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On increasing rapidly varying sequences

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Abstract. The class $\mathbb{R}_{\infty,s}$ of increasing rapidly varying sequences is investigated in connection with selection principles and game theory. We show that this class satisfies a Rothberger-type selection property as well as a game-theoretical property.

Keywords: rapidly varying sequence, rapidly varying function, numerical function, selection principles, α_i-properties, infinite game

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Introduction

We investigate some classes of sequences of positive real numbers which are important for analysis of divergent asymptotic processes. The main goal of this investigation is to establish some topological properties of these classes of sequences.

A function $f : [a, +\infty) \to (0, +\infty)$, $a > 0$, is called *slowly varying* (in the sense of Karamata) [1] if it is measurable and satisfies the following asymptotic condition

$$\lim_{x\to+\infty} \frac{f(\lambda x)}{f(x)} = 1,\ \lambda > 0. \tag{1}$$

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The class of slowly varying functions we denote by SV_f. These functions were introduced and firstly studied by J. Kamarata in his famous, pioneering 1930 paper [8], and are a part of the so-called Karamata theory (see [1] for a detail exposition).

A sequence $(c_n)_{n\in\mathbb{N}}$ of positive real numbers is said to be *slowly varying* (in the sense of Karamata) [2] if

$$\lim_{n\to+\infty}\frac{c_{[\lambda n]}}{c_n}=1,\ \ \lambda>0. \tag{2}$$

The class of slowly varying sequences we denote by SV_s.

Slow variability in the sense of Karamata is an important and widely investigated asymptotic property in analysis of divergent processes and is of fundamental importance in the theory of Tauberian theorems [1].

In [2], R. Bojanić and E. Seneta found (see also [6]) a nice qualitative relation between sequential property (2) and functional property (1) and established a unique concept of interpretation and development of the theory of slow variability in the sense of Karamata.

1 Theorem. *For a sequence $(c_n)_{n\in\mathbb{N}}$ of positive real numbers the following are equivalent:*

(*a*) *$(c_n)_{n\in\mathbb{N}}$ belongs to the class SV_s;*

(*b*) *The function f defined by $f(x)=c_{[x]}$, $x\geq 1$, is in the class SV_f.*

Some results based on Theorem 1 and related to O-regular variability, extended regular variability and SO-regular variability can be found in the papers [3, 4, 5].

1 Rapid variability

Let $(c_n)_{n\in\mathbb{N}}$ be a strictly increasing, unbounded sequence of positive real numbers. The *numerical function of* $(c_n)_{n\in\mathbb{N}}$ is the function $\delta_c:[c_1,+\infty)\to\mathbb{N}$ defined by

$$\delta_c(x)=\max\{\,n\in\mathbb{N}\mid c_n\leq x\,\}.$$

The numerical function of a sequence is an important characteristic of divergent sequences (see [9]). We have the following result.

2 Theorem. *Let $(c_n)_{n\in\mathbb{N}}$ be a strictly increasing, unbounded sequence of positive real numbers. Then:*

(a) $(c_n)_{n\in\mathbb{N}}$ belongs to the class SV_s if and only if the function δ_c satisfies

$$\lim_{x\to+\infty}\frac{\delta_c(\lambda x)}{\delta_c(x)}=0,\ \ \lambda\in(0,1). \tag{3}$$

(b) The sequence $(c_n)_{n\in\mathbb{N}}$ *satisfies the condition*

$$\lim_{n\to+\infty}\frac{c_{[\lambda n]}}{c_n}=0,\quad \lambda\in(0,1) \tag{4}$$

if and only if the function δ_c *belongs to the class* SV_f.

Proof. (a): Consider a continuous strictly increasing function $f:[1,+\infty)\to\mathbb{R}$, such that $f(n)=c_n$ for each $n\in\mathbb{N}$. Observe that f is a positive, measurable function, and that for $x\geq 1$ there exists $n_x\in\mathbb{N}$ such that $c_{n_x}\leq f(x)<c_{n_x+1}$, e.g. $n_x=[x]$, $x\geq 1$.

$(a.1)$ Let $(c_n)_{n\in\mathbb{N}}$ belongs to the class SV_s. Then by Theorem 1, the function φ defined by $\varphi(x)=c_{[x]}$, $x\geq 1$, is in the class SV_f. So, for $x\geq 1$ we have

$$1\leq\frac{f(x)}{\varphi(x)}\leq\frac{c_{[x]+1}}{c_{[x]}},$$

and thus by [4], $f(x)\sim\varphi(x)$, $x\to+\infty$. In other words, $f\in\mathsf{SV}_f$. By [7], the function $f^{-1}(x)$, $x\geq c_1$, is in the class of functions satisfying the asymptotic condition (3). Since $\delta_c(x)=[f^{-1}(x)]$, $x\geq c_1$, (see [9]), it follows $\delta_c(x)\sim f^{-1}(x)$, $x\to+\infty$, so that the function δ_c satisfies (3).

$(a.2)$ If the function δ_c satisfies the condition (3), then according to $(a.1)$, also f^{-1} satisfies the same condition. By [7] the function f belongs to the class SV_f, and thus for $\lambda>0$ (see [4])

$$\lim_{n\to+\infty}\frac{c_{[\lambda n]}}{c_n}=\lim_{n\to+\infty}\frac{f([\lambda n])}{f(n)}=\lim_{n\to+\infty}\frac{f([\lambda n])}{f(\lambda n)}\cdot\lim_{n\to+\infty}\frac{f(\lambda n)}{f(n)}=1.$$

This just means that the sequence $(c_n)_{n\in\mathbb{N}}$ is in the class SV_s.

(b): $(b.1)$ For the sequence $(c_n)_{n\in\mathbb{N}}$ consider the function f as in (a). Fix $\lambda\in(0,1)$. If we suppose that $(c_n)_{n\in\mathbb{N}}$ satisfies (4), then

$$\lim_{x\to+\infty}\sup_{t\in(0,\lambda]}\frac{c_{[tn]}}{c_n}=\lim_{n\to+\infty}\frac{c_{[\lambda n]}}{c_n}=0,$$

i.e. if $(c_n)_{n\in\mathbb{N}}$ satisfies (4), then for $\alpha\in(0,1)$ we have

$$0\leq\limsup_{x\to+\infty}\frac{f(\alpha x)}{f(x)}\leq\limsup_{x\to+\infty}\frac{c_{[\alpha x]+1}}{c_{[x]}}=0,$$

because for $x\geq x_0(\alpha)$ it holds $\frac{[\alpha x]+1}{[x]}\in(0,\lambda]$, where $\lambda=\frac{1}{2}(1+\alpha)\in(0,1)$. Then by [7] the function $f^{-1}(x)$, $x\geq c_1$, belongs to the class SV_f. By the same arguments as in (a), we have $\delta_c(x)\sim f^{-1}(x)$, $x\to+\infty$, and so $\delta_c\in\mathsf{SV}_f$.

$(b.2)$ If $\delta_c \in \mathsf{SV}_f$, then also $f^{-1} \in \mathsf{SV}_f$ (see (b.1)), and therefore, by [7], the function f satisfies the condition (3). This means that for $\lambda \in (0,1)$ it holds

$$0 \leq \limsup_{n\to+\infty} \frac{c_{[\lambda n]}}{c_n} = \limsup_{n\to+\infty} \frac{f([\lambda n])}{f(n)} \leq \limsup_{n\to+\infty} \frac{f([\lambda n])}{f(\lambda n)} \cdot \limsup_{n\to+\infty} \frac{f(\lambda n)}{f(n)} = 0,$$

i.e. the sequence $(c_n)_{n\in\mathbb{N}}$ satisfies (4). $\boxed{QED}$

A sequence $(c_n)_{n\in\mathbb{N}}$ of positive real numbers is said to be *rapidly varying* (in the sense of de Haan) with index of variability $+\infty$, if it satisfies the condition (4) for each $\lambda \in (0,1)$ (see [1]).

The class of such sequences is denoted by $\mathsf{R}_{\infty,s}$.

A function $f : [a, +\infty) \to (0, +\infty)$, $a > 0$, is said to be *rapidly varying* (in the sense of de Haan) of index $+\infty$ [7] (see also [1]) if it is measurable and satisfies the asymptotic condition (3) for each $\lambda \in (0,1)$.

The class of rapidly varying functions we denote by $\mathsf{R}_{\infty,f}$.

The classes $\mathsf{R}_{\infty,f}$ and $\mathsf{R}_{\infty,s}$ are important objects of asymptotic analysis (see [1]), and in particular their subclasses of increasing functions and increasing sequences; denote this subclasses by $\mathsf{IR}_{\infty,f}$ and $\mathsf{IR}_{\infty,s}$, respectively. A justification for this notation we can find in the following fact.

3 Proposition. *For each sequence $(c_n)_{n\in\mathbb{N}}$ from the class $\mathsf{IR}_{\infty,s}$ it holds*

$$\lim_{n\to+\infty} c_n = +\infty.$$

Proof. Since the sequence $(c_n)_{n\in\mathbb{N}}$ is increasing, it has a limit in $\overline{\mathbb{R}}$. If it holds $\lim_{n\to+\infty} c_n = A \in \mathbb{R}$, then we have $(c_n)_{n\in\mathbb{N}} \in \mathsf{SV}_s$. But this is impossible, because $\mathsf{R}_{\infty,s} \cap \mathsf{SV}_s = \emptyset$. $\boxed{QED}$

4 Proposition. *For an increasing sequence $(c_n)_{n\in\mathbb{N}}$ of positive real numbers the following are equivalent:*

(a) $(c_n)_{n\in\mathbb{N}} \in \mathsf{R}_{\infty,s}$;

(b) The function $f(x) = c_{[x]}$, $x \geq 1$, belongs to the class $\mathsf{R}_{\infty,f}$;

(c) $\lim_{n\to+\infty} \frac{c_{[\lambda n]}}{c_n} = +\infty$, *for each $\lambda > 1$.*

Proof. $(a) \Rightarrow (b)$: Let $(c_n)_{n\in\mathbb{N}} \in \mathsf{R}_{\infty,s}$. Then for any $\lambda \in (0,1)$

$$0 \leq \limsup_{x\to+\infty} \frac{c_{[\lambda x]}}{c_{[x]}} \leq \limsup_{n\to+\infty} \frac{c_{[pn]}}{c_n} = 0,$$

where $p = \frac{1+\lambda}{2} \in (0,1)$, because $\lambda x = \frac{\lambda x}{[x]} \cdot [x] \leq p \cdot [x]$ for sufficiently large $x \geq x_0(\lambda)$. Therefore, the increasing and positive function $f(x) = c_{[x]}$, $x \geq 1$, satisfies the condition (3) and thus belongs to the class $\mathsf{R}_{\infty,f}$.

$(b) \Rightarrow (a)$: It is trivial.

$(b) \Rightarrow (c)$: Suppose $f(x) = c_{[x]}$, $x \geq 1$, is an element of the class $\mathsf{R}_{\infty,f}$. Then by [7]

$$\lim_{x \to +\infty} \frac{c_{[\lambda x]}}{c_{[x]}} = +\infty,$$

for every $\lambda > 1$, and (c) holds.

$(c) \Rightarrow (b)$: From (c) it follows that for $\lambda > 1$ we have

$$\liminf_{x \to +\infty} \frac{c_{[\lambda x]}}{c_{[x]}} \geq \liminf_{n \to +\infty} \frac{c_{[\lambda n]}}{c_n} = +\infty,$$

since $\lambda x = \frac{\lambda x}{[x]} \cdot [x] \geq \lambda \cdot [x]$ for every $x \geq 1$. According to [7], then f belongs to the class $\mathsf{R}_{\infty,f}$. $\boxed{QED}$

From the proofs of Theorem 2 and Proposition 4, we conclude that the asymptotic conditions expressed by (4) and (c) in Proposition 4 are uniform on $(0, p]$, $p \in (0, 1)$, and $[q, +\infty)$, $q > 1$, respectively.

2 Selections, games and $\mathbb{R}_{\infty,s}$

In this section we show that the class of increasing rapidly varying sequences satisfies a selection principle and a game-theoretical condition.

Let $\mathcal{A}$ and $\mathcal{B}$ be sets whose elements are families of subsets of an infinite set X. Then (see [11]):
$\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis:

> For each sequence $(A_n \mid n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(b_n \mid n \in \mathbb{N})$ such that for each n $b_n \in A_n$ and $\{ b_n \mid n \in \mathbb{N} \}$ is an element of $\mathcal{B}$.

There is an infinitely long game associated to $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$.

The symbol $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$ denotes the infinitely long game for two players, ONE and TWO, who play a round for each positive integer. In the n-th round ONE chooses a set $A_n \in \mathcal{A}$, and TWO responds by choosing an element $b_n \in A_n$. TWO wins a play $(A_1, b_1; \ldots; A_n, b_n; \ldots)$ if $\{ b_n \mid n \in \mathbb{N} \} \in \mathcal{B}$; otherwise, ONE wins.

It is evident that if ONE does not have a winning strategy in the game $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$, then the selection hypothesis $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ is true. The converse implication need not be always true, but many properties described by selection principles can be characterized by the corresponding game (see [11]). (It is also clear that "TWO has a winning strategy in $\mathsf{G}_1(\mathcal{A},\mathcal{B})$" implies "ONE has no winning strategy in $\mathsf{G}_1(\mathcal{A},\mathcal{B})$".)

A strategy σ for player TWO is a *coding strategy* if TWO remembers only the most recent move by ONE and by TWO before deciding how to play the next move. More precisely the moves of TWO are: $b_1 = \sigma(A_1,\emptyset)$; $b_n = \sigma(A_n, b_{n-1})$, $n \geq 2$.

5 Theorem. *The player* TWO *has a winning coding strategy in the game* $\mathsf{G}_1(\mathbb{R}_{\infty,s},\mathbb{R}_{\infty,s})$.

PROOF. Let us define a strategy σ for TWO in the following way. Suppose that in the first round ONE plays $s_1 = (c_{1,m})_{m\in\mathbb{N}}$ from $\mathbb{R}_{\infty,s}$. Then TWO responds by choosing $\sigma(s_1,\emptyset) = c_{1,m_1}$, where c_{1,m_1} is *any* element in s_1. Let in the second round ONE play $s_2 = (c_{2,m})_{m\in\mathbb{N}}$; TWO (can apply Proposition 3) finds $c_{2,m_2} \in s_2$ such that $c_{2,m_2} > 2\cdot c_{1,m_1}$ and responds by $\sigma(s_2, c_{1,m_1}) = c_{2,m_2}$. If in the n-th round ONE has played $s_n = (c_{n,m})_{m\in\mathbb{N}}$, then TWO chooses $c_{n,m_n} \in s_n$ such that $c_{n,m_n} > 2\cdot c_{n-1,m_{n-1}}$ and plays $\sigma(s_n, c_{n-1,m_{n-1}}) = c_{n,m_n}$. And so on.

We claim that $(a_n)_{n\in\mathbb{N}} \in \mathbb{R}_{\infty,s}$. It is evident that the sequence $(a_n)_{n\in\mathbb{N}}$ is strictly increasing. Let us show that it is rapidly varying.

Let $\lambda > 1$. For $n \in \mathbb{N}$ we have

$$\frac{a_{[\lambda n]}}{a_n} = \frac{a_{[\lambda n]}}{a_{[\lambda n]-1}}\cdots\frac{a_{n+1}}{a_n}.$$

On the right side of the previous equality we have $[\lambda n] - n$ factors, and also $[\lambda n] - n > (\lambda n - 1) - n = (\lambda - 1)n - 1$. So,

$$\frac{a_{[\lambda n]}}{a_n} > 2^{[\lambda n]-n} > 2^{(\lambda-1)n-1},\ n \in \mathbb{N},$$

and therefore

$$\lim_{n\to+\infty}\frac{a_{[\lambda n]}}{a_n} = +\infty.$$

By Proposition 4 one concludes that $(a_n)_{n\in\mathbb{N}} \in \mathsf{R}_{\infty,s}$. QED

6 Corollary. *The class* $\mathbb{R}_{\infty,s}$ *satisfies the selection principle* $\mathsf{S}_1(\mathbb{R}_{\infty,s},\mathbb{R}_{\infty,s})$.

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are as above, $n, k \in \mathbb{N}$, and let for a set A the symbol $[A]^n$ denote the set of all n-element subsets of A. We are going now to show that the class $\mathbb{R}_{\infty,s}$ satisfies a combinatorial principle from Ramsey theory known as the *ordinary partition relation*

$$\mathcal{A} \to (\mathcal{B})^n_k$$

which is the statement:

> For each $A \in \mathcal{A}$ and for each function $f : [A]^n \to \{ 1, \dots, k \}$ there are a set $B \in \mathcal{B}$ with $B \subset A$ and some $i \in \{ 1, \dots, k \}$ such that for each $Y \in [B]^n$, $f(Y) = i$.

Note that several selection principles of the form $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ have been characterized by the ordinary partition relation (see [10]).

7 Theorem. *The class $\mathsf{R}_{\infty,s}$ satisfies the ordinary partition relation*

$$\mathsf{R}_{\infty,s} \to (\mathsf{R}_{\infty,s})^n_k, \ \ n, k \in \mathbb{N}.$$

Proof. We prove the theorem for $n = k = 2$; by a standard induction argument on n and k, the usual method for proving Ramsey theoretical statements for $n > 2$, $k > 2$ (see, for example, Theorem 1 in [13]), we can prove the general case. Let $s = (c_1, c_2, \dots)$ be a sequence in $\mathsf{R}_{\infty,s}$ and let $f : [s]^2 \to \{ 1, 2 \}$ be a coloring. It is easy to verify that one of the sets $s_1 := \{ c_i \in s \mid f(\{c_1, c_i\}) = 1 \}$ and $s_2 := \{ c_i \in s \mid f(\{c_1, c_i\}) = 2 \}$ is in $\mathsf{R}_{\infty,s}$. Denote by i_1 the element from $\{ 1, 2 \}$ for which s_{i_1} is in $\mathsf{R}_{\infty,s}$ and put $q_1 = s_{i_1}$. Inductively define q_n and i_n, $n \geq 2$, such that $q_n := \{ c_i \in q_{n-1} \mid f(\{c_n, c_i\}) = i_n \}$ is an increasing rapidly varying sequence. Apply now $\mathsf{S}_1(\mathsf{R}_{\infty,s}, \mathsf{R}_{\infty,s})$ to the sequence $(q_n)_{n \in \mathbb{N}}$ to choose for each n an element $a_n \in q_n$ such that $a = (a_n)_{n \in \mathbb{N}} \in \mathsf{R}_{\infty,s}$. We may assume that $a_n \neq a_m$ for $n \neq m$ and that there exists $i \in \{ 1, 2 \}$ satisfying: for each $a_m \in a$, $i_m = i$. It follows that $f(\{a_l, a_m\}) = i$ for each $\{a_l, a_m\} \in [a]^2$ and the theorem is shown. $\boxed{QED}$

3 α_i-properties and rapidly varying sequences

In [12], new selection principles were introduced in the following way; $\mathcal{A}$ and $\mathcal{B}$ are as above.

8 Definition. The symbol $\alpha_i(\mathcal{A}, \mathcal{B})$, $i = 2, 3, 4$, denotes the following selection hypothesis:

> For each sequence $(A_n \mid n \in \mathbb{N})$ of infinite elements of $\mathcal{A}$ there is an element $B \in \mathcal{B}$ such that:
>
> $\alpha_2(\mathcal{A}, \mathcal{B})$: for each $n \in \mathbb{N}$ the set $A_n \cap B$ is infinite;
>
> $\alpha_3(\mathcal{A}, \mathcal{B})$: for infinitely many $n \in \mathbb{N}$ the set $A_n \cap B$ is infinite;
>
> $\alpha_4(\mathcal{A}, \mathcal{B})$: for infinitely many $n \in \mathbb{N}$ the set $A_n \cap B$ is nonempty.

Evidently,

$$\alpha_2(\mathcal{A}, \mathcal{B}) \Rightarrow \alpha_3(\mathcal{A}, \mathcal{B}) \Rightarrow \alpha_4(\mathcal{A}, \mathcal{B}).$$

It is also clear that

$$\mathsf{S}_1(\mathbb{R}_{\infty,s}, \mathbb{R}_{\infty,s}) \Rightarrow \alpha_4(\mathbb{R}_{\infty,s}, \mathbb{R}_{\infty,s}).$$

We prove that each of properties $\alpha_i(\mathbb{R}_{\infty,s}, \mathbb{R}_{\infty,s})$, $i = 2, 3, 4$, is satisfied.

9 Theorem. *The class $\mathbb{R}_{\infty,s}$ satisfies the following:*

(1) $\mathsf{S}_1(\mathbb{R}_{\infty,s}, \mathbb{R}_{\infty,s})$;

(2) $\alpha_2(\mathbb{R}_{\infty,s}, \mathbb{R}_{\infty,s})$;

(3) $\alpha_3(\mathbb{R}_{\infty,s}, \mathbb{R}_{\infty,s})$;

(4) $\alpha_4(\mathbb{R}_{\infty,s}, \mathbb{R}_{\infty,s})$.

PROOF. We should prove only (2) because (1) holds by Corollary 6.

Let $(s_n \mid n \in \mathbb{N})$ be a sequence of elements of $\mathbb{R}_{\infty,s}$. For each $n \in \mathbb{N}$ consider a sequence $(s_{n,m} \mid m \in \mathbb{N})$ of pairwise disjoint subsequences of s_n. Evidently each $s_{n,m} \in \mathbb{R}_{\infty,s}$. Apply (1) to the sequence $(s_{n,m} \mid n, m \in \mathbb{N})$ and find a sequence $(c_{n,m})_{n,m\in\mathbb{N}}$ such that for each $(n, m) \in \mathbb{N} \times \mathbb{N}$, $c_{n,m} \in s_{n,m}$ and the sequence $\sigma := (c_{n,m})_{n,m\in\mathbb{N}} \in \mathbb{R}_{\infty,s}$. It is clear that for each $n \in \mathbb{N}$ the set $s_n \cap \sigma$ is infinite, i.e. σ is a selector for the sequence $(s_n \mid n \in \mathbb{N})$ showing that $\alpha_2(\mathbb{R}_{\infty,s}, \mathbb{R}_{\infty,s})$ holds. $\boxed{QED}$

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