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Rate of convergence of the conjugate Fourier integral and its Nörlund means

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Abstract. In this paper, we give the rate of convergence of the conjugate Fourier integral of functions of bounded variation over $(-\infty, \infty)$. Further, we give the rate of convergence of Nörlund means of the conjugate Fourier integral of f, where f is of bounded variation over every finite interval $[a, b]$, $-\infty < a < b < \infty$. Also, in particular as corollary, we get the rate of convergence of $(C, 1)$ means of the conjugate Fourier integral.

Keywords: Nörlund summability, Cesàro summability, functions of bounded variation on $\mathbb{R}$, Conjugate Fourier integral.

MSC 2020 classification: primary 40G05, 41A25; secondary 42A38, 42A10.

Introduction

The Dirichlet-Jordan theorem (see, e.g., [11, p. 57, Theorem 8.1]) asserts that the Fourier series of 2π-periodic function f of bounded variation on $[-\pi, \pi]$ converges at each point and the convergence is uniform on closed intervals of continuity of f. The conjugate analogue of the Dirichlet-Jordan theorem for Fourier series was given by W. H. Young [10]. Then, S. M. Mazhar and A. Al-Budaiwi [6] quantified the result of W. H. Young [10] by estimating the rate of convergence of the conjugate Fourier series at that point, which is also the conjugate analogue of the result of R. Bojanić [3]. Further, S. M. Mazhar [5] also gave the rate of convergence for Fourier-stieltjes series. Also, for non-periodic functions, the Dirichlet-Jordan theorem for Fourier integrals was given in (see, e.g., [9, p. 13, Theorem 3]). In our earlier work [2], we have quantified this result by estimating the rate of convergence of the Fourier integral of functions

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of bounded variation over $\mathbb{R} := (-\infty, \infty)$ at that point. Furthermore, we have also established the rate of convergence of the Nörlund means of the Fourier integral of f, where f is of bounded variation over every finite interval $[a, b]$, $-\infty < a < b < \infty$. Here, we give the conjugate analogue of our results by giving the rate of convergence of the conjugate Fourier integral of functions of bounded variation over $\mathbb{R}$ and giving the rate of convergence of Nörlund means of the conjugate Fourier integral of f, where f is of bounded variation over every finite interval $[a, b]$, $-\infty < a < b < \infty$.

1 Notations and Definitions

First, we recall the definition of (N, p_n) summability of the series $\sum a_n$. Let $\{p_n\}$ be a sequence of constants, real or complex and let $\{P_n\}$ denote the sequence of partial sums of $\sum_{n=0}^{\infty} p_n$. We assume that $P_n \neq 0$ for all $n \in \mathbb{N} \cup \{0\}$, $P_{-1} = p_{-1} = 0$, $P_{-2} = p_{-2} = 0$. For an arbitrary sequence $\{S_n\}$, we put

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} S_k, \ n = 0, 1, 2, \ldots,$$

and call $\{t_n\}$ the Nörlund transform of $\{S_n\}$. The sequence $\{S_n\}$ is said to be summable (N, p_n) (or Nörlund summable with respect to $\{p_n\}$) to s, if $\lim_{n\to\infty} t_n = s$. The series $\sum a_n$ with the sequence of partial sums S_n is said to be summable (N, p_n) if the sequence $\{S_n\}$ is summable (N, p_n).
Next, we recall the definition of Nörlund summability of the integral $\int_0^{\infty} f(u)du$. Let q be a real valued locally integrable function on $[0, \infty)$. If f is a locally integrable function in $\mathbb{R}^+ := (0, \infty)$, then the Nörlund transform $T(y)$ of f with respect to q is defined as

$$T(y) = \frac{1}{Q(y)} \int_0^y Q(y-u) f(u) du \quad \text{or} \quad T(y) = \frac{1}{Q(y)} \int_0^y q(y-u) S(u) du,$$

where $Q(y) = \int_0^y q(u)du \neq 0$, for each $y > 0$ and $S(u) = \int_0^u f(t)dt$. The integral $\int_0^{\infty} f(u)du$ is said to be Nörlund summable to s with respect to $q(y)$ (or summable $(N, q(y))$ to s) if $\lim_{y\to\infty} T(y)$ exists and is equal to s. In the special case, if $q(u) = \beta u^{\beta-1} (\beta > 0)$ then the Nörlund transform reduces to the (C, β) transform (see [9, p. 26]) defined as

$$T(y) = \frac{1}{y^{\beta}} \int_0^y (y-u)^{\beta} f(u) du, \quad y > 0.$$

Remark 1. Note that, if $q(y) = 1$ for $y \in [0,1]$ and $q(y) = 0$ for $y > 1$, then $Q(y)$=1 for all $y \geq 1$ and hence $T(y) = \int_0^y f(u)du$, $y \geq 1$ and $T(y) = \frac{1}{y}\int_0^y (y-u)f(u)du$, $0 < y < 1$. Therefore the integral

$$\begin{aligned}\int_0^\infty f(u)du \text{ is } (N, q(y)) \text{ summable} &\iff \lim_{y\to\infty} T(y) < \infty \\ &\iff \lim_{y\to\infty}\int_0^y f(u)du < \infty \\ &\iff \int_0^\infty f(u)du \text{ converges.}\end{aligned}$$

So, the summability $(N, q(y))$ of the integral $\int_0^\infty f(u)du$ is equivalent to its ordinary convergence in this case.

For a function $f \in L^1[-\pi, \pi]$, its Fourier series is defined as

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$

and its conjugate Fourier series is defined as

$$\sum_{n=1}^{\infty}(a_n \sin nt - b_n \cos nt),$$

where a_n and b_n are given by the usual Euler-Fourier formulae. Their k-th partial sums are

$$S_k(f;x) := S_k(x) = \frac{1}{2}a_0 + \sum_{n=1}^{k}(a_n \cos nt + b_n \sin nt)$$

$$\text{and} \qquad \tilde{S}_k(f;x) := \tilde{S}_k(x) = \sum_{n=1}^{k}(a_n \sin nt - b_n \cos nt).$$

The Fourier integral of a function $f \in L^1(\mathbb{R})$, is given by (see [11, Vol. 2, p. 244])

$$\int_0^\infty (a(u)\cos ux + b(u)\sin ux)du := \int_0^\infty A(u,x)du, \text{ say},$$

and conjugate Fourier integral of a function $f \in L^1(\mathbb{R})$, is given by (see [11, Vol. 2, p. 244])

$$\int_0^\infty (a(u)\sin ux - b(u)\cos ux)du := \int_0^\infty B(u,x)du, \text{ say},$$

where

$$a(u) = \frac{1}{\pi}\int_{\mathbb{R}} f(t)\cos ut dt \quad \text{and} \quad b(u) = \frac{1}{\pi}\int_{\mathbb{R}} f(t)\sin ut dt.$$

Their k-th partial integrals are

$$S(f;y,x) := S(y,x) = \int_0^y A(u,x)du$$

$$\text{and} \qquad \tilde{S}(f;y,x) := \tilde{S}(y,x) = \int_0^y B(u,x)du.$$

A function $f : [a,b] \to \mathbb{R}$ is of bounded variation over the interval $[a,b]$, in symbols: $f \in BV[a,b]$, if

$$V(f;[a,b]) := \sup\left\{\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|\right\} < \infty,$$

where supremum is taken over all partitions $a = x_0 < x_1 < ... < x_n = b$ of $[a,b]$.

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be of bounded variation over $\mathbb{R}$, in symbols: $f \in BV(\mathbb{R})$, if

$$V(f,\mathbb{R}) := \sup_T\left\{\sum_n |f(y_n) - f(x_n)|\right\} < \infty,$$

where T is a finite collection of non-overlapping subintervals $[x_n, y_n]$ in $\mathbb{R}$. Equivalently, a function $f : \mathbb{R} \to \mathbb{R}$ is of bounded variation if $f \in BV[a,b]$ for all $-\infty < a < b < \infty$ and $\sup\{V(f,[a,b]) : -\infty < a < b < \infty\}$ is finite. Furthermore, if this is the case, then the total variation $V(f) := V(f,\mathbb{R})$ is given by

$$V(f) = \sup\{V(f,[a,b]) : -\infty < a < b < \infty\}.$$

Similarly, we can define functions of bounded variation over $[a,\infty)$, for any $a \in \mathbb{R}$.

For $x \in \mathbb{R}$, we define

$$\phi_x(t) = \begin{cases} f(x+t) + f(x-t) - f(x+0) - f(x-0), & t \in (0,\infty), \\ 0, & t = 0, \end{cases}$$

$$\psi_x(t) = \frac{1}{2}(f(x+t) - f(x-t)),\ t \in [0,\infty),$$

$$\tilde{f}(x) = \lim_{h\to 0_+} \tilde{f}(x,h) = \lim_{h\to 0_+}\left\{\frac{-1}{\pi}\int_h^\infty \frac{\psi_x(t)}{t}dt\right\},$$

and

$$\tilde{f}_1(x) = \lim_{h\to 0_+} \tilde{f}_1(x,h) = \lim_{h\to 0_+} \left\{ \frac{-2}{\pi} \int_h^\pi \frac{\psi_x(t)}{2\tan\frac{t}{2}} dt \right\}.$$

Throughout this paper, A is an absolute constant, which may not have the same value at each occurrence. We note that if $f \in L^1(\mathbb{R})$, then $\tilde{f}(x)$ exists for almost all $x \in \mathbb{R}$ (see, e.g., [9, p. 132, Theorem 100]).

Jordan proved that if f is a 2π-periodic function of bounded variation on $[-\pi,\pi]$, then its Fourier series converges to $\frac{1}{2}[f(x+0)+f(x-0)]$ at each point of x (see, e.g., [11, p. 57, Theorem 8.1]). W. H. Young ([10] or see, e.g., [6]) gave the conjugate analogue of Jordan's theorem for Fourier series as the following theorem.

Theorem 1. *If f is a $2\pi-$periodic function and is of bounded variation on $[-\pi,\pi]$, then a necessary and sufficient condition for the convergence of the conjugate Fourier series of $f(x)$ at x is the existence of the integral*

$$\frac{-2}{\pi} \int_0^\pi \frac{\psi_x(t)}{2\tan\frac{t}{2}} dt = \lim_{h\to 0_+} \tilde{f}_1(x,h),$$

which represents then the sum of the series.

S. M. Mazhar and A. Al-budaiwi [6] quantified the above result as the following theorem.

Theorem 2. *If f is a $2\pi-$periodic function and is of bounded variation on $[-\pi,\pi]$, then for $x \in [-\pi,\pi]$ and $n \in \mathbb{N}$ we have*

$$\left| \tilde{S}_n(x) - \tilde{f}_1\left(x, \frac{\pi}{n}\right) \right| \le \frac{2+\frac{4}{\pi}}{n} \sum_{k=1}^{n} V\left(\psi_x(t), \left[0, \frac{\pi}{k}\right]\right),$$

where $\tilde{S}_n(x)$ is the n-th partial sum of the conjugate Fourier series of f.

For non-periodic functions, Jordan's theorem states that if $f \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, then its Fourier integral converges to $\frac{1}{2}[f(x+0)+f(x-0)]$ at each point of x (see, e.g., [9, p. 13, Theorem 3]). We quantified this result as the following theorem [2, Theorem 3.1].

Theorem 3. *Let $f \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$. Then for any $a, y \in \mathbb{R}^+$ with $n \le y < n+1$; $n \in \mathbb{N} \cup \{0\}$ and for any $x \in \mathbb{R}$, we have*

$$\left| S(y,x) - \frac{1}{2}[f(x+0)+f(x-0)] \right| \le \frac{A}{n+1} \sum_{k=0}^{n} V\left(\phi_x, \left[0, \frac{a}{k+1}\right]\right) + \frac{A}{ay},$$

where $S(y,x)$ is the y-th partial integral of the Fourier integral of f.

We have also give the rate of convergence of Nörlund means of the Fourier integral as the following theorem [2, Theorem 3.2].

Theorem 4. *Let $q(y)$ be a non-negative and non-increasing function for $y \geq 0$ such that $Q(y) \to \infty$ as $y \to \infty$. Further, let $f \in L^1(\mathbb{R})$, $\frac{\phi_x(t)}{t} \in L^1(\mathbb{R}^+)$, and $x \in \mathbb{R}$. Suppose f is of bounded variation in every finite interval $[a, b]$, $-\infty < a < b < \infty$. If $T_x(y)$ is a Nörlund transform of Fourier integral of f for $y > 0$, then for any fixed $a \in \mathbb{R}^+$ and for any $y \geq 1$ with $n \leq y < n + 1$; $n \in \mathbb{N}$, we have*

$$\left|T_x(y) - \frac{1}{2}[f(x+0) + f(x-0)]\right| \leq \frac{A}{Q(y)} \sum_{k=1}^{n} \left[\int_{k-1}^{k} q(t)dt \cdot V\left(\phi_x, \left[0, \frac{Q(1)}{Q(k)}a\right]\right)\right] + \frac{A}{aQ(y)}.$$

In this paper, we give the conjugate analogues of Theorem 3 and Theorem 4. Also, as a corollary, we get the Dirichlet-Jordan theorem for conjugate Fourier integrals, as well as the rate of convergence of $(C, 1)$ means of Fourier integrals of functions of bounded variation over every finite interval $[a, b]$, $-\infty < a < b < \infty$.

2 Results

In this section, we state our main theorems on the rate of convergence of conjugate Fourier integrals and their Nörlund means. We first consider the partial integrals of the conjugate Fourier integral of a function of bounded variation over $\mathbb{R}$. The following theorem establishes their rate of convergence and is the conjugate analogue of Theorem 3.

Theorem 5. *Let $f \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$. Then for any $a, y \in \mathbb{R}^+$ with $n \leq y < n + 1$; $n \in \mathbb{N} \cup \{0\}$ and for any $x \in \mathbb{R}$, we have*

$$\left|\tilde{S}(y, x) - \tilde{f}\left(x, \frac{a}{n+1}\right)\right| \leq \frac{\frac{a}{\pi} + \frac{4}{a\pi}}{n} \sum_{k=1}^{n} V\left(\psi_x, \left[0, \frac{a}{k}\right]\right) + \frac{A}{ay}, \tag{1}$$

where $\tilde{S}(y, x)$ is the y-th partial integral of the conjugate Fourier integral of f .

Next, we investigate the Nörlund means of conjugate Fourier integrals. For functions of bounded variation on every finite interval $[a, b]$, $-\infty < a < b < \infty$, we derive the rate of convergence of these means. This result provides the conjugate analogue of Theorem 4.

Theorem 6. *Let $q(y)$ be a non-negative and non-increasing function for $y \geq 0$ such that $Q(y) \to \infty$ as $y \to \infty$ and $\frac{y}{Q(y)}$ is bounded for $y \geq 1$. Further,* *let $f \in L^1(\mathbb{R})$ and $x \in \mathbb{R}$. Suppose f is of bounded variation in every finite interval $[a, b]$, $-\infty < a < b < \infty$. If $\tilde{T}_x(y)$ is a Nörlund transform of the conjugate Fourier integral of f for $y > 0$, then for any fixed $a \in \mathbb{R}^+$ and for any $y \geq 1$ with $n \leq y < n+1$; $n \in \mathbb{N}$, we have*

$$\begin{aligned}&\left|\tilde{T}_x(y) - \tilde{f}\left(x, \frac{Q(1)}{Q(n+1)}a\right)\right| \\ &\leq \frac{A}{Q(y)} \sum_{k=1}^{n} \int_{k-1}^{k} q(t)dt \cdot V\left(\psi_x, \left[0, \frac{Q(1)}{Q(k)}a\right]\right) + \frac{A}{a^2 Q(y)}. \qquad (2)\end{aligned}$$

Remark 2. By Remark 1, it follows that the Nörlund summability of the integral $\int_0^\infty f(x)dx$ is equivalent to its ordinary convergence when we put $q(y) = 1$ for $0 < y < 1$ and $q(y) = 0$ for $y \geq 1$. In this case, $Q(y) = y$ for $0 < y < 1$ and $Q(y) = 1$ for $y \geq 1$. Therefore, $Q(y) \to 1$ as $y \to \infty$ and $\frac{y}{Q(y)} = y$ for $y \geq 1$. So, all the conditions of Theorem 6 are not satisfied in this case and hence we cannot derive Theorem 5 from Theorem 6.

Remark 3. Although Theorem 5 cannot derived from Theorem 6, but we critically observe here that we obtained a result concerning the rate of convergence of Nörlund means of conjugate Fourier intergals by taking weaker condition "f is of bounded variation in every finite interval $[a, b], -\infty < a < b < \infty$" instead of "$f \in BV(\mathbb{R})$" which is used in Theorem 5 to obtain the rate of convergence of the partial integrals of conjugate Fourier integrals.

Putting $q(y) = 1$ in Theorem 6, we get the following corollary concerning the $(C, 1)$ summability of conjugate Fourier integrals.

Corollary 1. *Let $f \in L^1(\mathbb{R})$ and $x \in \mathbb{R}$. Suppose f is of bounded variation in every finite interval $[a, b]$, $-\infty < a < b < \infty$. If $\tilde{\sigma}_x(y)$ is a $(C, 1)$ transform of the conjugate Fourier integral of f for $y > 0$, then for any fixed $a \in \mathbb{R}^+$ and for any $y \geq 1$ with $n \leq y < n+1$; $n \in \mathbb{N}$, we have*

$$\left|\tilde{\sigma}_x(y) - \tilde{f}\left(x, \frac{a}{n+1}\right)\right| \leq \frac{A}{y} \sum_{k=1}^{n} V\left(\psi_x, \left[0, \frac{a}{k}\right]\right) + \frac{A}{a^2 y}. \qquad (3)$$

Note that, $\psi_x(t)$ is continuous at $t = 0$ for $x \in \mathbb{R}$ whenever $f(x)$ is continuous at x. Also, continuity of $\psi_x(t)$ at $t = 0$ implies that

$$V(\psi_x, [0, \delta]) \to 0 \quad \text{as} \quad \delta \to 0.$$

Hence, the right hand side of (1), (2), and (3) tend to 0 as $y \to \infty$ whenever $f(x)$ is continuous at x. Since the set of discontinuties of functions of bounded varition is countable and $\tilde{f}(x)$ exists for almost all $x \in \mathbb{R}$, we get the following corollaries from above results.

Corollary 2. *Let $f \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$. Then the conjugate Fourier integral of f converges to $\tilde{f}(x)$, for almost all $x \in \mathbb{R}$.*

Above corollary is the conjugate analogue of the Dirichlet-Jordan theorem for Fourier integrals (see, e.g., [9, p. 13, Theorem 3]).

Corollary 3. *Let $q(y)$ be a non-negative and non-increasing function for $y \geq 0$ such that $Q(y) \to \infty$ as $y \to \infty$ and $\frac{y}{Q(y)}$ is bounded. Further, let $f \in L^1(\mathbb{R})$ and $x \in \mathbb{R}$. Suppose f is of bounded variation in every finite interval $[a, b]$, $-\infty < a < b < \infty$. Then the conjugate Fourier integral of f is summable $(N, q(y))$ to $\tilde{f}(x)$, for almost all $x \in \mathbb{R}$.*

Corollary 4. *Let $f \in L^1(\mathbb{R})$ and $x \in \mathbb{R}$. Suppose f is of bounded variation in every finite interval $[a, b]$, $-\infty < a < b < \infty$. Then the conjugate Fourier integral of f is summable $(C, 1)$ to $\tilde{f}(x)$, for almost all $x \in \mathbb{R}$.*

Remark 4. We note that, if an integral is summable (C, α); $\alpha \geq 0$, it is summable (C, β) for $\beta > \alpha$, to the same value (see, [9, p. 27]). So, we can extend above corollary to (C, β) summability of the conjugate Fourier integral of f where $\beta \geq 1$.

3 Proof Of The Results

We need the following lemmas to prove our theorems.

Lemma 1. *Let $\tilde{\lambda}(x, a, y) = \int_x^a \frac{\cos yt}{t} dt$ with $a \in \mathbb{R}^+$ be any fixed constant, $0 < x < a$, and $y > 0$. Then $|\tilde{\lambda}(x, a, y)| \leq \frac{2}{xy}$.*

Proof. Let $x, a, y > 0$ be fixed and $0 < x < a$. Since $\frac{1}{t}$ is non-negative and non-increasing for $t > 0$, using second mean value theorem (see, e.g., [1, p. 3]) there exists $\eta \in [x, a]$ such that

$$\int_x^a \frac{\cos yt}{t} dt = \frac{1}{x} \int_x^\eta \cos yt dt = \frac{1}{x} \left[\frac{\sin y\eta}{y} - \frac{\sin yx}{y} \right].$$

Therefore, we get

$$|\tilde{\lambda}(x, a, y)| = \left| \int_x^a \frac{\cos yt}{t} dt \right| \leq \frac{2}{xy}.$$

QED

Lemma 2. *Let $\tilde{K}(y, t) = \frac{1}{\pi Q(y)} \int_0^y q(y - u) \frac{\cos ut}{t} du$, where $q(y)$ is non-negative and non-increasing for $y \geq 0$, and $Q(y) = \int_0^y q(u) du$. Then $|\tilde{K}(y, t)| \leq \frac{2q(0)}{\pi} \cdot \frac{1}{Q(y)t^2}$, for any $y, t > 0$.*

Proof. Observe that

$$\begin{aligned}|\tilde{K}(y,t)| &= \left|\frac{1}{\pi Q(y)}\int_0^y q(y-u)\frac{\cos ut}{t}du\right| \\ &= \frac{1}{\pi Q(y)t}\left|\int_0^y q(u)\cos(y-u)tdu\right|. \qquad (4)\end{aligned}$$

Since $q(y)$ is non-negative and non-increasing, using second mean value theorem (see, e.g., [1, p. 3]) there exists $\eta_1 \in [0, y]$ such that

$$\left|\int_0^y q(u)\cos(y-u)tdu\right| = q(0)\left|\int_0^{\eta_1}\cos(y-u)tdu\right|. \qquad (5)$$

Using (5) in (4), we get

$$\begin{aligned}|\tilde{K}(y,t)| = \frac{q(0)}{\pi Q(y)t}\left|\int_0^{\eta_1}\cos(y-u)tdu\right| &= \frac{q(0)}{\pi Q(y)t}\left|\int_{y-\eta_1}^{y}\cos utdu\right| \\ &= \frac{q(0)}{\pi Q(y)t}\left|\frac{\sin yt}{t} - \frac{\sin(y-\eta_1)t}{t}\right| \\ &\leq \frac{2q(0)}{\pi}\cdot\frac{1}{Q(y)t^2}.\end{aligned}$$

QED

Lemma 3. *Let $q(y)$ be a non-negative and non-increasing function for $y \geq 0$ such that $Q(y) = \int_0^y q(t)dt$. Define*

$$\tilde{\lambda}_1(x,a,y) = \int_x^a \tilde{K}(y,t)dt = \int_x^a \Big(\frac{1}{\pi Q(y)}\int_0^y q(y-u)\frac{\cos ut}{t}du\Big)dt,$$

where $0 < x < a$ and $y > 0$. Then $|\tilde{\lambda}_1(x,a,y)| < \frac{2q(0)}{\pi}\cdot\frac{1}{xQ(y)}$.

Proof. Let $y > 0$ and $0 < x < a$. Using Lemma 2, we have

$$\begin{aligned}|\tilde{\lambda}_1(x,a,y)| &= \left|\int_x^a\left(\frac{1}{\pi Q(y)}\int_0^y q(y-u)\frac{\cos ut}{t}du\right)dt\right| \\ &\leq \int_x^a \frac{2q(0)}{\pi}\cdot\frac{1}{Q(y)t^2}dt \\ &= \frac{2q(0)}{\pi}\cdot\frac{1}{Q(y)}\left[\frac{1}{x}-\frac{1}{a}\right] < \frac{2q(0)}{\pi}\cdot\frac{1}{xQ(y)}.\end{aligned}$$

QED

Proof of Theorem 5. Let $x \in \mathbb{R}$ and $a, y \in \mathbb{R}^+$ be fixed. Using the formulas of $a(u)$ and $b(u)$, we have

$$\begin{aligned}
\tilde{S}(y,x) &= \int_0^y B(u,x)du = \int_0^y [a(u)\sin ux - b(u)\cos ux]du \\
&= \int_0^y \left[\left(\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos utdt\right)\sin ux - \left(\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin utdt\right)\cos ux\right]du \\
&= \int_0^y \left[\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin(x-t)udt\right]du \\
&= \int_0^y \left[\frac{1}{\pi}\int_{-\infty}^{\infty} f(x-t)\sin utdt\right]du \\
&= \int_0^y \left[\frac{1}{\pi}\int_0^{\infty} f(x-t)\sin utdt - \frac{1}{\pi}\int_0^{\infty} f(x+t)\sin utdt\right]du \\
&= -\int_0^y \left[\frac{1}{\pi}\int_0^{\infty} [f(x+t)-f(x-t)]\sin utdt\right]du.
\end{aligned}$$

Since

$$\begin{aligned}
&\int_0^y \left[\frac{1}{\pi}\int_0^{\infty} |f(x+t)-f(x-t)||\sin ut|dt\right]du \\
&\leq \int_0^y \left[\frac{1}{\pi}\int_0^{\infty} |f(x+t)-f(x-t)|dt\right]du \leq Ay,
\end{aligned}$$

using Tonelli-Hobson Theorem (see, e.g., [4, p. 3]) in above equality, we have

$$\begin{aligned}
\tilde{S}(y,x) &= -\frac{1}{\pi}\int_0^{\infty}[f(x+t)-f(x-t)]\left[\int_0^y \sin utdu\right]dt \\
&= -\frac{1}{\pi}\int_0^{\infty}\psi_x(t)\frac{1-\cos yt}{t}dt. \qquad (6)
\end{aligned}$$

Since $y \in \mathbb{R}^+$, there exists $n \in \mathbb{N}\cup\{0\}$ such that $n \leq y < n+1$. Now, using (6) we have

$$\begin{aligned}
&\tilde{S}(y,x) - \tilde{f}\left(x, \frac{a}{n+1}\right) \\
&= -\frac{1}{\pi}\int_0^{\infty}\psi_x(t)\frac{1-\cos yt}{t}dt + \frac{1}{\pi}\int_{\frac{a}{n+1}}^{\infty}\frac{\psi_x(t)}{t}dt \\
&= -\frac{1}{\pi}\int_0^{\frac{a}{n+1}}\psi_x(t)\frac{1-\cos yt}{t}dt + \frac{1}{\pi}\int_{\frac{a}{n+1}}^{\infty}\psi_x(t)\frac{\cos yt}{t}dt \\
&= -\frac{1}{\pi}\int_0^{\frac{a}{n+1}}\psi_x(t)\frac{1-\cos yt}{t}dt + \frac{1}{\pi}\int_{\frac{a}{n+1}}^{a}\psi_x(t)\frac{\cos yt}{t}dt + \frac{1}{\pi}\int_a^{\infty}\psi_x(t)\frac{\cos yt}{t}dt \\
&= \tilde{S}_1(y,x) + \tilde{S}_2(y,x) + \tilde{S}_3(y,x), \text{ say.} \qquad (7)
\end{aligned}$$

Since $|\sin t| \leq t$ for $t \geq 0$, we have

$$
\begin{aligned}
|\tilde{S}_1(y,x)| \leq \frac{1}{\pi}\int_0^{\frac{a}{n+1}} |\psi_x(t)|\frac{|1-\cos yt|}{t}dt &= \frac{1}{\pi}\int_0^{\frac{a}{n+1}} |\psi_x(t)-\psi_x(0)|\frac{\left|2\sin^2\frac{yt}{2}\right|}{t}dt \\
&\leq \frac{1}{\pi}\int_0^{\frac{a}{n+1}} V(\psi_x,[0,t])\frac{2yt}{2t}dt \\
&\leq \frac{y}{\pi}V\left(\psi_x,\left[0,\frac{a}{n+1}\right]\right)\frac{a}{n+1} \\
&\leq \frac{a}{\pi}V\left(\psi_x,\left[0,\frac{a}{n+1}\right]\right). \qquad (8)
\end{aligned}
$$

Since $\cos yt$ and t are continuous over $\left[\frac{a}{n+1},a\right]$, $\frac{\cos yt}{t}$ is integrable over $\left[\frac{a}{n+1},a\right]$ and hence $\int_{\frac{a}{n+1}}^t \frac{\cos yt'}{t'}dt$ is absolutely continuous over $\left[\frac{a}{n+1},a\right]$. Therefore, using relation between Lebesgue-Stieltjes integral and Lebesgue integral (see, e.g., [8, p. 127]), we have

$$
\begin{aligned}
\int_{\frac{a}{n+1}}^a \psi_x(t)\frac{\cos yt}{t}dt &= \int_{\frac{a}{n+1}}^a \psi_x(t)d\left(\int_{\frac{a}{n+1}}^t \frac{\cos yx'}{x'}dx'\right) \\
&= \int_{\frac{a}{n+1}}^a \psi_x(t)d\left(\int_{\frac{a}{n+1}}^a \frac{\cos yx'}{x'}dx' - \int_t^a \frac{\cos yx'}{x'}dx'\right) \\
&= \int_{\frac{a}{n+1}}^a \psi_x(t)d\left(\tilde{\lambda}\left(\frac{a}{n+1},a,y\right) - \tilde{\lambda}(t,a,y)\right) \\
&= -\int_{\frac{a}{n+1}}^a \psi_x(t)d\tilde{\lambda}(t,a,y).
\end{aligned}
$$

Using integration by parts in above equality, we get

$$
\begin{aligned}
\int_{\frac{a}{n+1}}^a \psi_x(t)\frac{\cos yt}{t}dt &= \psi_x\left(\frac{a}{n+1}\right)\tilde{\lambda}\left(\frac{a}{n+1},a,y\right) - \psi_x(a)\tilde{\lambda}(a,a,y) \\
&\quad + \int_{\frac{a}{n+1}}^a \tilde{\lambda}(t,a,y)d\psi_x(t) \\
&= \psi_x\left(\frac{a}{n+1}\right)\tilde{\lambda}\left(\frac{a}{n+1},a,y\right) + \int_{\frac{a}{n+1}}^a \tilde{\lambda}(t,a,y)d\psi_x(t).
\end{aligned}
$$

Using above equality and Lemma 1 in $\tilde{S}_2(y,x)$, we get

$$\begin{aligned}
\pi\left|\tilde{S}_2(y,x)\right| &= \left|\psi_x\left(\frac{a}{n+1}\right)\tilde{\lambda}\left(\frac{a}{n+1},a,y\right)+\int_{\frac{a}{n+1}}^{a}\tilde{\lambda}(t,a,y)d\psi_x(t)\right| \\
&\le \left|\psi_x\left(\frac{a}{n+1}\right)-\psi_x(0)\right|\frac{2(n+1)}{ay}+\int_{\frac{a}{n+1}}^{a}\frac{2}{ty}dV(\psi_x,[0,t]) \\
&\le V\left(\psi_x,\left[0,\frac{a}{n+1}\right]\right)\frac{2(n+1)}{ay}+\int_{\frac{a}{n+1}}^{a}\frac{2}{ty}dV(\psi_x,[0,t]).
\end{aligned}$$

Now using integration by parts, we get

$$\begin{aligned}
\pi\left|\tilde{S}_2(y,x)\right| \le{}& V\left(\psi_x,\left[0,\frac{a}{n+1}\right]\right)\frac{2(n+1)}{ay}+\frac{2}{ay}V(\psi_x,[0,a]) \\
&-\frac{2(n+1)}{ay}V\left(\psi_x,\left[0,\frac{a}{n+1}\right]\right)-\frac{2}{y}\int_{\frac{a}{n+1}}^{a}V(\psi_x,[0,t])d\left(\frac{1}{t}\right) \\
={}&\frac{2}{ay}V(\psi_x,[0,a])+\frac{2}{y}\int_{\frac{a}{n+1}}^{a}\frac{V(\psi_x,[0,t])}{t^2}dt \\
={}&\frac{2}{ay}V(\psi_x,[0,a])+\frac{2}{ay}\int_{1}^{n+1}V\left(\psi_x,\left[0,\frac{a}{t}\right]\right)dt \\
={}&\frac{2}{ay}V(\psi_x,[0,a])+\frac{2}{ay}\sum_{k=1}^{n}\int_{k}^{k+1}V\left(\psi_x,\left[0,\frac{a}{t}\right]\right)dt \\
\le{}&\frac{2}{ay}V(\psi_x,[0,a])+\frac{2}{ay}\sum_{k=1}^{n}V\left(\psi_x,\left[0,\frac{a}{k}\right]\right) \\
\le{}&\frac{4}{an}\sum_{k=1}^{n}V\left(\psi_x,\left[0,\frac{a}{k}\right]\right). \qquad (9)
\end{aligned}$$

Since $f \in BV(\mathbb{R})$, $\psi_x(t) \in BV[0,\infty)$. Therefore, there exist two non-increasing bounded functions $\alpha_x(t)$ and $\beta_x(t)$ on $[0,\infty)$ such that $\psi_x(t) = \alpha_x(t) - \beta_x(t)$ for $t \in [0,\infty)$ (see, e.g., [7, p. 239]). Hence, we get

$$\begin{aligned}
|\tilde{S}_3(y,x)| &= \left|\frac{1}{\pi}\int_a^\infty \psi_x(t)\frac{\cos yt}{t}dt\right| \\
&\le \left|\frac{1}{\pi}\int_a^\infty \alpha_x(t)\frac{\cos yt}{t}dt\right|+\left|\frac{1}{\pi}\int_a^\infty \beta_x(t)\frac{\cos yt}{t}dt\right| \\
&= \left|\frac{1}{\pi}\lim_{b\to\infty}\int_a^b \alpha_x(t)\frac{\cos yt}{t}dt\right|+\left|\frac{1}{\pi}\lim_{b\to\infty}\int_a^b \beta_x(t)\frac{\cos yt}{t}dt\right|. \qquad (10)
\end{aligned}$$

Since $\frac{\alpha_x(t)}{t}$ is non-increasing, by second mean value theorem (see, e.g., [1, p. 3]) there exists η_b in $[a,b]$; for every $b>a$ such that

$$\int_a^b \alpha_x(t)\frac{\cos yt}{t}dt = \frac{\alpha_x(a)}{a}\int_a^{\eta_b}\cos ytdt + \frac{\alpha_x(b)}{b}\int_{\eta_b}^b \cos ytdt. \tag{11}$$

Therefore, using (11) and since $\alpha_x(t)$ and $\beta_x(t)$ are bounded, we have

$$\begin{aligned}\left|\lim_{b\to\infty}\int_a^b \alpha_x(t)\frac{\cos yt}{t}dt\right| &= \left|\lim_{b\to\infty}\left[\frac{\alpha_x(a)}{a}\int_a^{\eta_b}\cos ytdt + \frac{\alpha_x(b)}{b}\int_{\eta_b}^b \cos ytdt\right]\right| \\ &\le \lim_{b\to\infty}\left[\frac{A}{a}\left|\int_a^{\eta_b}\cos ytdt\right| + \frac{A}{b}\left|\int_{\eta_b}^b \cos ytdt\right|\right] \\ &= \lim_{b\to\infty}\left[\frac{A}{a}\left|\frac{\sin y\eta_b - \sin ya}{y}\right| + \frac{A}{b}\left|\frac{\sin yb - \sin y\eta_b}{y}\right|\right] \\ &\le \lim_{b\to\infty}\left[\frac{2A}{ay} + \frac{2A}{ay}\right] \le \frac{A}{ay}, \end{aligned} \tag{12}$$

where A is an upper bound for $\alpha_x(t)$ and $\beta_x(t)$. Similarly, we can prove that

$$\left|\lim_{b\to\infty}\int_a^b \beta_x(t)\frac{\cos yt}{t}dt\right| \le \frac{A}{ay}. \tag{13}$$

Using (12) and (13) in (10), we get

$$|\tilde{S}_3(y,x)| \le \frac{A}{ay\pi}. \tag{14}$$

Using (7)–(9) and (14), we get

$$\begin{aligned}&\left|\tilde{S}(y,x) - \tilde{f}\left(x, \frac{a}{n+1}\right)\right| \\ &\le \frac{a}{\pi}V\left(\psi_x, \left[0, \frac{a}{n+1}\right]\right) + \frac{4}{an\pi}\sum_{k=1}^{n} V\left(\psi_x, \left[0, \frac{a}{k}\right]\right) + \frac{A}{ay\pi}. \end{aligned} \tag{15}$$

Observe that

$$\begin{aligned}V\left(\psi_x, \left[0, \frac{a}{n+1}\right]\right) = \frac{n}{n}V\left(\psi_x, \left[0, \frac{a}{n+1}\right]\right) &= \frac{1}{n}\sum_{k=1}^{n} V\left(\psi_x, \left[0, \frac{a}{n+1}\right]\right) \\ &\le \frac{1}{n}\sum_{k=1}^{n} V\left(\psi_x, \left[0, \frac{a}{k}\right]\right). \end{aligned} \tag{16}$$

Using (16) in (15), we get

$$\left|\tilde{S}(y,x)-\tilde{f}\left(x,\frac{a}{n+1}\right)\right|\le\frac{\frac{a}{\pi}+\frac{4}{a\pi}}{n}\sum_{k=1}^{n}V\left(\psi_x,\left[0,\frac{a}{k}\right]\right)+\frac{A}{ay}.$$

QED

Proof of Theorem 6. Let $x\in\mathbb{R}$, $a\in\mathbb{R}^+$, and $y\ge 1$ be fixed. Let $\tilde{T}_x(y)$ be the Nörlund transform of $B(u,x)$ with respect to $q(y)$. Then using (6), we have

$$\begin{aligned}
&\tilde{T}_x(y)-\tilde{f}\left(x,\frac{Q(1)}{Q(n+1)}a\right)\\
&=\frac{1}{Q(y)}\int_0^y q(y-u)\tilde{S}(u,x)du+\frac{1}{\pi}\int_{\frac{Q(1)}{Q(n+1)}a}^{\infty}\frac{\psi_x(t)}{t}dt\\
&=\frac{1}{Q(y)}\int_0^y q(y-u)\left(-\frac{1}{\pi}\int_0^{\infty}\psi_x(t)\frac{1-\cos ut}{t}dt\right)du\\
&\qquad+\frac{1}{Q(y)}\int_0^y q(y-u)\left\{\frac{1}{\pi}\int_{\frac{Q(1)}{Q(n+1)}a}^{\infty}\frac{\psi_x(t)}{t}dt\right\}du\\
&=\frac{1}{Q(y)}\int_0^y q(y-u)\left(-\frac{1}{\pi}\int_0^{\frac{Q(1)}{Q(n+1)}a}\psi_x(t)\frac{1-\cos ut}{t}dt\right.\\
&\qquad\left.+\frac{1}{\pi}\int_{\frac{Q(1)}{Q(n+1)}a}^{\infty}\psi_x(t)\frac{\cos ut}{t}dt\right)du\\
&=-\frac{1}{\pi Q(y)}\int_0^y q(y-u)\left(\int_0^{\frac{Q(1)}{Q(n+1)}a}\psi_x(t)\frac{1-\cos ut}{t}dt\right)du\\
&\qquad+\frac{1}{\pi Q(y)}\int_0^y q(y-u)\left(\int_{\frac{Q(1)}{Q(n+1)}a}^{\infty}\psi_x(t)\frac{\cos ut}{t}dt\right)du\\
&=\tilde{I}_1(y,x)+\tilde{I}_2(y,x),\ \text{say}.
\end{aligned}\tag{17}$$

Since $f \in L^1(\mathbb{R})$, we have

$$\begin{aligned}
&\frac{1}{\pi|Q(y)|}\int_0^y |q(y-u)|\bigg(\int_0^{\frac{Q(1)}{Q(n+1)}a} |\psi_x(t)|\frac{|1-\cos ut|}{|t|}dt\bigg)du \\
&= \frac{1}{\pi Q(y)}\int_0^y q(y-u)\bigg(\int_0^{\frac{Q(1)}{Q(n+1)}a} |\psi_x(t)|\frac{|2\sin^2\left(\frac{ut}{2}\right)|}{t}dt\bigg)du \\
&\leq \frac{1}{\pi Q(y)}\int_0^y q(y-u)\bigg(\int_0^{\frac{Q(1)}{Q(n+1)}a} |\psi_x(t)|\frac{2ut}{2t}dt\bigg)du \\
&\leq \frac{y}{\pi Q(y)}\int_0^y q(y-u)\bigg(\int_0^{\frac{Q(1)}{Q(n+1)}a} |\psi_x(t)|dt\bigg)du \\
&\leq \frac{Ay}{Q(y)}\int_0^y q(y-u)du = Ay.
\end{aligned}$$

Hence, using Tonelli-Hobson Theorem (see, e.g., [4, p. 3]) in $\tilde{I}_1(y,x)$, we get

$$\tilde{I}_1(y,x) = \int_0^{\frac{Q(1)}{Q(n+1)}a} \psi_x(t)\bigg(\frac{1}{\pi Q(y)}\int_0^y q(y-u)\frac{1-\cos ut}{t}du\bigg)dt.$$

Since $|\sin t| \leq t$ for $t \geq 0$ and $\frac{y}{Q(y)}$ is bounded for $y \geq 1$, we get

$$\begin{aligned}
|\tilde{I}_1(y,x)| &\leq \int_0^{\frac{Q(1)}{Q(n+1)}a} |\psi_x(t)|\bigg(\frac{1}{\pi Q(y)}\int_0^y q(y-u)\frac{|2\sin^2\frac{ut}{2}|}{t}du\bigg)dt \\
&\leq \int_0^{\frac{Q(1)}{Q(n+1)}a} |\psi_x(t)-\psi_x(0)|\bigg(\frac{1}{\pi Q(y)}\int_0^y uq(y-u)du\bigg)dt \\
&\leq \frac{y}{\pi}\int_0^{\frac{Q(1)}{Q(n+1)}a} V\big(\psi_x,[0,t]\big)dt \\
&\leq \frac{yQ(1)a}{\pi Q(n+1)}V\bigg(\psi_x,\bigg[0,\frac{Q(1)}{Q(n+1)}a\bigg]\bigg)
\end{aligned}$$

$$\leq \frac{Q(1)ay}{\pi Q(y)}V\bigg(\psi_x,\bigg[0,\frac{Q(1)}{Q(n+1)}a\bigg]\bigg) \leq \frac{AQ(1)a}{\pi}V\bigg(\psi_x,\bigg[0,\frac{Q(1)}{Q(n+1)}a\bigg]\bigg), \quad (18)$$

where A is an upper bound for $\frac{y}{Q(y)}$. Again, since $f \in L^1(\mathbb{R})$, we have

$$\begin{aligned}
&\frac{1}{\pi|Q(y)|}\int_0^y |q(y-u)|\bigg(\int_{\frac{Q(1)}{Q(n+1)}a}^{\infty} |\psi_x(t)|\frac{|\cos ut|}{t}dt\bigg)du \\
&\leq \frac{Q(n+1)}{a\pi Q(1)Q(y)}\int_0^y q(y-u)\bigg(\int_{\frac{Q(1)}{Q(n+1)}a}^{\infty} |\psi_x(t)|dt\bigg)du \leq AQ(y).
\end{aligned}$$

Hence, using Tonelli-Hobson Theorem (see, e.g., [4, p. 3]) in $\tilde{I}_2(y,x)$, we get

$$\begin{aligned}\tilde{I}_2(y,x) &= \int_{\frac{Q(1)}{Q(n+1)}a}^{\infty} \psi_x(t)\bigg(\frac{1}{\pi Q(y)}\int_0^y q(y-u)\frac{\cos ut}{t}du\bigg)dt \\ &= \int_{\frac{Q(1)}{Q(n+1)}a}^{a} \psi_x(t)\bigg(\frac{1}{\pi Q(y)}\int_0^y q(y-u)\frac{\cos ut}{t}du\bigg)dt \\ &\qquad + \int_a^{\infty} \psi_x(t)\bigg(\frac{1}{\pi Q(y)}\int_0^y q(y-u)\frac{\cos ut}{t}du\bigg)dt \\ &= \tilde{I}_{21}(y,x) + \tilde{I}_{22}(y,x), \text{ say.} \end{aligned} \tag{19}$$

Using Lemma 2 and the fact that, $f \in L^1(\mathbb{R})$ in $\tilde{I}_{22}(y,x)$, we have

$$\begin{aligned}|\tilde{I}_{22}(y,x)| &= \bigg|\int_a^{\infty} \psi_x(t)\bigg(\frac{1}{\pi Q(y)}\int_0^y q(y-u)\frac{\cos ut}{t}du\bigg)dt\bigg| \\ &\le \int_a^{\infty} |\psi_x(t)|\frac{2q(0)}{\pi Q(y)t^2}dt \le \frac{2q(0)}{\pi a^2 Q(y)}\int_a^{\infty}|\psi_x(t)|dt \le \frac{A2q(0)}{\pi a^2 Q(y)}. \end{aligned} \tag{20}$$

Since $\tilde{K}(y,t) = \frac{1}{\pi Q(y)}\int_0^y q(y-u)\frac{\cos ut}{t}du$ is continuous over $\big[\frac{Q(1)}{Q(n+1)}a, a\big]$ for fixed $y>0$, it is integrable over $\big[\frac{Q(1)}{Q(n+1)}a, a\big]$ and hence $\int_{\frac{Q(1)}{Q(n+1)}a}^{t}\tilde{K}(y,t')dt'$ is absolutely continuous over $\big[\frac{Q(1)}{Q(n+1)}a, a\big]$. Therefore, using relation between Lebesgue-Stieljes integral and Lebesgue integral (see, e.g., [8, p. 127]) in $\tilde{I}_{21}(y,x)$, we get

$$\begin{aligned}\tilde{I}_{21}(y,x) &= \int_{\frac{Q(1)}{Q(n+1)}a}^{a} \psi_x(t)\tilde{K}(y,t)dt \\ &= \int_{\frac{Q(1)}{Q(n+1)}a}^{a} \psi_x(t)d\bigg(\int_{\frac{Q(1)}{Q(n+1)}a}^{t}\tilde{K}(y,t')dt'\bigg) \\ &= \int_{\frac{Q(1)}{Q(n+1)}a}^{a} \psi_x(t)d\bigg(\int_{\frac{Q(1)}{Q(n+1)}a}^{a}\tilde{K}(y,t')dt' - \int_t^a \tilde{K}(y,t')dt'\bigg) \\ &= -\int_{\frac{Q(1)}{Q(n+1)}a}^{a} \psi_x(t)d\bigg(\int_t^a \tilde{K}(y,t')dt'\bigg) = -\int_{\frac{Q(1)}{Q(n+1)}a}^{a}\psi_x(t)d\tilde{\lambda}_1(t,a,y). \end{aligned}$$

Now, using integration by parts, we get

$$\begin{aligned}\tilde{I}_{21}(y,x) &= -\psi_x(a)\tilde{\lambda}_1(a,a,y) + \psi_x\bigg(\frac{Q(1)}{Q(n+1)}a\bigg)\tilde{\lambda}_1\bigg(\frac{Q(1)}{Q(n+1)}a,a,y\bigg) \\ &\qquad + \int_{\frac{Q(1)}{Q(n+1)}a}^{a}\tilde{\lambda}_1(t,a,y)d\psi_x(t) \\ &= \psi_x\bigg(\frac{Q(1)}{Q(n+1)}a\bigg)\tilde{\lambda}_1\bigg(\frac{Q(1)}{Q(n+1)}a,a,y\bigg) + \int_{\frac{Q(1)}{Q(n+1)}a}^{a}\tilde{\lambda}_1(t,a,y)d\psi_x(t). \end{aligned}$$

Further, using Lemma 3 and integration by parts in $\tilde{I}_{21}(y,x)$, we get

$$\begin{aligned}
|\tilde{I}_{21}(y,x)| &\le \left|\psi_x\left(\frac{Q(1)}{Q(n+1)}a\right)-\psi_x(0)\right|\frac{2q(0)Q(n+1)}{\pi a Q(1)Q(y)}+\\
&\quad+\int_{\frac{Q(1)}{Q(n+1)}a}^{a}\frac{2q(0)}{\pi Q(y)t}dV(\psi_x,[0,t]) \qquad (21)\\
&\le \frac{2q(0)}{\pi}\left[\frac{Q(n+1)}{aQ(1)Q(y)}V\left(\psi_x,\left[0,\frac{Q(1)}{Q(n+1)}a\right]\right)+\int_{\frac{Q(1)}{Q(n+1)}a}^{a}\frac{1}{Q(y)t}dV(\psi_x,[0,t])\right]\\
&\le \frac{2q(0)}{\pi}\left[\frac{Q(n+1)}{aQ(1)Q(y)}V\left(\psi_x,\left[0,\frac{Q(1)}{Q(n+1)}a\right]\right)+\frac{1}{aQ(y)}V(\psi_x,[0,a])\right.\\
&\quad\left.-\frac{Q(n+1)}{aQ(1)Q(y)}V\left(\psi_x,\left[0,\frac{Q(1)}{Q(n+1)}a\right]\right)-\frac{1}{Q(y)}\int_{\frac{Q(1)}{Q(n+1)}a}^{a}V(\psi_x,[0,t])d\left(\frac{1}{t}\right)\right]\\
&= \frac{2q(0)}{\pi}\left[\frac{1}{aQ(y)}V(\psi_x,[0,a])+\frac{1}{Q(y)}\int_{\frac{Q(1)}{Q(n+1)}a}^{a}\frac{V(\psi_x,[0,t])}{t^2}dt\right]. \qquad (22)
\end{aligned}$$

Now, observe that

$$\begin{aligned}
\int_{\frac{Q(1)}{Q(n+1)}a}^{a}\frac{V(\psi_x,[0,t])}{t^2}dt &= \frac{1}{aQ(1)}\int_{Q(1)}^{Q(n+1)}V\left(\psi_x,\left[0,\frac{Q(1)}{t}a\right]\right)dt\\
&= \frac{1}{aQ(1)}\sum_{k=1}^{n}\int_{Q(k)}^{Q(k+1)}V\left(\psi_x,\left[0,\frac{Q(1)}{t}a\right]\right)dt\\
&\le \frac{1}{aQ(1)}\sum_{k=1}^{n}[Q(k+1)-Q(k)]\cdot V\left(\psi_x,\left[0,\frac{Q(1)}{Q(k)}a\right]\right)\\
&= \frac{1}{aQ(1)}\sum_{k=1}^{n}\int_{k}^{k+1}q(t)dt\cdot V\left(\psi_x,\left[0,\frac{Q(1)}{Q(k)}a\right]\right)\\
&\le \frac{1}{aQ(1)}\sum_{k=1}^{n}\int_{k-1}^{k}q(t)dt\cdot V\left(\psi_x,\left[0,\frac{Q(1)}{Q(k)}a\right]\right). \qquad (23)
\end{aligned}$$

Using (23) in (21), we get

$$\begin{aligned}
\tilde{I}_{21}(y,x) &\le \frac{2q(0)}{\pi}\left[\frac{1}{aQ(1)Q(y)}\int_0^1 q(t)dtV(\psi_x,[0,a])\right.\\
&\quad\left.+\frac{1}{aQ(1)Q(y)}\sum_{k=1}^{n}\int_{k-1}^{k}q(t)dt\cdot V\left(\psi_x,\left[0,\frac{Q(1)}{Q(k)}a\right]\right)\right]\\
&\le \frac{4q(0)}{a\pi Q(1)}\cdot\frac{1}{Q(y)}\sum_{k=1}^{n}\int_{k-1}^{k}q(t)dt\cdot V\left(\psi_x,\left[0,\frac{Q(1)}{Q(k)}a\right]\right). \qquad (24)
\end{aligned}$$

Using (20) and (24) in (19), we get

$$\tilde{I}_2(y,x) \le \frac{4q(0)}{a\pi Q(1)} \cdot \frac{1}{Q(y)} \sum_{k=1}^{n} \int_{k-1}^{k} q(t)dt \cdot V\left(\psi_x, \left[0, \frac{Q(1)}{Q(k)}a\right]\right) + \frac{2Aq(0)}{\pi a^2 Q(y)}. \tag{25}$$

Using (18) and (25) in (17), we get

$$\begin{aligned} \left|\tilde{T}_x(y) - \tilde{f}\left(x, \frac{Q(1)}{Q(n+1)}a\right)\right| &\le \frac{AQ(1)a}{\pi} V\left(\psi_x, \left[0, \frac{Q(1)}{Q(n+1)}a\right]\right) \\ &+ \frac{4q(0)}{a\pi Q(1)} \cdot \frac{1}{Q(y)} \sum_{k=1}^{n} \int_{k-1}^{k} q(t)dt \cdot V\left(\psi_x, \left[0, \frac{Q(1)}{Q(k)}a\right]\right) + \frac{2Aq(0)}{\pi a^2 Q(y)}. \end{aligned} \tag{26}$$

Since $q(y)$ is non-negative and non-increasing, $Q(y) = \int_a^b q(u)du \le (b-a)q(a)$ and $Q(y) = \int_a^b q(u)du \ge (b-a)q(b)$ for any $b > a \ge 0$. Hence, for $0 < z \le 1$ and $y \ge 1$ we have

$$\frac{Q(y+z)}{Q(y)} = 1 + \frac{\int_y^{y+z} q(u)du}{\int_0^y q(u)du} \le 1 + \frac{zq(y)}{yq(y)} \le 1 + \frac{1}{y} \le 2. \tag{27}$$

Using (27), we have

$$\begin{aligned} V\left(\psi_x, \left[0, \frac{Q(1)}{Q(n+1)}a\right]\right) &= \frac{Q(y)}{Q(y)} V\left(\psi_x, \left[0, \frac{Q(1)}{Q(n+1)}a\right]\right) \\ &\le \frac{2Q(n)}{Q(y)} V\left(\psi_x, \left[0, \frac{Q(1)}{Q(n+1)}a\right]\right) \\ &= \frac{2}{Q(y)} \sum_{k=1}^{n} \int_{k-1}^{k} q(t)dt \cdot V\left(\psi_x, \left[0, \frac{Q(1)}{Q(n+1)}a\right]\right) \\ &\le \frac{2}{Q(y)} \sum_{k=1}^{n} \left[\int_{k-1}^{k} q(t)dt \cdot V\left(\psi_x, \left[0, \frac{Q(1)}{Q(k)}a\right]\right)\right]. \end{aligned} \tag{28}$$

Using (28) in (26), we get

$$\begin{aligned} &\left|\tilde{T}_x(y) - \tilde{f}\left(x, \frac{Q(1)}{Q(n+1)}a\right)\right| \\ &\le \left[\frac{A2Q(1)a}{\pi} + \frac{4q(0)}{a\pi Q(1)}\right] \frac{1}{Q(y)} \sum_{k=1}^{n} \int_{k-1}^{k} q(t)dt \cdot V\left(\psi_x, \left[0, \frac{Q(1)}{Q(k)}a\right]\right) + \frac{2Aq(0)}{\pi a^2 Q(y)} \\ &\le \frac{A}{Q(y)} \sum_{k=1}^{n} \int_{k-1}^{k} q(t)dt \cdot V\left(\psi_x, \left[0, \frac{Q(1)}{Q(k)}a\right]\right) + \frac{A}{a^2 Q(y)}. \end{aligned}$$

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