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Families of trigonometric thin sets and related exceptional sets

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Abstract. We present a survey of some recent results concerning thin sets of trigonometric series. We focus on those that are related to some set theoretical problems and either depend on the assumed set theory or used a typical set theoretic tool.

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Let

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi nx + b_n \sin 2\pi nx)\,, \tag{1}$$

be a trigonometric series, a_n, b_n, $n \in \omega$, $b_0 = 0$, being reals. In Comptes Rendus des l'Académie des Sciences de Paris 1912 two papers by A. Denjoy [12] and N. N. Luzin [13] with the same title *Sur l'absolue convergence des séries trigonométriques* have independently appeared. In both papers the same result has been proved:

1 Theorem (A. Denjoy – N. N. Luzin)**.** *If the series* (1) *absolutely converges on a set of positive Lebesgue measure, then*

$$\sum_{n=0}^{\infty}(|a_n| + |b_n|) < \infty,$$

i.e. the series (1) *absolutely converges everywhere.*

Note that the set of points in which the series (1) absolutely converges is Borel and therefore Lebesgue measurable. Later on N. N. Luzin proved:

2 Theorem (N. N. Luzin)**.** *If the series* (1) *absolutely converges on a non-meager set, then*

$$\sum_{n=0}^{\infty}(|a_n| + |b_n|) < \infty,$$

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i.e. the series (1) *absolutely converges everywhere.*

In accordance with these results a notion of an AC-set — see e.g. [2] — was introduced: a set $A \subseteq \mathbb{T} = \mathbb{R}/\mathbb{Z}$ is an **AC-set** if every trigonometric series (1) converging absolutely on A converges absolutely everywhere. Thus the Denjoy-Luzin theorem says that any set of positive Lebesgue measure is an AC-set. Similarly the Luzin theorem says that any non-meager set is an AC-set. A set that is not an AC-set is called an **N-set** — see [14]. That was essentially R. Salem (see e.g. [18]) who proved that a set A is an N-set if and only if there exists a sequence $\{a_n\}_{n=0}^{\infty}$ of nonnegative reals such that $\sum_{n=0}^{\infty} a_n = \infty$ and $\sum_{n=0}^{\infty} a_n |\sin(\pi n x)| < \infty$ for every $x \in A$.

We shall work on the compact topological group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Usually we identify $\mathbb{T}$ with the interval $\langle 0, 1\rangle$ with identified end points 0 and 1. For any real $x \in \mathbb{R}$ we denote by $\|x\|$ the distance of x to the nearest integer. Since

$$\|x\| \leq |\sin(\pi x)| \leq \pi \|x\|$$

for every $x \in \mathbb{R}$ we shall often use $\|x\|$ instead of $|\sin(\pi x)|$.

In harmonic analysis another thin sets are considered. Let G be a locally compact topological group, $\hat{G}$ being its dual group. Elements of $\hat{G}$ are continuous homomorphism of G into $\mathbb{T}$ and are called **characters**. Let us consider the Banach space $\mathrm{C}^*(X)$ of continuous bounded real functions on a subset $X \subseteq G$ with the supremum norm $\|f\| = \sup\{\, |f(x)| : x \in X\,\}$. Evidently $\hat{G}|X \subseteq \mathrm{C}^*(X)$. A set X is said to be a **Dirichlet set** if 1_X belongs to the topological closure of $\hat{G}|X$ in $\mathrm{C}^*(X)$. One can easily see that X is a Dirichlet set if and only if there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ of positive integers such that $\|n_k x\| \rightrightarrows 0$ on X. See [8] for bibliographical sources concerning introduced notion.

Denote by $\mathrm{C}^*(X)^*$ the dual space of $\mathrm{C}^*(X)$. Thus $\mathrm{C}^*(X)^*$ is the space of all continuous linear functionals on $\mathrm{C}^*(X)$. There is another important topology on $\mathrm{C}^*(X)$ — the weak topology: the weakest topology in which every $F \in \mathrm{C}^*(X)^*$ is continuous. Thus it seems to be natural to say that a set $X \subseteq G$ is a "weak Dirichlet set" if 1_x belongs to the closure of $\hat{G}|X$ in the weak topology. If X is a locally compact space then by the Riesz theorem $\mathrm{C}^*(X)^*$ is the space of all Borel measures on X. In accordance with this we introduce a notion. A set $X \subseteq \mathbb{T}$ is a **weak Dirichlet set** (shortly **wD-set**) if there exists a universally measurable set $B \subseteq \mathbb{T}$ such that $X \subseteq B$ and for every positive Borel measure μ on B there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ such that

$$\lim_{k\to\infty} \int_B \|n_k x\| \, d\mu(x) = 0.$$

One can easily prove that

$$\text{Dirichlet set} \rightarrow \text{N-set} \rightarrow \text{weak Dirichlet set}.$$

The families of all Dirichlet sets, N-sets and weak Dirichlet sets will be denoted by $\mathcal{D}$, $\mathcal{N}$, and $w\mathcal{D}$, respectively. All those families have important similar properties and their members are called **trigonometric thin sets**.

Important tool in investigation of trigonometric thin sets is (for a proof see e.g. [8]):

3 Theorem (Dirichlet – Minkowski theorem)**.** *Assume that* $\{n_i\}_{i=0}^{\infty}$ *is an increasing sequence of natural numbers. For any reals* $x_1, \dots, x_k \in \mathbb{T}$ *and any* $\varepsilon > 0$*, there are* $i, j \in \omega$ *such that* $0 \leq i < j \leq (2/\varepsilon)^k$ *and*

$$\|(n_j - n_i)x_l\| < \varepsilon \quad \textit{for } l = 1, 2, \dots, k. \tag{2}$$

By this theorem we can omit the atomic measures (such that $\mu(\{x\}) > 0$ for some x) in the definition of a weak Dirichlet set.

Now we introduce a notion (see e.g. [6]). A family $\mathcal{F} \subseteq \mathcal{P}(\mathbb{T})$ is a **family of thin sets** if

(a) $\{x\} \in \mathcal{F}$ for every $x \in \mathbb{T}$;

(b) if $B \subseteq A \in \mathcal{F}$ then $B \in \mathcal{F}$;

(c) no non-empty open interval belongs to $\mathcal{F}$.

A set $\mathcal{G} \subset \mathcal{F}$ is a **base of** $\mathcal{F}$ if for every $A \in \mathcal{F}$ there exists a set $B \in \mathcal{G}$ such that $A \subseteq B$. Evidently $\{A \in \mathcal{N} : A \text{ is a } \mathrm{F}_\sigma \text{ set}\}$ is a base of $\mathcal{N}$. Similarly, $\mathcal{D}$ has a base consisting of closed sets.

The arithmetic difference of two set is defined as

$$A - B = \{z \in \mathbb{T} : (\exists x \in A)(\exists y \in B)\, z = x - y\}.$$

The basic result concerning families of thin sets is based on the famous theorem.

4 Theorem (H. Steinhaus)**.** *If* $A \subseteq \mathbb{T}$ *has positive Lebesgue measure or has the Baire property and is not meager, then the arithmetic difference* $A - A$ *contains a non-empty open intervals.*

As a consequence we obtain:

5 Theorem. *If a family of thin sets* $\mathcal{F}$ *is closed under arithmetic difference, then every set from* $\mathcal{F}$ *is meager and has Lebesgue measure zero.*

Let us remark that both Denjoy-Luzin and Luzin theorems are easy consequence of Theorem 5.

A set $A \subseteq \mathbb{T}$ is **permitted** for the family $\mathcal{F}$ if for any $B \in \mathcal{F}$ also $A \cup B \in \mathcal{F}$. $\mathrm{Perm}(\mathcal{F})$ denotes the family of all permitted sets for $\mathcal{F}$. One can easily see that $\mathrm{Perm}(\mathcal{F}) \subseteq \mathcal{F}$ is an ideal and the equality $\mathrm{Perm}(\mathcal{F}) = \mathcal{F}$ holds true if and only if $\mathcal{F}$ is an ideal. The notion was essentially introduced in [1]. By Theorem 3 every finite set is permitted for $\mathcal{D}$.

For a proof of the next classical result see [1] or [18].

6 Theorem (J. Arbault – P. Erdös)**.** *Every countable subset of $\mathbb{T}$ is permitted for the family $\mathcal{N}$.*

Let us recall the definitions of another thin sets of harmonic analysis as formulated e.g. in [8]. A set $A \subseteq \mathbb{T}$ is called:

- a **pseudo Dirichlet set** (shortly **pD-set**), if there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ of positive integers such that $\{\|n_k x\|\}_{k=0}^{\infty}$ converges quasinormally to 0 on A;
- an **A-set** if there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ of positive integers such that $\{\|n_k x\|\}_{k=0}^{\infty}$ converges pointwise to 0 on A;
- an **N$_0$-set** if there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ of positive integers such that $\sum_{k=0}^{\infty} \|n_k x\| < \infty$ on A;
- a **B$_0$-set** if there exist a real $c > 0$ and an increasing sequence $\{n_k\}_{k=0}^{\infty}$ of positive integers such that $\sum_{k=0}^{\infty} \|n_k x\| < c$ on A;
- a **B-set** if there exist a real $c > 0$ and a sequence $\{a_n\}_{n=0}^{\infty}$ of nonnegative reals, such that $\sum_{n=0}^{\infty} a_n = \infty$ and $\sum_{n=0}^{\infty} a_n \|nx\| < c$ on A.

The corresponding families will be denoted $p\mathcal{D}$, $\mathcal{A}$, $\mathcal{N}_0$, $\mathcal{B}_0$, $\mathcal{B}$, respectively. Of course, in all the definitions the function $\|x\|$ may be replaced by $|\sin \pi x|$.

7 Theorem.

(i) *Every family $\mathcal{D}$, $p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{B}_0$, $\mathcal{N}$, $\mathcal{B}$, $\mathcal{A}$, $w\mathcal{D}$ is a family of thin sets and the following inclusions hold true:*

$$\begin{tikzcd}
 & \mathcal{A} \arrow[r] & w\mathcal{D} \\
p\mathcal{D} \arrow[r] & \mathcal{N}_0 \arrow[u] \arrow[r] & \mathcal{N} \arrow[u] \\
\mathcal{D} \arrow[u] \arrow[r] & \mathcal{B}_0 \arrow[u] \arrow[r] & \mathcal{B} \arrow[u]
\end{tikzcd}$$

(ii) *Every family $\mathcal{D}$, $p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{B}_0$, $\mathcal{N}$, $\mathcal{B}$, $\mathcal{A}$, $w\mathcal{D}$ is closed under arithmetic difference and therefore contains only meager sets of Lebesgue measure zero.*

(iii) *Every family $\mathcal{D}$, $p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{B}_0$, $\mathcal{N}$, $\mathcal{B}$, $\mathcal{A}$, $w\mathcal{D}$ has a Borel basis.*

(iv) *Every family $p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{N}$, $\mathcal{A}$, $w\mathcal{D}$ contains each countable subset of $\mathbb{T}$.*

(v) *Every finite subset of $\mathbb{T}$ is permitted for the families $\mathcal{D}$, $p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{B}_0$, $\mathcal{N}$, $\mathcal{B}$, $\mathcal{A}$, $w\mathcal{D}$.*

(vi) *Every countable subset of $\mathbb{T}$ is permitted for the families $p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{N}$, $\mathcal{A}$, $w\mathcal{D}$.*

(vii) *None of the families $\mathcal{D}$, $p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{B}_0$, $\mathcal{N}$, $\mathcal{B}$, $\mathcal{A}$, $w\mathcal{D}$ is closed under set union.*

In the next $f, g : \mathbb{T} \longrightarrow \langle 0, 1\rangle$ are supposed to be continuous and $f(0) = g(0) = 0$.

Replacing the $\| \ \|$ function in the above definitions by a function f we obtain notions of a **f-Dirichlet set** (shortly **$\mathbf{D}_f$-set**), **pseudo f-Dirichlet set** (shortly **$\mathbf{pD}_f$-set**), **$\mathbf{A}_f$-set**, **$\mathbf{N}_{0f}$-set**, **$\mathbf{B}_{0f}$-set**, **$\mathbf{N}_f$-set**, **$\mathbf{B}_f$-set**, and **weak f-Dirichlet set** (shortly **$\mathbf{wD}_f$-set**).

Similar inclusions as above hold true for those families and every countable set is a pD_f-set, i.e. the conclusions (i), (iii) and (iv) of Theorem 7 hold true. For the conclusion (ii) one needs some additional condition, see [3].

Let $\mathrm{Z}(f)$ denote the **zero-set** $\{x \in \mathbb{T} : f(x) = 0\}$ of f. The main result [5], [4] in the study of the relationships between families $\mathcal{F}_f$ and $\mathcal{F}_g$ is:

8 Theorem. *If $n \cdot \mathrm{Z}(f) \subseteq \mathrm{Z}(g)$ for some positive integer n, then $\mathcal{F}_f \subseteq \mathcal{F}_g$ for $\mathcal{F} = \mathcal{D}$, $p\mathcal{D}$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{N}$, $w\mathcal{D}$.*

9 Corollary. *If $\mathrm{Z}(f)$ is a finite set of rationals, then $\mathcal{F} = \mathcal{F}_f$ for $\mathcal{F} = \mathcal{D}$, $p\mathcal{D}$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{N}$, $w\mathcal{D}$.*

10 Corollary. *$\mathcal{D} \subseteq \mathcal{F}_f$ for any $\mathcal{F} = \mathcal{D}$, $p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{B}_0$, $\mathcal{B}$, $\mathcal{N}$, $\mathcal{A}$, $w\mathcal{D}$.*

The relationship of N_0-sets is a little more complicated. We have [5] a strengthening of a classical result by J. Arbault [1]:

11 Theorem. *Assume that $(\forall x, |x| < 1/2)\, f(x/m) \leq f(x)$ for any positive integer m and $\mathrm{Z}(g)$ is a finite set of rationals. Then $\mathcal{N}_{0f} \subseteq \mathcal{N}_{0g}$ if and only if*

$$(\forall \{x_k\}_{k=0}^{\infty}) \left(\sum_{k=0}^{\infty} f(x_k) < \infty \to \sum_{k=0}^{\infty} g(x_k) < \infty \right).$$

A long time the problem of the least size of a base of trigonometric families was open. We give a solution [7].

Following the idea of J. Marcinkiewicz [14] we can construct a family $\mathcal{M}$ of Dirichlet sets, $|\mathcal{M}| = \mathfrak{c}$ such that $A - B$ contains a non-trivial interval for any $A, B \in \mathcal{M}$, $A \neq B$. Thus:

12 Theorem. *Let $\mathcal{F}$ be a family of thin sets such that $\mathcal{D} \subseteq \mathcal{F}$ and there exists a family of thin sets $\mathcal{H}$ closed under arithmetic difference and such that $\mathcal{F} \subseteq \mathcal{H}$. Then any base of the family $\mathcal{F}$ has cardinality at least $\mathfrak{c}$.*

13 Corollary. *Every base of any trigonometric family of thin sets has cardinality at least $\mathfrak{c}$.*

Let us recall that a topological space X is called a γ**-space** — for further information see [8] — if the topological space $\mathrm{C}_p(X)$ of continuous real functions on X with the topology inherited from the product space ${}^X\mathbb{R}$ is Fréchet, i.e. for any set $A \subseteq \mathrm{C}_p(X)$ and any $f \in \overline{A}$ there exists a sequence $\{f_n\}_{n=0}^{\infty}$ of elements of A converging pointwise to f on X. In [8] the authors present recent results on the size of permitted sets for trigonometric families of thin sets and prove:

14 Theorem. *Every γ-set is permitted for any of the families $p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{N}$, $\mathcal{A}$, and $w\mathcal{D}$.*

Since it is well known that it is consistent with ZFC that $\mathfrak{c} > \aleph_1$ and there exists a gamma set of reals of size $\mathfrak{c}$ — for details see [8], the main consequence of Theorem 14 is:

15 Metatheorem. It is consistent with ZFC that $\mathfrak{c} > \aleph_1$ and there exists a permitted set for any of the families $p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{N}$, $\mathcal{A}$, and $w\mathcal{D}$ of size $\mathfrak{c}$.

J. Arbault presented a "proof" of the existence of a perfect permitted set for $\mathcal{N}$. N. K. Bary in [2] has found a gap in the proof. Improved result was proved by M. Repický. He introduced [16] a notion of a set of perfect measure zero. A set A has **perfect measure zero** if for every sequence of positive reals $\{\varepsilon_n\}_{n=1}^{\infty}$ there is an increasing sequence of integers $\{n_k\}_{k-0}^{\infty}$ and a sequence of finite families of intervals $\{\mathcal{I}_n\}_{n=1}^{\infty}$ such that $|\mathcal{I}_n| \leq n$, $|I| < \varepsilon_n$ for every $I \in \mathcal{I}_n$, and $A \subseteq \bigcup_m \bigcap_{k>m} \bigcup \mathcal{I}_{n_k}$. Every γ-set has perfect measure zero and every set of perfect measure zero has strong measure zero. Main result of [17] is:

16 Theorem (M. Repický)**.** *Let $\mathcal{F}$ be any of the families $\mathcal{N}$, $\mathcal{A}$, $\mathcal{N}_0$, and $p\mathcal{D}$. The unions of less than $\mathfrak{t}$ sets having perfect measure zero are permitted for $p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{N}$,and $\mathcal{A}$.*

In July 2003, L. Bukovský conjectured that:

17 Conjecture. Every $\mathcal{A}$-permitted and $\mathcal{N}$-permitted set is perfectly meager.

P. Eliaš [9, 10] successively proved the conjecture for both classes $\mathcal{A}$-permitted and $\mathcal{N}$-permitted sets. We sketch briefly the idea of his proof.

In [11] the authors proved the following

18 Theorem (P. Erdös – K. Kunen – R. D. Mauldin)**.** *If $P \subseteq \mathbb{T}$ is perfect set then there exists a perfect set Q of measure zero such that $P + Q = \mathbb{T}$.*

P. Eliaš improved it as:

19 Theorem (P. Eliaš). *Let $P \subseteq \mathbb{T}$ be a perfect set. Then there exists a pseudo Dirichlet set Q such that the set $P \cap (x+Q)$ is dense in P for every $x \in \mathbb{T}$.*

20 Corollary (P. Eliaš). *Let $P \subseteq \mathbb{T}$ be a perfect set. Then there exists a pseudo Dirichlet set Q such that $P + Q = \mathbb{T}$.*

Now, one can easily prove:

21 Theorem (P. Eliaš). *Assume that $\mathcal{F}$ is a family of thin sets with a F_σ basis containing every pseudo Dirichlet set. If $\mathcal{F}$ is closed under arithmetic difference then every $\mathcal{F}$–permitted set is perfectly meager.*

Proof. Assume that $A \subseteq \mathbb{T}$ is an $\mathcal{F}$–permitted set, $P \subseteq \mathbb{T}$ is perfect. By Theorem 19 there exists a pseudo Dirichlet set Q such that $P \cap (x-Q)$ is dense in P for every $x \in \mathbb{T}$. By assumption about $\mathcal{F}$ we have $Q \in \mathcal{F}$ and therefore $A \cup Q \in \mathcal{F}$. Thus, there exists an F_σ set $B \in \mathcal{F}$, $B \supseteq A \cup Q$. Since $\mathcal{F}$ is closed under arithmetic difference we have $B - B \neq \mathbb{T}$. Then there exists an $x \in \mathbb{T}$ such that $B \cap (x+B) = \emptyset$. Then also $B \cap (x+Q) = \emptyset$ and therefore $P \cap (x+Q)$ is a subset of G_δ set $P \setminus B$ dense in P. Hence $P \cap A$ is meager. $\boxed{QED}$

22 Theorem. *Every set permitted for any of the families $p\mathcal{D}$, $\mathcal{N}$, $\mathcal{N}_0$, and $\mathcal{A}$ is perfectly meager.*

Proof. For any of the families $p\mathcal{D}$, $\mathcal{N}_0$ and $\mathcal{N}$ the assertion follows directly from Theorem 21, since all of them have an F_σ base.

For A-sets we must a little change the proof. Let A, P, Q be as above. Since $A \cup Q$ is an A-set there exists an increasing sequence $\{n_k\}_{k=0}^\infty$ such that $A \cup Q \subseteq \{ x \in \mathbb{T} : \|n_k x\| \to 0 \}$. Denote

$$B_i = \{ x \in \mathbb{T} : (\forall k \geq i) \|n_k x\| \leq 1/8 \}, \quad B = \bigcup_i B_i.$$

Then B is an F_σ set and $A \cup Q \subseteq B$. If $x \in B - B$ then there are i_1, i_2 and $x_1 \in B_{i_1}$, $x_2 \in B_{i_2}$ such that $x = x_1 - x_2$. If $i_0 = \max\{ i_1, i_2 \}$ then $x_1, x_2 \in B_{i_0}$ and therefore $x \in B_{i_0} - B_{i_0}$. Thus $B - B = \bigcup_i (B_i - B_i)$. On the other hand we have $B_i - B_i \subseteq B_{i+1} - B_{i+1}$ and $B_i - B_i \subseteq \{ x \in \mathbb{T} : \|n_i x\| \leq 1/4 \}$. One can easily see that $\lambda(\{ x \in \mathbb{T} : \|nx\| \leq 1/4 \}) = 1/2$ and therefore $\lambda(B_n - B_n) \leq 1/2$ for any $n > 0$.[1] Thus $\lambda(B - B) \leq 1/2$. Hence $B - B \neq \mathbb{T}$ and we can continue as in the proof of Theorem 21. $\boxed{QED}$

A. Miller [15] has shown that **ZFC** is consistent with $\mathfrak{c} > \aleph_1$ and "every perfectly meager set of reals has cardinality at most $\aleph_1$". Hence we obtain:

23 Metatheorem. **ZFC**+"every set permitted for any of the families $p\mathcal{D}$, $\mathcal{N}$, $\mathcal{N}_0$, and $\mathcal{A}$ has cardinality $\leq \aleph_1$" is consistent.

[1] Actually $\lambda(B_n) = 0$.

24 Metatheorem. "Every set of cardinality $< \mathfrak{c}$ is permitted for the families $p\mathcal{D}$, $\mathcal{N}$, $\mathcal{N}_0$, and $\mathcal{A}$" is undecidable in **ZFC**.

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