \in \tau\} : X \in \mathbf{S}\}$$

is a co-mark of $\mathbf{S}$ and $\mathrm{T}(\mathbf{M}, \mathrm{R}_{\mathbf{M}}) \in I\!\!P(\mathbf{S})$. Note that $U_\delta^{\mathbf{S}}$ is the δ-component of $\mathbf{M}$.

First, we prove that for every $\eta \in \tau$, any component of $\mathbf{M}(\overline{\mathrm{s}}_\eta)$ is a component of $\mathbf{M}$. Indeed, let $\varepsilon \in \tau$. There exists an odd ordinal $\delta \in \tau$ such that $\psi^{-1}(\delta) = (\varepsilon, \eta)$. By construction

$$U_\delta^{\mathbf{S}} = \{U_\varepsilon^{\eta,X} : X \in \mathbf{S}\}.$$

This means that the ε-component of $\mathbf{M}(\bar{\mathrm{s}}_\eta)$ is a δ-component of $\mathbf{M}$. Thus, $\mathbf{M}$ is a co-mark of $\mathbf{S}$ and $\mathbf{M}(\bar{\mathrm{s}}_\eta) \prec^{\mathrm{cm}} \mathbf{M}$ for $\eta \in \delta$. The last relation implies that

$$(\mathbf{M}(\bar{\mathrm{s}}_\eta), \mathrm{R}_{\mathbf{M}(\bar{\mathrm{s}}_\eta)}) \prec^{cm}_{af} (\mathbf{M}, \mathrm{R}_{\mathbf{M}})$$

and, therefore,

$$\{(\mathbf{M}(\bar{\mathrm{s}}_\eta), \mathrm{R}_{\mathbf{M}(\bar{\mathrm{s}}_\eta)})\} \prec^{\mathbf{S}} \{(\mathbf{M}, \mathrm{R}_{\mathbf{M}})\},\ \eta \in \tau.$$

Now, suppose that there exists $\{(\mathbf{M}', \mathrm{R}')\} \in C(\mathbf{S})$ such that

$$\{(\mathbf{M}(\bar{\mathrm{s}}_\eta), \mathrm{R}(\bar{\mathrm{s}}_\eta))\} \prec^{\mathbf{S}} \{(\mathbf{M}', \mathrm{R}')\}.$$

Then,

$$(\mathbf{M}(\bar{\mathrm{s}}_\eta), \mathrm{R}_{\mathbf{M}(\bar{\mathrm{s}}_\eta)}) \prec^{cm}_{af} (\mathbf{M}', \mathrm{R}').$$

In particular, this imply that any component of $\mathbf{M}(\bar{s}_\eta)$ is a component of $\mathbf{M}'$ and, therefore, any component of $\mathbf{M}$ is a component of $\mathbf{M}'$, which means $\mathbf{M} \prec^{cm} \mathbf{M}'$. Thus, in order to prove that

$$\{(\mathbf{M}, \mathrm{R}_{\mathbf{M}})\} \prec^{\mathbf{S}} \{(\mathbf{M}', \mathrm{R}')\}$$

it is suffices to prove that R' is a final refinement of $\mathrm{R}_{\mathbf{M}}$. Suppose that

$$\mathrm{R}_{\mathbf{M}} = \{ \sim^{\mathrm{s}}_{\mathbf{M}} : \mathrm{s} \in \mathcal{F} \},$$

$$\mathrm{R}_{\mathbf{M}'} = \{ \sim^{\mathrm{s}}_{\mathbf{M}'} : \mathrm{s} \in \mathcal{F} \},$$

$$\mathrm{R}(\bar{\mathrm{s}}_\eta) = \{ \sim^{\mathrm{s}}_{\eta} : \mathrm{s} \in \mathcal{F} \}, \quad \eta \in \tau,$$

and

$$\mathrm{R}' = \{ \sim^{\prime,\mathrm{s}} : \mathrm{s} \in \mathcal{F} \}.$$

Let $s \equiv \{ \delta_0, \dots, \delta_n \} \in \mathcal{F}$. For every $i \in \{ 0, \dots, n \}$, there exist $\varepsilon_i, \eta_i \in \tau$ such that ε_i-component of $\mathbf{M}(\bar{\mathrm{s}}_{\eta_{\mathrm{i}}})$ coincides with δ_i-component of $\mathbf{M}$. Also, there exist ε'_i such that ε'_i-component of M' coincides with δ_i-component of $\mathbf{M}$. Let $t' = \{ \varepsilon'_0, \dots, \varepsilon'_n \} \in \mathcal{F}$. Since R' is final refinement of $\mathrm{R}_{\mathbf{M}'}$, there exists $t \in \mathcal{F}$ such that $\sim^{\prime,t} \subseteq \sim^{t'}_{\mathbf{M}'}$. On the other hand, $\sim^{s}_{\mathbf{M}} = \sim^{t'}_{\mathbf{M}'}$ and, therefore, $\sim^{\prime,t}_{\mathbf{M}'} \subseteq \sim^{s}_{\mathbf{M}}$, which means that R' is final refinement of $\mathrm{R}_{\mathbf{M}}$. Hence,

$$\{(\mathbf{M}, \mathrm{R}_{\mathbf{M}})\} \prec^{\mathbf{S}} \{(\mathbf{M}', \mathrm{R}')\}$$

which means that

$$c = \{(\mathbf{M}, \mathrm{R}_{\mathbf{M}})\} = \sup\{ \{(\mathbf{M}(\bar{\mathrm{s}}_\eta), \mathrm{R}(\bar{\mathrm{s}}_\eta))\} : \eta \in \delta \}.$$

Since A is τ-closed, $c \in A$ and, therefore, $T(c) \in I\!\!P(\mathbf{S})$. Thus, the defined strategy of Player II is winning. $\boxed{QED}$

3 Corollary. *Let* $\mathbf{S}$ *be an indexed collection of spaces such that* $I\!P(\mathbf{S})$ *is one of the following classes of spaces:*

(1) The class of all spaces.

(2) The class of countably-dimensional spaces.

(3) The class of all strongly countable dimensional spaces.

(4) The class of all locally finite-dimensional spaces.

(5) The class of all spaces of dimension ind $\leq \alpha$*, where* $\alpha \in \tau^{+}$*.*

Then, Player II has a winning strategy in the games $G_{\nu}(\mathbf{S})$ *and* $G^{cm}(\mathbf{S})$*.*

PROOF. The proof of this corollary follows by the fact that the mentioned classes of spaces are second type saturated classes (see [1]). QED

4 Problem. Let $\mathbf{S}$ be an indexed collection of spaces such that Player II has a winning strategy in the game $G_{\nu}(\mathbf{S})$ (respectively, in the game $G^{cm}(\mathbf{S})$). Is the class $I\!P(\mathbf{S})$ second type saturated?

In our opinion the answer to Problem 1 is negative. In this case, it is interesting to study topological properties of the class $I\!P(\mathbf{S})$. One of such property is the existence of universal elements. We have the following proposition.

5 Proposition. *Let* $\mathbf{S}$ *be an indexed collection of spaces such that Player II has a winning strategy in the game* $G_{\nu}(\mathbf{S})$ *(respectively, in the game* $G^{cm}(\mathbf{S})$*). Then, the class* $I\!P(\mathbf{S})$ *of spaces has a universal element.*

PROOF. Since the Player II has a winning strategy in the game $G_{\nu}(\mathbf{S})$ (respectively, in the game $G^{cm}(\mathbf{S})$) there exists an element $c \in \mathrm{C}(\mathbf{S})$ (respectively, a co-mark

$$\mathbf{M} \equiv \{ \{ U_{\delta}^{X} : \delta \in \tau \} : X \in \mathbf{S} \}$$

of $\mathbf{S}$) such that $\mathrm{T}(c) \in I\!P(\mathbf{S})$ (respectively, $\mathrm{T}(\mathbf{M}, \mathrm{R}_{\mathbf{M}}) \in I\!P(\mathbf{S})$). Since $\mathrm{T}(c)$ and $\mathrm{T}(\mathbf{M}, \mathrm{R}_{\mathbf{M}})$ are Containing Spaces, each element X of $\mathbf{S}$ and, therefore, each element X of $I\!P(\mathbf{S})$ is contained topologically in these spaces, which means that they are universal elements in $I\!P(\mathbf{S})$. QED

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