

# A Maximal Characterization of Capacities

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Received: 17.04.25; accepted: 19.11.25.

**Abstract.** We show that the Bessel capacities can be characterized in terms of maximal function. The boundedness of maximal function associated with Radon-Nikodym type Bessel capacities will also be addressed.

**Keywords:** Bessel capacities, maximal function, Radon-Nikodym

**MSC 2020 classification:** primary 31C15, secondary 42B25

## 1 Introduction and Statements of Main Results

Let  $n \in \mathbb{N}$  and  $\mathcal{P}(\mathbb{R}^n)$  be the power set of  $\mathbb{R}^n$ . A set function  $\mathcal{C} : \mathcal{P}(\mathbb{R}^n) \rightarrow [0, \infty]$  is called a capacity provided that

- (1)  $\mathcal{C}(\emptyset) = 0$ .
- (2) Monotonicity. For any  $A, B \in \mathcal{P}(\mathbb{R}^n)$ , if  $A \subseteq B$ , then  $\mathcal{C}(A) \leq \mathcal{C}(B)$ .
- (3) Finitely Subadditivity. For any  $A, B \in \mathcal{P}(\mathbb{R}^n)$ , it holds that

$$\mathcal{C}(A \cup B) \leq \mathcal{C}(A) + \mathcal{C}(B).$$

Given a capacity  $\mathcal{C}$ , we can define the associated Choquet integrals, as the following shown. Let  $E \in \mathcal{P}(\mathbb{R}^n)$  and  $f : \mathbb{R}^n \rightarrow [-\infty, \infty]$  be an arbitrary function, where the Lebesgue measurability is not assumed. We define the Choquet integral of  $f$  on  $E$  by

$$\int_E |f| d\mathcal{C} = \int_0^\infty \mathcal{C}(\{x \in E : |f(x)| > t\}) dt.$$

The monotonicity property (2) ensures that the Choquet integrals exist in  $[0, \infty]$ .

The concrete capacities that primarily addressed in the present paper are the Bessel capacities  $\text{Cap}_{\alpha,s}(\cdot)$  defined as follows. Let  $0 < \alpha < \infty$  and  $1 < s < n/\alpha$ . Define

$$\text{Cap}_{\alpha,s}(E) = \inf\{\|\varphi\|_{L^s(\mathbb{R}^n)}^s : \varphi \geq 0, G_\alpha * \varphi \geq \chi_E\}, \quad E \subseteq \mathbb{R}^n,$$

where  $G_\alpha(x) = \mathcal{F}^{-1}[(1+4\pi^2|\cdot|^2)^{-\frac{\alpha}{2}}](x)$ ,  $x \in \mathbb{R}^n$ , is the Bessel kernel, and  $\mathcal{F}^{-1}$  is the inverse distributional Fourier transform on  $\mathbb{R}^n$ . According to [2, Proposition 2.3.9],  $\text{Cap}_{\alpha,s}(\cdot)$  can be equivalently defined as

$$\text{Cap}_{\alpha,s}(E) = \inf\{\|\varphi\|_{L^s(\mathbb{R}^n)}^s : \varphi \geq 0, G_\alpha * \varphi \geq \chi_E \text{ quasi-everywhere}\},$$

where we say that a property  $P(\cdot)$  holds quasi-everywhere provided that the set

$$N = \{x \in \mathbb{R}^n : P(x) \text{ is false}\}$$

satisfies that  $\text{Cap}_{\alpha,s}(N) = 0$ .

The main result in [4] gives a functional generalization of  $\text{Cap}_{\alpha,s}(\cdot)$ . More precisely, given any function  $f : \mathbb{R}^n \rightarrow [-\infty, \infty]$ , which is not necessarily Lebesgue measurable, one can consider the functional  $\mathcal{C}$  that

$$\mathcal{C}(f) = \inf\{\|\varphi\|_{L^s(\mathbb{R}^n)}^s : \varphi \geq 0, G_\alpha * \varphi \geq |f|^{\frac{1}{s}} \text{ quasi-everywhere}\}.$$

As noted before,  $\text{Cap}_{\alpha,s}(E) = \mathcal{C}(\chi_E)$ ,  $E \subseteq \mathbb{R}^n$ . Moreover, it is proved in [4, Proposition 2.1] that  $\mathcal{C}$  is subadditive, i.e.,

$$\mathcal{C}(f + g) \leq \mathcal{C}(f) + \mathcal{C}(g)$$

holds for any functions  $f, g : \mathbb{R}^n \rightarrow [-\infty, \infty]$ . Further, [4, Theorem 2.8] gives

$$\mathcal{C}(f) \approx \int_{\mathbb{R}^n} |f| d\text{Cap}_{\alpha,s}.$$

Here,  $A \approx B$  abbreviates for  $C^{-1}B \leq A \leq CB$  for a positive constant  $C$  independent of the main parameters in  $A, B$ .

The present paper gives a characterization of  $\mathcal{C}$  in terms of a maximal function. To this end, for any function  $f : \mathbb{R}^n \rightarrow [-\infty, \infty]$ , define  $\mathbf{M}_{\mathcal{C}}^{\text{loc}}(f)$  by

$$\mathbf{M}_{\mathcal{C}}^{\text{loc}} f(x) = \sup_{0 < r \leq 1} \frac{1}{\text{Cap}_{\alpha,s}(B_r(x))} \int_{B_r(x)} |f| d\text{Cap}_{\alpha,s}, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all balls  $B_r(x)$  centered at  $x$  with radii  $r \leq 1$ . Such a maximal function  $\mathbf{M}_{\mathcal{C}}^{\text{loc}}$  is also considered in [5], where its boundedness

on function spaces is tackled therein. Now we consider two functionals  $\mathcal{C}$  and  $\mathfrak{C}$  defined by

$$\mathcal{C}(f) = \inf\{\|\varphi\|_{L^s(\text{Cap}_{\alpha,s})}^s : \varphi \geq 0, \mathbf{M}_{\mathcal{C}}^{\text{loc}}\varphi \geq |f|^{\frac{1}{s}} \text{ quasi-everywhere}\},$$

and

$$\mathfrak{C}(f) = \inf\{\|\varphi\|_{L^s(\mathbb{R}^n)}^s : \varphi \geq 0, \mathbf{M}_{\mathcal{C}}^{\text{loc}}(G_{\alpha} * \varphi) \geq |f|^{\frac{1}{s}} \text{ quasi-everywhere}\}.$$

Our first result is the following.

**Theorem 1.** *Let  $\mathcal{C}$ ,  $\mathcal{C}$ , and  $\mathfrak{C}$  be the functionals defined as above. Then*

$$\mathcal{C}(f) \approx \mathcal{C}(f) \approx \mathfrak{C}(f)$$

*holds for every function  $f : \mathbb{R}^n \rightarrow [-\infty, \infty]$ , where the implicit constants depend only on  $n$ ,  $\alpha$ , and  $s$ . In particular, it holds that*

$$\text{Cap}_{\alpha,s}(E) = \mathcal{C}(\chi_E) \approx \mathcal{C}(\chi_E) \approx \mathfrak{C}(\chi_E)$$

*for arbitrary set  $E \subseteq \mathbb{R}^n$ .*

There are various characterizations of  $\mathcal{C}$  in terms of other functionals in [6]. On the other hand, we note that [1, Section 4, Lemma 3] also gives a maximal characterization of the functional associated with Hausdorff content.

As an application of Theorem 1, we consider the boundedness of maximal function  $\mathbf{M}_{\mathcal{C}}^{\text{loc}}$  on the Choquet integral spaces associated to Radon-Nikodym type capacities  $\omega \cdot \text{Cap}_{\alpha,s}(\cdot)$ , where  $\omega : \mathbb{R}^n \rightarrow [0, \infty]$ . To this end, for any  $0 < q < \infty$  and  $f : \mathbb{R}^n \rightarrow [-\infty, \infty]$ , denote by

$$\|f\|_{L^q(\omega \cdot \text{Cap}_{\alpha,s})} = \left( \int_{\mathbb{R}^n} |f|^q \omega d\text{Cap}_{\alpha,s} \right)^{\frac{1}{q}}.$$

To be more specific, the above quantity is interpreted as

$$\|f\|_{L^q(\omega \cdot \text{Cap}_{\alpha,s})} = \left( \int_0^{\infty} \text{Cap}_{\alpha,s}(\{x \in \mathbb{R}^n : |f(x)|^q \omega(x) > t\}) dt \right)^{\frac{1}{q}}.$$

The second result reads as the following.

**Theorem 2.** *For  $0 < q < s$ , there is a positive constant  $C_{n,\alpha,s,\omega,q}$  such that the estimate*

$$\|\mathbf{M}_{\mathcal{C}}^{\text{loc}} f\|_{L^q(\omega \cdot \text{Cap}_{\alpha,s})} \leq C_{n,\alpha,s,\omega,q} \|f\|_{L^s(\text{Cap}_{\alpha,s})} \quad (1)$$

holds for every function  $f : \mathbb{R}^n \rightarrow [-\infty, \infty]$  if and only if

$$\|\omega\|_{L^{\frac{s}{s-q}}(\text{Cap}_{\alpha,s})} < \infty. \quad (2)$$

In which case, it holds that

$$C_{n,\alpha,s,\omega,q}^q \approx \|\omega\|_{L^{\frac{s}{s-q}}(\text{Cap}_{\alpha,s})},$$

where the implicit constants depend only on  $n$ ,  $\alpha$ ,  $s$ , and  $q$ .

Needless to say, the  $\omega$  in Theorem 2 needs not to be Lebesgue measurable, it is merely a nonnegative function defined on  $\mathbb{R}^n$ . However, in the end point case  $q = s$ , we need to restrict  $\omega$  to be a weight which belongs to the Muckenhoupt  $A_1$  class. By a weight  $\omega$ , we mean that  $\omega$  is Lebesgue measurable, locally integrable on  $\mathbb{R}^n$ , and  $\omega > 0$  almost everywhere. Further, we say that  $\omega \in A_1$  provided that

$$[\omega]_{A_1} = \sup_B \left( \frac{1}{|B|} \int_B \omega(x) dx \right) \|\omega^{-1}\|_{L^\infty(B)} < \infty,$$

where the supremum is taken over all balls  $B$  in  $\mathbb{R}^n$ . Given such a weight  $\omega$ , we define the weighted Bessel capacities  $\text{Cap}_{\alpha,s}^\omega(\cdot)$  by

$$\text{Cap}_{\alpha,s}^\omega(E) = \inf \{ \|\varphi\|_{L^s(\omega)}^s : \varphi \geq 0, G_\alpha * \varphi \geq \chi_E \},$$

where, as usual, we denote by

$$\|\varphi\|_{L^s(\omega)} = \left( \int_{\mathbb{R}^n} |\varphi(x)|^s \omega(x) dx \right)^{\frac{1}{s}}$$

for measurable functions  $\varphi : \mathbb{R}^n \rightarrow [-\infty, \infty]$ . The weighted capacities are greatly studied in the literature [7]. The Choquet integrals associated to  $\text{Cap}_{\alpha,s}^\omega(\cdot)$  are then given by

$$\|f\|_{L^q(\text{Cap}_{\alpha,s}^\omega)} = \left( \int_{\mathbb{R}^n} |f|^q d\text{Cap}_{\alpha,s}^\omega \right)^{\frac{1}{q}},$$

equivalently,

$$\|f\|_{L^q(\text{Cap}_{\alpha,s}^\omega)} = \left( \int_0^\infty \text{Cap}_{\alpha,s}^\omega(\{x \in \mathbb{R}^n : |f(x)|^q > t\}) dt \right)^{\frac{1}{q}},$$

where  $f : \mathbb{R}^n \rightarrow [-\infty, \infty]$ . The end point case of Theorem 2 reads as the following.

**Theorem 3.** *Let  $\omega \in A_1$ . There is a positive constant  $C_{n,\alpha,s,\omega}$  depending on  $n, \alpha, s$ , and the  $A_1$  constant of  $\omega$  such that*

$$\|\mathbf{M}_{\mathcal{C}}^{\text{loc}} f\|_{L^s(\text{Cap}_{\alpha,s}^{\omega})} \leq C_{n,\alpha,s,\omega} \|f\|_{L^s(\text{Cap}_{\alpha,s}^{\omega})} \quad (3)$$

holds for every function  $f : \mathbb{R}^n \rightarrow [-\infty, \infty]$ .

In view of Theorems 2 and 3, the Radon-Nikodym type  $\omega \cdot \text{Cap}_{\alpha,s}(\cdot)$  and the weighted capacities  $\text{Cap}_{\alpha,s}^{\omega}(\cdot)$  are not directly comparable, where [1, Theorem 15] gives a set of highly technical sufficient conditions for the comparison of those two capacities. It seems that there has been no further discussion on this topic in the literature to date.

## 2 Proof of Main Results

Let  $\mathcal{C}$  be either  $\text{Cap}_{\alpha,s}(\cdot)$  or  $\text{Cap}_{\alpha,s}^{\omega}(\cdot)$ . The monotonicity and  $\mathcal{C}(\emptyset) = 0$  that mentioned in the first section are valid. Moreover, the following two extra properties about  $\mathcal{C}$  also hold (see [2, Section 2.3] and [7, Section 3.2]).

- (1) Fatou's property. If  $\{E_N\}_{N=1}^{\infty}$  is an increasing sequence of arbitrary sets in  $\mathbb{R}^n$ , then

$$\mathcal{C}\left(\bigcup_{N \in \mathbb{N}} E_N\right) = \sup_{N \in \mathbb{N}} \mathcal{C}(E_N).$$

- (2) Countably Subadditivity. If  $\{E_N\}_{N=1}^{\infty}$  is a sequence of arbitrary sets in  $\mathbb{R}^n$ , then

$$\mathcal{C}\left(\bigcup_{N \in \mathbb{N}} E_N\right) \leq \sum_{N \in \mathbb{N}} \mathcal{C}(E_N).$$

Below we collect some elementary facts regarding the Choquet integrals. Readers can easily derive the facts by using the aforementioned properties of capacities. Suppose that  $f, g : \mathbb{R}^n \rightarrow [-\infty, \infty]$ .

- (1) If  $f = g$   $\mathcal{C}$ -quasi-everywhere, then

$$\int_{\mathbb{R}^n} |f| d\mathcal{C} = \int_{\mathbb{R}^n} |g| d\mathcal{C}.$$

- (2) The following monotonicity holds. If  $|f| \leq |g|$ , then

$$\int_{\mathbb{R}^n} |f| d\mathcal{C} \leq \int_{\mathbb{R}^n} |g| d\mathcal{C}.$$

(3) For every  $\lambda \in \mathbb{R}$ , it holds that

$$\int_{\mathbb{R}^n} |\lambda f| d\mathcal{C} = |\lambda| \int_{\mathbb{R}^n} |f| d\mathcal{C}.$$

(4) The following quasi-subadditive holds.

$$\int_{\mathbb{R}^n} |f + g| d\mathcal{C} \leq 2 \left( \int_{\mathbb{R}^n} |f| d\mathcal{C} + \int_{\mathbb{R}^n} |g| d\mathcal{C} \right).$$

(5) The following monotonicity holds. If  $\{f_N\}_{N=1}^{\infty}$  is an increasing sequence of nonnegative functions on  $\mathbb{R}^n$ , then

$$\int_{\mathbb{R}^n} \left( \sup_{N \in \mathbb{N}} f_N \right) d\mathcal{C} = \sup_{N \in \mathbb{N}} \int_{\mathbb{R}^n} f_N d\mathcal{C}.$$

We will frequently use the following capacitary strong type inequality (CSI) in the sequel.

$$\int_{\mathbb{R}^n} |G_\alpha * \varphi|^s d\text{Cap}_{\alpha,s}^\omega \leq C_{n,\alpha,s,\omega} \int_{\mathbb{R}^n} |\varphi(x)|^s \omega(x) dx$$

holds for any function  $\varphi : \mathbb{R}^n \rightarrow [-\infty, \infty]$  and  $\omega \in A_1$ . The unweighted version of CSI can be found in [2, Theorem 7.1.1], where the weighted version is recorded in [3, Proposition 2.4].

On the other hand, we have the strong type estimate that

$$\|\mathbf{M}_{\mathcal{C}}^{\text{loc}} f\|_{L^q(\text{Cap}_{\alpha,s})} \leq C_{n,\alpha,s,q} \|f\|_{L^q(\text{Cap}_{\alpha,s})}, \quad 1 < q < \infty, \quad (4)$$

where  $f : \mathbb{R}^n \rightarrow [-\infty, \infty]$  is arbitrary (see [5, (1.5) of Theorem 1.1]). By using Lebesgue differentiation theorem, one has

$$|f(x)| \leq \mathbf{M}_{\mathcal{C}}^{\text{loc}} f(x) \quad \text{quasi-everywhere} \quad (5)$$

whenever  $f$  is quasi-continuous on  $\mathbb{R}^n$  (see [5, Proposition 1.6]). The definition of quasi-continuity is given as follows.

- (1)  $f$  is defined quasi-everywhere on  $\mathbb{R}^n$ , i.e.,  $f : \mathbb{R}^n \setminus N \rightarrow [-\infty, \infty]$ , where  $\text{Cap}_{\alpha,s}(N) = 0$  for some  $N \subseteq \mathbb{R}^n$ .
- (2)  $f$  is finite quasi-everywhere.
- (3) Given any  $\varepsilon > 0$ , there is an open set  $G \subseteq \mathbb{R}^n$  such that  $\text{Cap}_{\alpha,s}(G) < \varepsilon$  and the restriction function  $f|_{G^c}$  is continuous on  $G^c$ .

Observe that  $G_\alpha * \varphi$  is quasi-continuous whenever  $\varphi \in L^s(\mathbb{R}^n)$  (see [2, Proposition 6.1.2]).

**Remark 1.** Let  $f : \mathbb{R}^n \rightarrow [-\infty, \infty]$  be given. We claim that if  $\|f\|_{L^1(\mathcal{C})} = 0$ , then  $f = 0$  quasi-everywhere. Indeed, for every  $0 < t < \infty$ , we have

$$t \cdot \mathcal{C}(\{x \in \mathbb{R}^n : |f(x)| > t\}) = \int_{\{|f|>t\}} t d\mathcal{C} \leq \int_{\{|f|>t\}} |f| d\mathcal{C} \leq \int_{\mathbb{R}^n} |f| d\mathcal{C}$$

By letting  $t = 1/N$ ,  $N = 1, 2, \dots$ , we have

$$\begin{aligned} \mathcal{C}(\{x \in \mathbb{R}^n : |f(x)| > 0\}) &= \bigcup_{N \in \mathbb{N}} \mathcal{C}\left(\left\{x \in \mathbb{R}^n : |f(x)| > \frac{1}{N}\right\}\right) \\ &\leq \sum_{N \in \mathbb{N}} \mathcal{C}\left(\left\{x \in \mathbb{R}^n : |f(x)| > \frac{1}{N}\right\}\right) \\ &= 0, \end{aligned}$$

where the first inequality follows by the countably subadditivity of  $\mathcal{C}$  and hence  $f = 0$  quasi-everywhere.

Moreover, if  $\|f\|_{L^q(\mathcal{C})} = 0$ ,  $0 < q < \infty$ , then  $\| |f|^q \|_{L^1(\mathcal{C})} = 0$ , and the above argument applied to  $|f|^q$  shows again that  $f = 0$  quasi-everywhere.

*Proof of Theorem 1.* Let  $f : \mathbb{R}^n \rightarrow [-\infty, \infty]$  be given. First, we show that

$$\mathcal{C}(f) \leq C_{n,\alpha,s} \mathcal{E}(f). \quad (6)$$

Let  $\varphi : \mathbb{R}^n \rightarrow [0, \infty]$  be such that  $\mathbf{M}_{\mathcal{C}}^{\text{loc}} \varphi \geq |f|^{1/s}$  quasi-everywhere. Recall that  $\mathcal{C}(f) \approx \|f\|_{L^1(\text{Cap}_{\alpha,s})}$ . Using (4) yields

$$\begin{aligned} \mathcal{C}(f) &\leq C_{n,\alpha,s} \|f\|_{L^1(\text{Cap}_{\alpha,s})} \\ &\leq C_{n,\alpha,s} \|(\mathbf{M}_{\mathcal{C}}^{\text{loc}} \varphi)^s\|_{L^1(\text{Cap}_{\alpha,s})} \\ &= C_{n,\alpha,s} \|\mathbf{M}_{\mathcal{C}}^{\text{loc}} \varphi\|_{L^s(\text{Cap}_{\alpha,s})}^s \\ &\leq C'_{n,\alpha,s} \|\varphi\|_{L^s(\text{Cap}_{\alpha,s})}^s. \end{aligned}$$

Taking infimum to all such  $\varphi$  gives (6). Now we show that

$$\mathcal{E}(f) \leq C_{n,\alpha,s} \mathfrak{C}(f). \quad (7)$$

Suppose that  $\varphi : \mathbb{R}^n \rightarrow [0, \infty]$  is such that  $\mathbf{M}_{\mathcal{C}}^{\text{loc}}(G_\alpha * \varphi) \geq |f|^{1/s}$  quasi-everywhere. Using CSI, one has

$$\mathcal{E}(f) \leq \|G_\alpha * \varphi\|_{L^s(\text{Cap}_{\alpha,s})}^s \leq C_{n,\alpha,s} \|\varphi\|_{L^s(\mathbb{R}^n)}^s.$$

Taking infimum to all such  $\varphi$  gives (7). Finally we show that

$$\mathfrak{C}(f) \leq \mathcal{C}(f). \quad (8)$$

If  $\varphi : \mathbb{R}^n \rightarrow [0, \infty]$  is such that  $G_\alpha * \varphi \geq |f|^{1/s}$  quasi-everywhere, then by considering  $\varphi_N = \min(\varphi, N)\chi_{B_N(0)}$ ,  $N = 1, 2, \dots$ , one has  $\varphi_N \uparrow \varphi$  pointwise everywhere, and hence

$$G_\alpha * \varphi(x) = \sup_{N \in \mathbb{N}} G_\alpha * \varphi_N(x), \quad x \in \mathbb{R}^n.$$

On the other hand, it is clear that  $\varphi_N \in L^s(\mathbb{R}^n)$ , hence  $G_\alpha * \varphi_N$  is quasi-continuous (see [2, Proposition 6.1.2]). Using (5), we have

$$G_\alpha * \varphi_N(x) \leq \mathbf{M}_C^{\text{loc}}(G_\alpha * \varphi_N)(x) \quad \text{quasi-everywhere,}$$

say,

$$G_\alpha * \varphi_N(x) \leq \mathbf{M}_C^{\text{loc}}(G_\alpha * \varphi_N)(x) \quad x \in \mathbb{R}^n \setminus E_N,$$

where  $E_N \subseteq \mathbb{R}^n$  is such that  $\text{Cap}_{\alpha,s}(E_N) = 0$ . By letting

$$E = \bigcup_{N \in \mathbb{N}} E_N.$$

Then the countably subadditivity of  $\text{Cap}_{\alpha,s}(\cdot)$  entails

$$\text{Cap}_{\alpha,s}(E) \leq \sum_{N \in \mathbb{N}} \text{Cap}_{\alpha,s}(E_N) = 0.$$

Taking supremum over all  $N \in \mathbb{N}$ , we conclude that

$$G_\alpha * \varphi(x) \leq \mathbf{M}_C^{\text{loc}}(G_\alpha * \varphi)(x), \quad x \in \mathbb{R}^n \setminus E.$$

which yields

$$G_\alpha * \varphi(x) \leq \mathbf{M}_C^{\text{loc}}(G_\alpha * \varphi)(x) \quad \text{quasi-everywhere.}$$

Then  $\mathfrak{C}(f) \leq \|\varphi\|_{L^s(\mathbb{R}^n)}^s$ , which gives (8).  $\square$

The following Hölder's type inequality will be used in the sequel.

**Lemma 1.** *Let  $\varphi, \psi; \mathbb{R}^n \rightarrow [0, \infty]$ ,  $q' = q/(q-1)$ , and  $1 < q < \infty$ . Then*

$$\int_{\mathbb{R}^n} \varphi \psi d\text{Cap}_{\alpha,s} \leq 2 \|\varphi\|_{L^q(\text{Cap}_{\alpha,s})} \|\psi\|_{L^{q'}(\text{Cap}_{\alpha,s})}. \quad (9)$$

*Proof.* Recall the standard Young's inequality that

$$ab \leq \frac{a^q}{q} + \frac{b^{q'}}{q'}, \quad a, b \geq 0.$$

If either of  $\|\varphi\|_{L^q(\text{Cap}_{\alpha,s})}$ ,  $\|\psi\|_{L^{q'}(\text{Cap}_{\alpha,s})}$  is zero, then  $\varphi$  or  $\psi$  is zero quasi-everywhere, and (9) follows immediately. Assume that

$$0 < \|\varphi\|_{L^q(\text{Cap}_{\alpha,s})}, \|\psi\|_{L^{q'}(\text{Cap}_{\alpha,s})} < \infty.$$

Then we have

$$\begin{aligned} & \int_{\mathbb{R}^n} \frac{\varphi}{\|\varphi\|_{L^q(\text{Cap}_{\alpha,s})}} \frac{\psi}{\|\psi\|_{L^{q'}(\text{Cap}_{\alpha,s})}} d\text{Cap}_{\alpha,s} \\ & \leq \int_{\mathbb{R}^n} \frac{1}{q} \left( \frac{\varphi}{\|\varphi\|_{L^q(\text{Cap}_{\alpha,s})}} \right)^q + \frac{1}{q'} \left( \frac{\psi}{\|\psi\|_{L^{q'}(\text{Cap}_{\alpha,s})}} \right)^{q'} d\text{Cap}_{\alpha,s} \\ & \leq 2 \left( \int_{\mathbb{R}^n} \frac{1}{q} \left( \frac{\varphi}{\|\varphi\|_{L^q(\text{Cap}_{\alpha,s})}} \right)^q d\text{Cap}_{\alpha,s} + \int_{\mathbb{R}^n} \frac{1}{q'} \left( \frac{\psi}{\|\psi\|_{L^{q'}(\text{Cap}_{\alpha,s})}} \right)^{q'} d\text{Cap}_{\alpha,s} \right) \\ & = 2 \left( \frac{1}{q} + \frac{1}{q'} \right) \\ & = 2, \end{aligned}$$

which gives the Hölder's inequality.  $\square$

*Proof of Theorem 2.* Assume that (1) holds for all such  $f$ . Let  $\varphi : \mathbb{R}^n \rightarrow [0, \infty]$  be such that  $\mathbf{M}_C^{\text{loc}} \varphi \geq (\omega^{s/(s-q)})^{1/s}$  quasi-everywhere. This gives

$$\begin{aligned} \|\omega\|_{L^{\frac{s}{s-q}}(\text{Cap}_{\alpha,s})}^{\frac{s}{s-q}} &= \int_{\mathbb{R}^n} \omega^{\frac{s}{s-q}} d\text{Cap}_{\alpha,s} \\ &= \int_{\mathbb{R}^n} \omega^{\frac{q}{s-q}} \cdot \omega d\text{Cap}_{\alpha,s} \\ &\leq \int_{\mathbb{R}^n} (\mathbf{M}_C^{\text{loc}} \varphi)^q \omega d\text{Cap}_{\alpha,s} \\ &\leq C_{n,\alpha,s,\omega,q}^q \|\varphi\|_{L^s(\text{Cap}_{\alpha,s})}^q. \end{aligned}$$

Using Theorem 1, we obtain

$$\begin{aligned}
\|\omega\|_{L^{\frac{s}{s-q}}(\text{Cap}_{\alpha,s})}^{\frac{s}{s-q}} &\leq C_{n,\alpha,s} C_{n,\alpha,s,\omega,q}^q \mathcal{C}(\omega^{\frac{s}{s-q}})^{\frac{q}{s}} \\
&\leq C_{n,\alpha,s,q} C_{n,\alpha,s,\omega,q}^q \mathcal{C}(\omega^{\frac{s}{s-q}})^{\frac{q}{s}} \\
&\leq C'_{n,\alpha,s,q} C_{n,\alpha,s,\omega,q}^q \|\omega^{\frac{s}{s-q}}\|_{L^1(\text{Cap}_{\alpha,s})}^{\frac{q}{s}} \\
&= C'_{n,\alpha,s,q} C_{n,\alpha,s,\omega,q}^q \|\omega\|_{L^{\frac{s}{s-q}}(\text{Cap}_{\alpha,s})}^{\frac{q}{s}}.
\end{aligned}$$

We conclude that

$$\|\omega\|_{L^{\frac{s}{s-q}}(\text{Cap}_{\alpha,s})} \leq C'_{n,\alpha,s,q} C_{n,\alpha,s,\omega,q}^q,$$

which gives (2). For the converse, we use Lemma 1 with the exponents  $s/q$  and  $s/(s-q)$ , which satisfy

$$\left(\frac{s}{q}\right)^{-1} + \left(\frac{s}{s-q}\right)^{-1} = 1.$$

As a consequence, we have

$$\begin{aligned}
\|\mathbf{M}_{\mathcal{C}}^{\text{loc}} f\|_{L^q(\omega \cdot \text{Cap}_{\alpha,s})}^q &= \int_{\mathbb{R}^n} (\mathbf{M}_{\mathcal{C}}^{\text{loc}} f)^q \omega d\text{Cap}_{\alpha,s} \\
&\leq 2 \|(\mathbf{M}_{\mathcal{C}}^{\text{loc}} f)^q\|_{L^{\frac{s}{q}}(\text{Cap}_{\alpha,s})} \|\omega\|_{L^{\frac{s}{s-q}}(\text{Cap}_{\alpha,s})} \\
&= 2 \|\omega\|_{L^{\frac{s}{s-q}}(\text{Cap}_{\alpha,s})} \|\mathbf{M}_{\mathcal{C}}^{\text{loc}} f\|_{L^s(\text{Cap}_{\alpha,s})}^q \\
&\leq C_{n,\alpha,s,q} \|\omega\|_{L^{\frac{s}{s-q}}(\text{Cap}_{\alpha,s})} \|f\|_{L^s(\mathbb{R}^n)}^q,
\end{aligned}$$

where the last inequality is due to (4). Now it is easy to see that (2) follows.  $\square$

*Proof of Theorem 3.* We first prove that

$$\sup_{0 < t < \infty} t \cdot \text{Cap}_{\alpha,s}^\omega(\{x \in \mathbb{R}^n : \mathbf{M}_{\mathcal{C}}^{\text{loc}}((G_\alpha * \varphi)^s) > t\}) \leq C_{n,\alpha,s,\omega} \|\varphi\|_{L^s(\omega)}^s \quad (10)$$

for every  $\varphi : \mathbb{R}^n \rightarrow [-\infty, \infty]$ . Fix a  $0 < t < \infty$ . Denote by

$$E_t = \{x \in \mathbb{R}^n : \mathbf{M}_{\mathcal{C}}^{\text{loc}}((G_\alpha * \varphi)^s) > t\}.$$

Suppose that  $x_0 \in E_t$ . Let  $r = r_{x_0} \in (0, 1]$  be such that

$$\int_{B_r(x_0)} (G_\alpha * \varphi)^s d\text{Cap}_{\alpha,s} > t.$$

We decompose  $\varphi = \phi + \psi$ , where  $\phi(x) = 0$  for  $|x - x_0| > 4r$ , and  $\phi(x) = \varphi(x)$  for  $|x - x_0| \leq 4r$ . Then

$$\begin{aligned} \text{Cap}_{\alpha,s}(B_r(x_0)) \cdot t &< \int_{B_r(x_0)} (G_\alpha * \varphi)^s d\text{Cap}_{\alpha,s} \\ &\leq \int_{B_r(x_0)} 2^{s-1} ((G_\alpha * \phi)^s + (G_\alpha * \psi)^s) d\text{Cap}_{\alpha,s} \\ &\leq 2^s \left( \int_{B_r(x_0)} (G_\alpha * \phi)^s d\text{Cap}_{\alpha,s} + \int_{B_r(x_0)} (G_\alpha * \psi)^s d\text{Cap}_{\alpha,s} \right). \end{aligned}$$

Therefore, we have either

$$\int_{B_r(x_0)} (G_\alpha * \phi)^s d\text{Cap}_{\alpha,s} > \frac{1}{2^{s+1}} \text{Cap}_{\alpha,s}(B_r(x_0)) \cdot t, \quad (11)$$

or

$$\int_{B_r(x_0)} (G_\alpha * \psi)^s d\text{Cap}_{\alpha,s} > \frac{1}{2^{s+1}} \text{Cap}_{\alpha,s}(B_r(x_0)) \cdot t. \quad (12)$$

Assume that (11) holds at the moment. We have by CSI that

$$\int_{\mathbb{R}^n} (G_\alpha * \phi)^s d\text{Cap}_{\alpha,s} \leq C_{n,\alpha,s} \int_{\mathbb{R}^n} \phi(x)^s dx = C_{n,\alpha,s} \int_{B_{4r}(x_0)} \varphi(x)^s dx,$$

which yields

$$\int_{B_{4r}(x_0)} \varphi(x)^s dx \geq C_{n,\alpha,s} \text{Cap}_{\alpha,s}(B_r(x_0)) \cdot t. \quad (13)$$

On the other hand, if (12) holds, then we claim that

$$G_\alpha * \varphi(x_0) \geq C_{n,\alpha,s} t^{\frac{1}{s}}. \quad (14)$$

In fact, for any  $x_1, x_2 \in B_r(x_0)$  and  $y \notin B_{4r}(x_0)$ , we have

$$\frac{1}{3}|x_2 - y| \leq |x_1 - y| \leq 3|x_2 - y|.$$

If  $|x_1 - y| < 3$ , then  $|x_2 - y| < 9$ , and hence

$$G_\alpha(x_1 - y) \leq C_{n,\alpha} \frac{1}{|x_1 - y|^{n-\alpha}} \leq C_{n,\alpha} \frac{1}{|x_2 - y|^{n-\alpha}} \leq C_{n,\alpha} G_\alpha(x_2 - y).$$

Furthermore, if  $|x_1 - y| \geq 3$ , then

$$G_\alpha(x_1 - y) \leq C_{n,\alpha} G_\alpha(x_1 - y + x_2 - x_1) = C_{n,\alpha} G_\alpha(x_2 - y),$$

where we have used the fact that  $|x_2 - x_1| < 2$  and

$$G_\alpha(u) \leq C_{n,\alpha} G_\alpha(u + v), \quad |u| \geq 3, \quad |v| \leq 2$$

(see [2, (1.2.24) of Section 1.2.4]). Therefore, for any  $x_1 \in B_r(x_0)$ ,

$$G_\alpha * \psi(x_1) \leq C_{n,\alpha} \inf_{x \in B_r(x_0)} G_\alpha * \psi(x) \leq C_{n,\alpha} \inf_{x \in B_r(x_0)} G_\alpha * \varphi(x).$$

Hence, (12) implies that

$$t^{\frac{1}{s}} \leq C_{n,\alpha,s} \inf_{x \in B_r(x_0)} G_\alpha * \varphi(x),$$

which yields (14). Let  $U$  be the set of all  $x \in E_t$  such that (13) holds, i.e.,

$$\int_{B_{4r_x}(x)} \varphi(z)^s dz \geq C_{n,\alpha,s,\omega} \text{Cap}_{\alpha,s}(B_{r_x}(x)) \cdot t.$$

Then, by (14),  $G_\alpha * \varphi(x) \geq C_{n,\alpha,s} t^{1/s}$  everywhere on  $E_t \setminus U$ . As a result,

$$\text{Cap}_{\alpha,s}^\omega(E_t \setminus U) \leq C_{n,\alpha,s} \frac{1}{t} \|\varphi\|_{L^s(\omega)}^s.$$

Note that

$$U \subseteq \bigcup_{x \in E_t} B_{r_x}(x).$$

Let  $W$  be an arbitrarily bounded subset of  $U$ . By Besicovitch covering theorem, there are collections of balls

$$A_1 = \{B_{r_{k_1}}(x_{k_1})\}, \dots, A_{c_n} = \{B_{r_{k_{c_n}}}(x_{k_{c_n}})\}$$

such that each  $A_i$  is disjoint,

$$W \subseteq \bigcup_{i=1}^{c_n} \bigcup_{B \in A_i} B,$$

and

$$\int_{B_{4r_{k_i}}(x_{k_i})} \varphi(x)^s dx \geq C_{n,\alpha,s} \text{Cap}_{\alpha,s}(B_{r_{k_i}}(x_{k_i})) \cdot t.$$

Using [7, Lemma 3.3.12], one has

$$\begin{aligned} \text{Cap}_{\alpha,s}^{\omega}(B_{r_{k_i}}(x_{k_i})) &\leq C_{n,\alpha,s,\omega} r_{k_i}^{-\alpha s} \omega(B_{r_{k_i}}(x_{k_i})) \\ &= C_{n,\alpha,s,\omega} r_{k_i}^{n-\alpha s} \frac{\omega(B_{r_{k_i}}(x_{k_i}))}{r_{k_i}^n} \\ &\leq C'_{n,\alpha,s,\omega} \text{Cap}_{\alpha,s}(B_{r_{k_i}}(x_{k_i})) \frac{\omega(B_{4r_{k_i}}(x_{k_i}))}{r_{k_i}^n}. \end{aligned}$$

Since  $\omega \in A_1$ , we have

$$\frac{\omega(B_{4r_{k_i}}(x_{k_i}))}{r_{k_i}^n} \leq C_n [\omega]_{A_1} \omega(x) \quad \text{almost everywhere } x \in B_{4r_{k_i}}(x_{k_i}).$$

Then

$$\begin{aligned} \text{Cap}_{\alpha,s}(B_{r_{k_i}}(x_{k_i})) \frac{\omega(B_{4r_{k_i}}(x_{k_i}))}{r_{k_i}^n} &\leq C_{n,\omega} \frac{1}{t} \int_{B_{4r_{k_i}}(x_{k_i})} \varphi(x)^s \frac{\omega(B_{4r_{k_i}}(x_{k_i}))}{r_{k_i}^n} dx \\ &\leq C_{n,\omega} \frac{1}{t} \int_{B_{4r_{k_i}}(x_{k_i})} \varphi(x)^s \omega(x) dx. \end{aligned}$$

As a result, we obtain

$$\begin{aligned} \text{Cap}_{\alpha,s}(W) &\leq \sum_{i=1}^{c_n} \sum_{B \in A_i} \text{Cap}_{\alpha,s}(B) \\ &= C_{n,\alpha,s} \sum_{i=1}^{c_n} \sum_{k_i} \text{Cap}_{\alpha,s}(B_{r_{k_i}}(x_{k_i})) \\ &\leq C_{n,\alpha,s,\omega} \frac{1}{t} \sum_{i=1}^{c_n} \sum_{k_i} \int_{B_{4r_{k_i}}(x_{k_i})} \varphi(x)^s \omega(x) dx \\ &\leq C_{n,\alpha,s,\omega} \frac{1}{t} \|\varphi\|_{L^s(\omega)}^s. \end{aligned}$$

Now we let  $W = U \cap B_N(0)$  for  $N = 1, 2, \dots$ , the Fatou's property of  $\text{Cap}_{\alpha,s}^{\omega}(\cdot)$  entails

$$\text{Cap}_{\alpha,s}^{\omega}(U) = \lim_{N \rightarrow \infty} \text{Cap}_{\alpha,s}^{\omega}(U \cap B_N(0)) \leq C_{n,\alpha,s,\omega} \frac{1}{t} \|\varphi\|_{L^s(\omega)}^s.$$

Since  $\text{Cap}_{\alpha,s}^{\omega}(E_t) \leq \text{Cap}_{\alpha,s}^{\omega}(E_t \setminus U) + \text{Cap}_{\alpha,s}^{\omega}(U)$ , the claim (10) follows.

Now we are ready to prove (3). Assume first that  $|f| \leq 1$  quasi-everywhere. Then for every  $x \in \mathbb{R}^n$ , it holds that

$$\begin{aligned} \mathbf{M}_{\mathcal{C}}^{\text{loc}} f(x) &= \sup_{0 < r \leq 1} \frac{1}{\text{Cap}_{\alpha,s}(B_r(x))} \int_{B_r(x)} |f| d\text{Cap}_{\alpha,s} \\ &\leq \sup_{0 < r \leq 1} \frac{1}{\text{Cap}_{\alpha,s}(B_r(x))} \int_{B_r(x)} 1 d\text{Cap}_{\alpha,s} \\ &= 1. \end{aligned}$$

Let  $\varphi : \mathbb{R}^n \rightarrow [0, \infty]$  be such that  $G_\alpha * \varphi \geq |f|^{1/s}$  quasi-everywhere. Using (10), we have

$$\begin{aligned} \int_{\mathbb{R}^n} (\mathbf{M}_{\mathcal{C}}^{\text{loc}} f)^s d\text{Cap}_{\alpha,s}^\omega &= \int_0^\infty \text{Cap}_{\alpha,s}^\omega(\{x \in \mathbb{R}^n : \mathbf{M}_{\mathcal{C}}^{\text{loc}} f(x)^s > t\}) dt \\ &= \int_0^1 \text{Cap}_{\alpha,s}^\omega(\{x \in \mathbb{R}^n : \mathbf{M}_{\mathcal{C}}^{\text{loc}} f(x) > t^{\frac{1}{s}}\}) dt \\ &\leq \int_0^1 \text{Cap}_{\alpha,s}^\omega(\{x \in \mathbb{R}^n : \mathbf{M}_{\mathcal{C}}^{\text{loc}}((G_\alpha * \varphi)^s)(x) > t^{\frac{1}{s}}\}) dt \\ &\leq C_{n,\alpha,s,\omega} \int_0^1 \frac{1}{t^{\frac{1}{s}}} \|\varphi\|_{L^s(\omega)}^s dt \\ &= C'_{n,\alpha,s,\omega} \|\varphi\|_{L^s(\omega)}^s. \end{aligned}$$

By considering the weighted version of  $\mathcal{C}$ , taking infimum over all such  $\varphi$ , one obtains (3) (see [3, Proposition 2.5]). Denote by

$$\|f\|_{L^\infty(\text{Cap}_{\alpha,s}^\omega)} = \inf\{a > 0 : |f| \leq a \text{ quasi-everywhere with respect to } \text{Cap}_{\alpha,s}^\omega(\cdot)\}.$$

One easily obtains  $\|f\|_{L^\infty(\text{Cap}_{\alpha,s}^\omega)} = 0$  if and only if  $f = 0$   $\text{Cap}_{\alpha,s}^\omega(\cdot)$ -quasi-everywhere, and  $|f| \leq \|f\|_{L^\infty(\text{Cap}_{\alpha,s}^\omega)}$   $\text{Cap}_{\alpha,s}^\omega(\cdot)$ -quasi-everywhere. Assume at the moment that

$$\begin{aligned} \text{Cap}_{\alpha,s}^\omega(\cdot)\text{-quasi-everywhere entails } \text{Cap}_{\alpha,s}(\cdot)\text{-quasi-everywhere, i.e,} \\ \text{Cap}_{\alpha,s}^\omega(E) = 0 \text{ entails } \text{Cap}_{\alpha,s}(E) = 0, \quad E \subseteq \mathbb{R}^n. \end{aligned} \quad (15)$$

Denote by  $M = \|f\|_{L^\infty(\text{Cap}_{\alpha,s}^\omega)}$  and assume first that  $M < \infty$ . Then  $f/(M+1) \leq 1$   $\text{Cap}_{\alpha,s}(\cdot)$ -quasi-everywhere. By the result just proved, we have

$$\int_{\mathbb{R}^n} \left( \mathbf{M}_{\mathcal{C}}^{\text{loc}} \left( \frac{f}{M+1} \right) \right)^s d\text{Cap}_{\alpha,s}^\omega \leq C_{n,\alpha,s,\omega} \int_{\mathbb{R}^n} \left( \frac{f}{M+1} \right)^s d\text{Cap}_{\alpha,s}^\omega.$$

Canceling both sides by the factor  $1/(M+1)^s$ , then the estimate (3) follows. For general  $f : \mathbb{R}^n \rightarrow [-\infty, \infty]$ , by considering  $f_N = \min(f, N)$ ,  $N = 1, 2, \dots$ , we have  $\|f_N\|_{L^\infty(\text{Cap}_{\alpha,s}^\omega)} < \infty$ . The previous result yields

$$\int_{\mathbb{R}^n} (\mathbf{M}_C^{\text{loc}} f_N)^s d\text{Cap}_{\alpha,s}^\omega \leq C_{n,\alpha,s,\omega} \int_{\mathbb{R}^n} f_N^s d\text{Cap}_{\alpha,s}^\omega.$$

Taking  $N \rightarrow \infty$ , we have  $f_N \uparrow f$  everywhere, and (3) follows by the Fatou's property of Choquet integrals.

To complete the proof of this theorem, it remains to justify (15). To this end, since  $\text{Cap}_{\alpha,s}(\cdot)$  is countably subadditive, we may assume that  $E \subseteq B$  for some ball  $B$  with radius  $1/2$ . Since  $E \subseteq B_1(x)$  for all  $x \in E$ , we have

$$\begin{aligned} \mathbf{M}_C^{\text{loc}} \chi_E(x) &\geq \frac{1}{\text{Cap}_{\alpha,s}(B_1(x))} \int_{B_1(x)} \chi_E d\text{Cap}_{\alpha,s} \\ &= \frac{\text{Cap}_{\alpha,s}(E \cap B_1(x))}{\text{Cap}_{\alpha,s}(B_1(x))} \\ &= \frac{\text{Cap}_{\alpha,s}(E)}{\text{Cap}_{\alpha,s}(B_1(x))} \\ &= \frac{\text{Cap}_{\alpha,s}(E)}{\text{Cap}_{\alpha,s}(B_1(0))}, \end{aligned}$$

where we have used the simple fact that  $\text{Cap}_{\alpha,s}(\cdot)$  is translation invariant in the last equality. Suppose the contrary that  $\text{Cap}_{\alpha,s}(E) > 0$ . We obtain by (10) that

$$\begin{aligned} &\frac{1}{2} \frac{\text{Cap}_{\alpha,s}(E)}{\text{Cap}_{\alpha,s}(B_1(0))} \text{Cap}_{\alpha,s}^\omega(B) \\ &\leq \frac{1}{2} \frac{\text{Cap}_{\alpha,s}(E)}{\text{Cap}_{\alpha,s}(B_1(0))} \text{Cap}_{\alpha,s}^\omega \left( \left\{ x \in \mathbb{R}^n : \mathbf{M}_C^{\text{loc}} \chi_E(x) > \frac{1}{2} \frac{\text{Cap}_{\alpha,s}(E)}{\text{Cap}_{\alpha,s}(B_1(0))} \right\} \right) \\ &\leq C_{n,\alpha,s,\omega} \text{Cap}_{\alpha,s}^\omega(E) \\ &= 0. \end{aligned}$$

Since  $\text{Cap}_{\alpha,s}^\omega(B) > 0$  (see [7, Lemma 3.3.12]), we deduce that  $\text{Cap}_{\alpha,s}(E) = 0$ , a contradiction. Therefore, the assertion (15) holds, which finishes the proof of this theorem.  $\square$

### 3 Final Remarks

The main results are all valid for Riesz capacities  $\text{cap}_{\alpha,s}(\cdot)$  defined by

$$\text{cap}_{\alpha,s}(E) = \inf \{ \|\varphi\|_{L^s(\mathbb{R}^n)}^s : \varphi \geq 0, I_\alpha * \varphi \geq \chi_E \}, \quad E \subseteq \mathbb{R}^n,$$

where  $I_\alpha(\cdot) = |\cdot|^{-(n-\alpha)}$  are the Riesz kernels. The corresponding functionals are

$$\mathcal{C}(f) = \inf\{\|\varphi\|_{L^s(\mathbb{R}^n)}^s : \varphi \geq 0, I_\alpha * \varphi \geq |f|^{\frac{1}{s}} \text{ quasi-everywhere}\},$$

$$\mathcal{C}(f) = \inf\{\|\varphi\|_{L^s(\text{cap}_{\alpha,s})}^s : \varphi \geq 0, \mathbf{M}_C \varphi \geq |f|^{\frac{1}{s}} \text{ quasi-everywhere}\},$$

$$\mathfrak{C}(f) = \inf\{\|I_\alpha * \varphi\|_{L^s(\mathbb{R}^n)}^s : \varphi \geq 0, \mathbf{M}_C(I_\alpha * \varphi) \geq |f|^{\frac{1}{s}} \text{ quasi-everywhere}\},$$

where

$$\mathbf{M}_C \varphi(x) = \sup_{r>0} \frac{1}{\text{cap}_{\alpha,s}(B_r(x))} \int_{B_r(x)} |\varphi| d\text{cap}_{\alpha,s}.$$

The weighted Riesz capacities are defined in a canonical way and we have

$$\|\mathbf{M}_C f\|_{L^s(\text{cap}_{\alpha,s}^\omega)} \leq C_{n,\alpha,s,\omega} \|f\|_{L^s(\text{cap}_{\alpha,s}^\omega)},$$

the proofs are almost verbatim to the arguments given in the previous section.

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