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On solutions of a difference equation driven by a sequence of identically distributed and weakly independent cylindrical random variables

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Abstract. We consider a sequence $\{Z_n\}_{n\in\mathbb{Z}}$ of weakly independent and identically distributed cylindrical random variables in a Banach space U and a bounded linear operator A on U and show that under suitable conditions, for each $n \in \mathbb{Z}$, the series $\sum_{k=0}^{\infty} Z_{n-k}((A^*)^k(\cdot))$ converges in $\mathfrak{C}_2$, where $\mathfrak{C}_2$ is a Banach space of cylindrical random variables to be defined and if we define $Y_n := \sum_{k=0}^{\infty} Z_{n-k}((A^*)^k(\cdot))$, then the cylindrical process $\{Y_n\}_{n\in\mathbb{Z}}$ is the unique cylindrical weakly stationary solution of the cylindrical difference equation $X_n = AX_{n-1} + Z_n$. We show that without additional conditions on the operator A and the cylindrical distribution of Z_1, the cylindrical distribution of Y_n is A-decomposable. Further, we prove that under mild conditions on the cylindrical distribution of Z_1, Y_n is induced by a U-valued random variable and finally we show that under certain conditions, the process $\{X_n\}$ is a cylindrical Markov process.

Keywords: Cylindrical random variable, Generalized Stochastic Process, Difference Equation, Stationary Process

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Introduction

Let X_0 be a random variable, $\{A_n\}_{n\in\mathbb{Z}}$ and $\{B_n\}_{n\in\mathbb{Z}}$ be sequences of random variables all defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\{X_n\}$ be the sequence defined recursively by

$$X_n = A_n X_{n-1} + B_n, \ n \geq 1. \tag{1}$$

Over the last decades, there has been significant interest in (1), which has been extensively studied in the literature and widely used as a model for the evolution of many random phenomena in discrete time including the decay of radioactive material, population growth and portfolio dynamics,(see Vervaat [34], Kesten [22], Embrechts and Goldie [11], Paulson and Uppuluri [26], Cavalli-Sforza and Feldman [9], Goldie and Grübel [14], Wolfe [35]) just to name a few, apart from the fact that it is an interesting mathematical object in its own right.

The study of (1) started probably with the works of Bharucha-Reid [2] and Grenander [16] in which such difference equations are mentioned, Masimov [25] on random walks on lines and Kesten [22] on renewal theory for products of random matrices.

Subsequently, the theory and applications of (1) in Euclidean space, especially in the one dimensional setting evolved rapidly with contributions from many authors.

In the one dimensional setting, various aspects of this equation studied in the literature include: conditions under which $\{X_n\}$ converges in distribution, tails of limit distributions, stationary distributions of $\{X_n\}$, exponential growth rate of $\{|X_n|\}$, existence of moments, etc.

Conditions under which $\{X_n\}$ converges in distribution, appear in Vervaat [34], Brandt [7], Kesten [22], Buraczewski et al. [8]. In particular if $\{X_n\}$ converges in distribution then the equation $X \stackrel{d}{=} AX + B$ has a solution (where (A, B) is a generic element of the sequence $\{(A_n, B_n)\}$, i.e. $(A, B) \stackrel{d}{=} (A_n, B_n)$ for any $n \geq 1$). An object which has been investigated frequently in this case is the distribution of X. When $\{(A_n, B_n)\}_{n\in\mathbb{Z}}$ is independent and identically distributed (i.i.d.), Grincevicjus [18] gives properties of the distribution of X if $\mathbb{P}(A_1 \neq 0) = 1$. In this case X can not have a non degenerate discrete distribution. Particular cases of this distribution are computed explicitly in Masimov [25]. Tails of the distribution of X are studied in Goldie [13], Goldie and Maller [15], Goldie and Greubel [14], Hitczenko and Wesolowski [19], Grincevicius [17], Resnick and Willekens [27], Kesten [22]. Essentially, under suitable conditions $0 < \lim\limits_{t\to\infty} t^\alpha \mathbb{P}(X > t) < \infty$ and $\limsup\limits_{t\to\infty} \frac{1}{t} \ln \mathbb{P}(|X| > t) \leq -\theta$ where α and θ are positive constants.

Stationary solutions are studied in Resnick and Willekens [27], Brandt [7], Bougerol and Picard [6] etc. In addition to giving conditions for the existence of stationary solutions, Bougerol and Picard [6] give representations for these solutions. In the one-dimensional case, the results of Brandt [7] are important for applications because they drop the requirement that $\{(A_n, B_n)\}_{n\in\mathbb{Z}}$ is i.i.d. They require simply stationarity and ergodicity.

Conditions for the existence of moments are given in Karlsen [21] and Vervaat [34].

Real multidimensional difference equations with i.i.d. coefficients appear in Bougerol and Picard [6] and Kesten [22], Buraczewski et. al. [8]. Some of the results presented in Bougerol and Picard [6] extend results in Brandt [7] to the multidimensional setting, when $\{(A_n, B_n)\}_{n\in\mathbb{Z}}$ is i.i.d.

Growth rates and questions of central limit theorems for autoregressive processes are presented in Shu [30] where the author provides conditions under which the almost sure limit of $\{\frac{1}{n}\ln|X_n|\}$ exists when $\mathbb{E}\ln|A_1| < 0$. A central limit theorem is given for a related autoregressive process. These results generalize earlier results of Sczekely [28], who considered the case where the random variables A_n are positive and $B_n = 1$ for all n.

In Hilbert and Banach spaces this difference equation has been studied amongst others by Bosq [4, 5, 3]. The results obtained are generalizations of the corresponding results in finite dimensional Euclidean space.

In recent years, there has been increasing interest in the theory of cylindrical processes, which play an important role in the theory of stochastic partial differential equations as a model for random noise.

In this theory, one encounters equations of the form

$$Lu(t,x) = \sigma(u(t,x))\,\dot{W}\,(t,x) + b(u(t,x)), \tag{2}$$

where $t \geq 0$, $x \in \mathbb{R}^d$, $d \geq 1$, L is a partial differential operator, b and σ are real valued functions and $\dot{W}$ represents noise. One possible model for the noise is a cylindrical process as in equation (4.1) in Dalang and Quer-Sardanyons [10].

If we set $L = \dfrac{\partial}{\partial t} - \triangle$, $\sigma \equiv 1$ and $b \equiv 0$ then (2) takes the form

$$\frac{\partial u(t,x)}{\partial t} = \triangle u(t,x) + \dot{W}\,(t,x). \tag{3}$$

If we now set $Y(t) := u(t,x)$ and $W(t) := W(t,x)$ independent of x then we get

$$dY(t) = \triangle Y(t)dt + \dot{W}\,(t)dt. \tag{4}$$

If we now replace $\dot{W}(t)\,dt$ by $dM(t)$ for a cylindrical process M in the sense of Applebaum and Riedle [1] and $\triangle$ by a linear operator T, then (4) becomes

$$dY(t) = TY(t)dt + dM(t). \tag{5}$$

The equation (5) is studied in section 5 of [1]. Other arguments for example in Kosmala [23] also lead to a study of (5).

In analogy to Wolfe [35, Section 3, Page 305], (see equations (3.2) and (3.3)) the difference equation

$$X_n = AX_{n-1} + Z_n, \tag{6}$$

where $\{Z_n\}$ is a sequence of identically distributed and weakly independent cylindrical random variables and A is a suitable linear operator, may be considered as a discrete analogue of (5). Apart from this, (6) is a very interesting mathematical object in its own right.

The object we study in the present note is (6). Our objective is to prove the existence and uniqueness of a cylindrical weakly stationary solution of (6) under fairly weak conditions on A, when $\{Z_k\}_{k\in\mathbb{Z}}$ is weakly independent and identically distributed and to give some properties of the solution. These results are presented in the remainder of this note as follows: In Section 1 we introduce most of the basic notation and definitions which we shall use. Our main theorems are presented in Section 2 and Section 3. In Section 2 we prove the existence and uniqueness of a cylindrical weakly stationary solution to the difference equation (6) with arbitrary weakly independent and identically distributed cylindrical driver. These results are a generalization of the results obtained in Shu [31] when the cylindrical driver is Gaussian. Properties of the solution are then presented in Section 3, followed by a conclusion.

1 Prerequisites

In what follows, U will denote a separable Banach space. Its dual is denoted by U^* and the dual pairing by $\langle\cdot,\cdot\rangle$. $\mathcal{B}(U)$ will denote the Borel σ-algebra on U relative to the norm topology and $\mathcal{L}(U)$ will denote the set of bounded linear operators from U to U. $\mathbb{N}$ will denote the set of natural numbers, $\mathbb{N} = \{1,\ 2,\ldots\}$ and $\mathbb{R}$ will denote the set of real numbers. If $n \in \mathbb{N}$, $a_1,\ldots,a_n \in U^*$ and $u \in U$, we define $\pi_{a_1,\ldots,a_n}(u) := (\,\langle u,a_1\rangle,\ldots,\langle u,a_n\rangle\,)$. For $\Lambda \subseteq U^*$ we define $\mathcal{Z}(\Lambda) := \{\pi^{-1}_{a_1,\ldots,a_n}(B) : B \in \mathcal{B}(\mathbb{R}^n), n \in \mathbb{N}, a_1,\ldots,a_n \in \Lambda\}$, $\mathcal{Z}(U^*)$ is an algebra with $\sigma(\mathcal{Z}(U^*)) \subseteq \mathcal{B}(U)$, where $\sigma(\mathcal{Z}(U^*))$ denotes the σ-algebra generated by $\mathcal{Z}(U^*)$ and the inclusion may be strict. Since U is assumed to be seperable, by Ito [20, Proposition 2.1], $\sigma(\mathcal{Z}(U^*)) = \mathcal{B}(U)$.

Definition 1. A cylindrical probability measure on $\mathcal{Z}(U^*)$ is a map $\mu : \mathcal{Z}(U^*) \to \mathbb{R}^+$ such that $\mu(U) = 1$ and $\mu|_{\mathcal{Z}(\Lambda)}$ is a probability, measure for each finite subset Λ of U^*.

Remark 1. We shall henceforth simply talk of a cylindrical measure and thereby mean a cylindrical probability measure.

Definition 2. Let μ be a cylindrical measure, then the map $\hat{\mu} : U^* \to \mathbb{C}$; $a \mapsto \hat{\mu}(a) := \int\limits_U e^{i\langle x,a\rangle}\mu(dx),\ a \in U^*$ is called the characteristic functional of μ.

Remark 2. In definition 2 we use the Lebesgue integral on $(U, \mathcal{Z}(\{a\}), \mu)$.

Definition 3. Let X be a metric space. A mapping $\varphi : \Omega \to X$ is called separably-valued if $\varphi(\Omega)$ is a separable subset of X.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We write $L_0(\Omega, \mathcal{F}, \mathbb{P})$ for the space of equivalence classes (modulo $\mathbb{P}$) of seperably valued real valued random variables.

Definition 4. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A linear transformation $X : U^* \to L_0(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a cylindrical random variable in U.

Remark 3. If X is a cylindrical random variable in U then we shall write Xa or $X(a)$ for the value of X at $a \in U^*$, whichever is more convenient in the given context.

Definition 5. The characteristic functional of a cylindrical random variable X in U is the map $\varphi_X : U^* \to \mathbb{C}$, $a \mapsto \varphi_X(a) := \mathbb{E}(e^{iXa})$.

Remark 4. The characteristic functional of a cylindrical random variable X in U is positive definite, continuous on finite dimensional subspaces of U^* and $\varphi_X(0) = 1$. By Vakhania and Tarieladze [33, Chapter VI Proposition 3.2] there exists a cylindrical measure μ on $\mathcal{Z}(U^*)$ whose characteristic functional coincides with that of X. We call μ the cylindrical distribution of X. Conversely by [33, Chapter VI Section 3.2, Pages 395-396], given a cylindrical measure μ on $\mathcal{Z}(U^*)$ there exists a cylindrical random variable X in U whose cylindrical distribution is μ. μ and X are related through $\mu_{\pi_{a_1,\ldots,a_n}} = \mathbb{P}_{Xa_1,\ldots,Xa_n}$, where for $n \in \mathbb{N},\ B \in \mathcal{B}(\mathbb{R}^n)$ and $a_1, \ldots, a_n \in U^*$,

$$\mu_{\pi_{a_1,\ldots,a_n}}(B) := \mu(\pi^{-1}_{a_1,\ldots,a_n}(B))$$

and

$$\mathbb{P}_{Xa_1,\ldots,Xa_n}(B) := \mathbb{P}((Xa_1, \ldots, Xa_n) \in B).$$

Definition 6. Cylindrical random variables X and Y in U with a common cylindrical distribution are said to be identically distributed.

For cylindrical(ordinary) random variables X and Y, we write $X \overset{d}{=} Y$ if X and Y have the same cylindrical(ordinary) distribution. By ordinary random variable we mean a random element in $\mathbb{R}^n,\ n \geq 1$ and by ordinary distribution we mean a probability distribution on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)),\ n \geq 1$. It is clear that if two cylindrical(ordinary) random variables have the same characteristic functional(function) then they are identically distributed. The converse of this statement is also true.

Definition 7. Let $\{Z_n\}_{n\in\mathbb{Z}}$ be cylindrical random variables in U. If for all $n \in \mathbb{N}$, distinct indices $i_1, \ldots, i_n \in \mathbb{Z}$ and $a_1, \ldots, a_n \in U^*$, the random variables $Z_{i_1}a_1, \ldots, Z_{i_n}a_n$ are independent then we say $\{Z_n\}_{n\in\mathbb{Z}}$ is weakly independent.

Definition 8. A sequence $\{X_n\}_{n\in I}$, $I = \mathbb{Z}$ *or* $\mathbb{N}$ of U valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is said to be weakly stationary if

i. for all $n \in I$, $\mathbb{E}\|X_n\|^2 < \infty$.

ii. $\mathbb{E}(X_n) = \mu$, independent of $n,\ \mu \in U$.

iii. For all $n,\ m,\ l \in I,\ l \geq 0$ and $x^*, y^* \in U^*$, $\mathbb{E}(\,\langle X_{n+l} - \mu, x^*\rangle\langle X_{m+l} - \mu, y^*\rangle\,) = \mathbb{E}(\,\langle X_n - \mu, x^*\rangle\langle X_m - \mu, y^*\rangle\,)$.

If X is a cylindrical random variable in U and $a_1, \ldots, a_n \in U^*,\ n \in \mathbb{N}$, we write $X(a_1, \ldots, a_n)$ for the $\mathbb{R}^n$-valued random vector $(\,Xa_1, \ldots, Xa_n\,)$.

Definition 9. A sequence $\{X_n\}_{n\in I}$ of cylindrical random variables in U is cylindrical weakly stationary if for all $k \geq 1$ and $a_1, \ldots, a_k \in U^*$, the process $\{X_n(a_1, \ldots, a_k)\}_{n\in I}$ is weakly stationary in $\mathbb{R}^k$.

We shall define

$$L^2(\Omega, \mathcal{F}, \mathbb{P}) := \{\text{random elements } \varphi \text{ in } \mathbb{R} \text{ defined on } \Omega : \mathbb{E}|\varphi|^2 < \infty\}$$

for a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and denote by $\mathfrak{C}_2$ the collection of all cylindrical random variables X in U for which:

(i) $X(U^*) := \{Xa : a \in U^*\}$ is contained in a closed and separable subspace of $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

(ii) the map $X : U^* \to L^2(\Omega, \mathcal{F}, \mathbb{P}),\ a \mapsto Xa$ is continuous.

By Mamporia [24, page 602], if we define

$$\|X\|_{\mathfrak{C}_2} := \sup\{\|Xa\|_{L^2(\Omega,\mathcal{F},\mathbb{P})} : a \in U^*, \|a\|_{U^*} \leq 1\}$$

for $X \in \mathfrak{C}_2$ then $\|\cdot\|_{\mathfrak{C}_2}$ is a norm relative to which $\mathfrak{C}_2$ is a Banach space. In the rest of what follows, we shall be concerned with norms on different spaces. For simplicity, we shall omit the subscripts when there is no risk of confusion; the relevant norm will be clear from the context.

2 Existence and uniqueness of a cylindrical weakly stationary solution

The following Lemma will be used repeatedly in the sequel.

Lemma 1. Let A be a bounded linear operator on U and $\{Z_n\}_{n\in\mathbb{Z}}$ be a sequence of identically distributed cylindrical random variables in $\mathfrak{C}_2$ which are weakly independent then:

(a) (i) $\sum_{m=1}^{\infty} Z_m((A^*)^{m-1}(\cdot))$ converges in $\mathfrak{C}_2$ if and only if $\sum_{m=0}^{\infty} Z_{n-m}((A^*)^m(\cdot))$ converges in $\mathfrak{C}_2$ for each $n \in \mathbb{Z}$.

(ii) for each $a \in U^*$, $\sum_{m=1}^{\infty} Z_m((A^*)^{m-1}a)$ converges in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ if and only if $\sum_{m=0}^{\infty} Z_{n-m}((A^*)^m a)$ converges in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ for all $n \in \mathbb{Z}$.

(b) If $\sum_{m=0}^{\infty} \|A^m\| < \infty$, then,

(i) $\sum_{m=1}^{\infty} Z_m((A^*)^{m-1}(\cdot))$ converges in $\mathfrak{C}_2$

(ii) $\sum_{m=1}^{\infty} Z_m((A^*)^{m-1}(a))$ converges in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ for each $a \in U^*$

(iii) for each $a \in U^*$, $\sum_{m=1}^{\infty} Z_m((A^*)^{m-1}a)$ converges absolutely almost surely.

(iv) for each $a \in U^*$ and $n \in \mathbb{Z}$, $\sum_{m=0}^{\infty} Z_{n-m}((A^*)^m a)$ converges absolutely almost surely.

Proof. To prove $(a)(i)$ and $(a)(ii)$, it is enough to observe that for every $l \geq k \geq 1$, $n \in \mathbb{Z}$ and $a \in U^*$, the following identities hold:

$$\sum_{m=k}^{l+k-1} Z_m((A^*)^{m-1}(\cdot)) \stackrel{d}{=} \sum_{m=k-1}^{l+k-2} Z_{n-m}((A^*)^m(\cdot)) \tag{7}$$

and

$$\sum_{m=k}^{l+k-1} Z_m((A^*)^{m-1}a) \stackrel{d}{=} \sum_{m=k-1}^{l+k-2} Z_{n-m}((A^*)^m a)). \tag{8}$$

The completeness of $\mathfrak{C}_2$ and (7) allow us to complete the proof of $(a)(i)$ while the completeness of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ and (8) allow us to complete the proof of $(a)(ii)$.
(b) (i) We note first that by Shu, [31, Lemma 2], for each fixed $a \in U^*$, $\{Z_k a\}_{k\in\mathbb{Z}}$ is a sequence of i.i.d. real random variables since $\{Z_k\}$ is a sequence of identically distributed and weakly independent cylindrical random variables. Therefore

$$Z_k((A^*)^m a) \stackrel{d}{=} Z_1((A^*)^m a) \tag{9}$$

for any $m \geq 0,\ k \in \mathbb{Z}$ and $a \in U^*$ and hence

$$\begin{aligned} \mathbb{E}|Z_k((A^*)^{k-1}a)|^2 &= \|Z_k((A^*)^{k-1}a)\|^2_{L^2(\Omega,\mathcal{F},\mathbb{P})} = \|Z_1((A^*)^{k-1}a)\|^2_{L^2(\Omega,\mathcal{F},\mathbb{P})} \\ &\leq \|Z_1\|^2_{\mathfrak{C}_2}\|a\|^2_{U^*}\|(A^*)^{k-1}\|^2_{\mathcal{L}(U^*)}. \end{aligned} \tag{10}$$

Let $S_n := \sum_{k=1}^{n} Z_k((A^*)^{k-1}(\cdot))$, then, $\|S_n - S_m\| = 0$ for $n = m$ and for $m > n$ it holds that

$$\begin{aligned} \|S_m - S_n\| &= \|\sum_{k=n+1}^{m} Z_k((A^*)^{k-1}(\cdot))\| \leq \sum_{k=n+1}^{m} \|Z_k((A^*)^{k-1}(\cdot))\| \\ &= \sum_{k=n+1}^{m} \|Z_1((A^*)^{k-1}(\cdot))\| \leq \|Z_1\|_{\mathfrak{C}_2} \sum_{k=n+1}^{m} \|(A^*)^{k-1}\|_{\mathcal{L}(U^*)}, \end{aligned}$$

where the last inequality follows from (10). We have that $\|Z_1\| < \infty$ since $Z_1 \in \mathfrak{C}_2$. Since $\sum_{k=1}^{\infty} \|A^{k-1}\| < \infty$, it follows that $\{S_n\}$ is a Cauchy sequence in $\mathfrak{C}_2$ and hence converges since $\mathfrak{C}_2$ is a Banach space.

(ii) This follows from the inequality,

$$\|S_m a - S_n a\|_{L_2(\Omega,\mathcal{F},\mathbb{P})} = \|(S_m - S_n)a\|_{L_2(\Omega,\mathcal{F},\mathbb{P})} \leq \|Z_1\|_{\mathfrak{C}_2}\|a\| \sum_{k=n+1}^{m} \|(A^*)^{k-1}\|$$

for $m > n$.

(iii) By the Hölder inequality,

$$\begin{aligned} \mathbb{E}|Z_k((A^*)^{k-1}a)| \leq & \quad \|Z_k((A^*)^{k-1}a)\|_{L^2(\Omega,\mathcal{F},\mathbb{P})} \\ \leq & \quad \|Z_1\|_{\mathfrak{C}_2}\|(A^*)^{k-1}\|_{\mathcal{L}(U^*)}\|a\|_{U^*}. \end{aligned} \tag{11}$$

Hence $\sum_{k=1}^{\infty} \mathbb{E}|Z_k((A^*)^{k-1}a)| \leq \|Z_1\|_{\mathfrak{C}_2}\|a\|_{U^*} \sum_{k=1}^{\infty} \|A^{k-1}\|_{\mathcal{L}(U^*)} < \infty$ and if we write $V(\xi)$ for the variance of the random variable ξ, then since $(\mathbb{E}(\xi))^2 \leq \mathbb{E}(\xi^2)$ by Jensen's inequality, it follows that

$$\begin{aligned} \sum_{k=1}^{\infty} V(|Z_k((A^*)^{k-1}a)|) & \leq 2\sum_{k=1}^{\infty} \mathbb{E}|Z_k((A^*)^{k-1}a)|^2 \\ & \leq 2\|Z_1\|^2_{\mathfrak{C}_2}\|a\|^2_{U^*} \sum_{k=1}^{\infty} \|A^{k-1}\|^2_{\mathcal{L}(U^*)}. \end{aligned}$$

Since $\sum_{k=1}^{\infty} \|(A^*)^{k-1}\| < \infty$, there exists N such that for all $k \geq N$, $\|(A^*)^{k-1}\|^2 \leq \|(A^*)^{k-1}\|$. Therefore $2\|Z_1\|^2_{\mathfrak{C}_2}\|a\|^2_{U^*} \sum_{k=1}^{\infty} \|A^{k-1}\|^2_{\mathcal{L}(U^*)} < \infty$. It then follows from Kolmogorov's two series Theorem, [29, Theorem 2, Chapter IV §2] that the series $\sum_{k=1}^{\infty} Z_k((A^*)^{k-1}a)$ converges absolutely with probability 1.

(iv) The proof of *(iv)* is similar to that of *(iii)*, if we take note of (9) and use the fact that $Z_k((A^*)^{k-1}a) \stackrel{d}{=} Z_{n-(k-1)}((A^*)^{k-1}a)$ for all $n \in \mathbb{Z}$, $a \in U^*$ and $k \geq 1$. *QED*

Let us recall from Applebaum and Riedle, [1, page 720] that if A is a bounded linear operator on U and X is a cylindrical random variable in U then AX defined by $a \mapsto (AX)a := X(A^*a), a \in U^*$ is a cylindrical random variable in U.

Definition 10. Let $\{Z_n\}_{n\in\mathbb{Z}}$ be a sequence of cylindrical random variables in U. A solution to the equation

$$X_n = AX_{n-1} + Z_n, \ n \in \mathbb{Z},$$

is understood to be a sequence $\{X_n\}_{n\in\mathbb{Z}}$ of cylindrical random variables in U, such that for all $a \in U^*$ and $n \in \mathbb{Z}$,

$$\mathbb{P}(X_n a = (AX_{n-1})a + Z_n a) = 1.$$

The following is the main Theorem of this section:

Theorem 1. Let A be a bounded linear operator on U such that $\sum_{k=0}^{\infty} \|A^k\| < \infty$. Let $\{Z_n\}_{n\in\mathbb{Z}}$ be identically distributed and weakly independent cylindrical random variables in U. For all $n \in \mathbb{Z},\ l \in \mathbb{N}$, let $\{Y_{n,l}\}$ be the cylindrical process in U defined by

$$Y_{n,l} := \sum_{k=0}^{l} Z_{n-k}((A^*)^k(\cdot)). \tag{12}$$

Then

(i) For each $n \in \mathbb{Z}$, the sequence $\{Y_{n,l}\}$ converges in $\mathfrak{C}_2$ as l tends to ∞. We denote its limit by Y_n.

(ii) For each $a \in U^*$, the sequence $\{Y_n a\}$ converges almost surely.

(iii) $\{Y_n\}$ is cylindrical weakly stationary.

(iv) $\{Y_n\}$ is the unique cylindrical weakly stationary process for which,

   (a) $Y_n \in \mathfrak{C}_2$ for all $n \in \mathbb{Z}$ and
   (b) for all $a \in U^*$, $Y_n a = (AY_{n-1})a + Z_n a$ almost surely.

(v) If $\{X_n\}_{n\in\mathbb{Z}}$ is any sequence of cylindrical random variables in U which satisfies $X_n = AX_{n-1} + Z_n$ for all $n \in \mathbb{Z}$, then

   (a) If $X_0 \stackrel{d}{=} Y_0$ then $X_n \stackrel{d}{=} Y_n$ for all $n \geq 1$.
   (b) If $X_0 = Y_0$ then $X_n = Y_n$ for all $n \geq 1$.
   (c) If $X_0 \in \mathfrak{C}_2$ then
$$\lim_{n\to\infty} \|X_n - Y_n\|_{\mathfrak{C}_2} = 0.$$

Proof. (i) follows from Lemma 1 $(a)(i)$ and $(b)(i)$.
(ii) follows from Lemma 1 $(b)(iv)$.
(iii) The idea of the proof of (iii) is similar to that of the proof of [31, Theorem 2 (iii)]. Let $n \in \mathbb{Z}$. By Lemma 1 (a) (ii) and (b) (ii), $\{Y_n a\}$ converges in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ for all $a \in U^*$, hence for all $a \in U^*$, $\mathbb{E}|Y_n a|^2 < \infty$. Therefore for

$m \geq 1$ and $a_1, \ldots, a_m \in U^*$, $\mathbb{E}\|Y_n(a_1, \ldots, a_m)\|^2 = \sum_{j=1}^{m} \mathbb{E}|Y_n a_j|^2 < \infty$. This proves (i) of Definition 8.

We now prove (ii) of that definition. By (11) and the fact that the cylindrical random variables are identically distributed, we have that

$$\sum_{k=0}^{\infty} \mathbb{E}|Z_{n-k}(A^*)^k a_j| \leq \|Z_1\|_{\mathfrak{C}_2} \|a_j\|_{U^*} \sum_{k=0}^{\infty} \|(A^*)^k\|_{\mathcal{L}(U^*)} < \infty$$

for all $j = 1, \ldots, m$, hence since $\sum_{k=0}^{\infty} Z_{n-k}((A^*)^k a_j)$ converges absolutely almost surely by Lemma 1 $(b)(iv)$, it follows from the monotone convergence theorem that

$$\mathbb{E}|\sum_{k=0}^{\infty} Z_{n-k}(A^*)^k a_j| \leq \mathbb{E} \sum_{k=0}^{\infty} |Z_{n-k}(A^*)^k a_j| = \sum_{k=0}^{\infty} \mathbb{E}|Z_{n-k}(A^*)^k a_j| < \infty$$

for all $j = 1, \ldots, m$.

Since $\sum_{k=0}^{\infty} Z_{n-k}((A^*)^k a_j)$ converges absolutely almost surely it converges almost surely, hence again using Lebesgues Theorem, we can interchange the sum and the expectation to get $\mathbb{E}Y_n a_j = \sum_{k=0}^{\infty} \mathbb{E}Z_{n-k}((A^*)^k a_j)$ for all $j = 1, \ldots, m$.

$$\begin{aligned} \text{By (9), } \mathbb{E}Y_n(a_1, \cdots, a_m) &= \left(\sum_{k=0}^{\infty} \mathbb{E}(Z_{n-k}(A^*)^k a_1), \ldots, \sum_{k=0}^{\infty} \mathbb{E}(Z_{n-k}(A^*)^k a_m) \right) \\ &= \left(\sum_{k=0}^{\infty} \mathbb{E}(Z_1(A^*)^k a_1), \ldots, \sum_{k=0}^{\infty} \mathbb{E}(Z_1(A^*)^k a_m) \right) \end{aligned}$$

for all n. Therefore $\mathbb{E}Y_n(a_1, \ldots, a_m)$ does not depend on n.

To prove the third property in Definition 8, we assume (to simplify the presentation), that $\mathbb{E}Y_n(a_1, \ldots, a_m) = 0$ for all n. If $m \geq 1$, $a_1, \ldots, a_m \in U^*$,

$s,\ t,\ h \in \mathbb{Z},\ h \geq 0,\ \alpha,\ \beta \in \mathbb{R}^m,\ \alpha := (\alpha_1, \ldots, \alpha_m)$ and $\beta := (\beta_1, \ldots, \beta_m)$, then

$$\begin{aligned}
&\mathbb{E}\langle Y_{s+h}(a_1, \ldots, a_m), \alpha\rangle\langle Y_{t+h}(a_1, \ldots, a_m), \beta\rangle = \\
= \;& \mathbb{E}[\sum_{j=1}^{m} \alpha_j Y_{s+h} a_j][\sum_{l=1}^{m} \beta_l Y_{t+h} a_l] \\
= \;& \sum_{j,l=1}^{m} \alpha_j \beta_l \mathbb{E}(Y_{s+h} a_j)(Y_{t+h} a_l) \\
= \;& \sum_{j,l=1}^{m} \alpha_j \beta_l \sum_{n,k=0}^{\infty} \mathbb{E} Z_{s+h-k}((A^*)^k a_j) Z_{t+h-n}((A^*)^n a_l),
\end{aligned}$$

where the last equality is justified by the fact that the series $Y_{s+h}a_j$ and $Y_{t+h}a_l$ are almost surely absolutely convergent by Lemma 1 (b) (iv), convergent in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ by Lemma 1 (b) (ii), Lemma 1 (a) (ii) and an application of Lebesgues theorem.

To facilitate presentation, let us define $S(s,t,j,h,l)_= := \{n, k \geq 0 : s+h-k = t+h-n, (A^*)^k a_j = (A^*)^n a_l\}$
$S(s,t,j,h,l)_{\neq} := \{n, k \geq 0 : s+h-k = t+h-n, (A^*)^k a_j \neq (A^*)^n a_l\}$ and $S(s,t,h) := \{n, k \geq 0 : s+h-k \neq t+h-n\}$

We have that

$$\sum_{j,l=1}^{m} \alpha_j \beta_l \sum_{n,k=0}^{\infty} \mathbb{E} Z_{s+h-k}((A^*)^k a_j) Z_{t+h-n}((A^*)^n a_l)$$

$$= \sum_{j,l=1}^{m} \alpha_j \beta_l \sum_{\{n,k \in S(s,t,j,h,l)_=\}} \mathbb{E} Z_{s+h-k}((A^*)^k a_j) Z_{t+h-n}((A^*)^n a_l) \tag{13}$$

$$+ \sum_{j,l=1}^{m} \alpha_j \beta_l \sum_{\{n,k \in S(s,t,j,h,l)_{\neq}\}} \mathbb{E} Z_{s+h-k}((A^*)^k a_j) Z_{t+h-n}((A^*)^n a_l) \tag{14}$$

$$+ \sum_{j,l=1}^{m} \alpha_j \beta_l \sum_{\{n,k \in S(s,t,h)\}} \mathbb{E} Z_{s+h-k}((A^*)^k a_j) Z_{t+h-n}((A^*)^n a_l) \tag{15}$$

Since the sequence $\{Z_n\}$ is weakly independent and identically distributed, we have that

$$\begin{aligned}
(15) &= \sum_{j,l=1}^{m} \alpha_j \beta_l \sum_{\{n,k \in S(s,t,h)\}} \mathbb{E} Z_{s+h-k}((A^*)^k a_j) \mathbb{E} Z_{t+h-n}((A^*)^n a_l) \\
&= \sum_{j,l=1}^{m} \alpha_j \beta_l \sum_{\{n,k \in S(s,t,h)\}} \mathbb{E} Z_{s-k}((A^*)^k a_j) \mathbb{E} Z_{t-n}((A^*)^n a_l) \\
&= \sum_{j,l=1}^{m} \alpha_j \beta_l \sum_{\{n,k \in S(s,t,h)\}} \mathbb{E} Z_{s-k}((A^*)^k a_j) Z_{t-n}((A^*)^n a_l). \qquad (16)
\end{aligned}$$

Since $\{Z_n\}$ is identically distributed, we have that

$$\begin{aligned}
(13) &= \sum_{j,l=1}^{m} \alpha_j \beta_l \sum_{\{n,k \in S(s,t,j,h,l)_=\}} \mathbb{E}(Z_{s+h-k}((A^*)^k a_j))^2 \\
&= \sum_{j,l=1}^{m} \alpha_j \beta_l \sum_{\{n,k \in S(s,t,j,h,l)_=\}} \mathbb{E}(Z_{s-k}((A^*)^k a_j))^2 \\
&= \sum_{j,l=1}^{m} \alpha_j \beta_l \sum_{\{n,k \in S(s,t,j,h,l)_=\}} \mathbb{E}(Z_{s-k}((A^*)^k a_j) Z_{t-n}((A^*)^k a_j)). \qquad (17)
\end{aligned}$$

Since $\{Z_n\}$ is a sequence of identically distributed, weakly independent random variables it follows from [31, Lemma 2] that for $s+h-k = t+h-n$,

$$(Z_{s+h-k}((A^*)^k a_j) \quad Z_{t+h-n}((A^*)^n a_l)) \stackrel{d}{=} (Z_{s-k}((A^*)^k a_j) \quad Z_{t-n}((A^*)^n a_l)).$$

Therefore

$$(14) = \sum_{j,l=1}^{m} \alpha_j \beta_l \sum_{\{n,k \in S(s,t,j,h,l)_{\neq}\}} \mathbb{E} Z_{s-k}((A^*)^k a_j) Z_{t-n}((A^*)^n a_l). \qquad (18)$$

From (16), (17) and (18) it follows that

$$\begin{aligned}
& \mathbb{E}\langle Y_{s+h}(a_1, \ldots, a_m), \alpha\rangle \langle Y_{t+h}(a_1, \ldots, a_m), \beta\rangle \\
&= \sum_{j,l=1}^{m} \alpha_j \beta_l \sum_{n,k=0}^{\infty} \mathbb{E} Z_{s+h-k}((A^*)^k a_j) Z_{t+h-n}((A^*)^n a_l) \\
&= \sum_{j,l=1}^{m} \alpha_j \beta_l \sum_{n,k=0}^{\infty} \mathbb{E} Z_{s-k}((A^*)^k a_j) Z_{t-n}((A^*)^n a_l) \\
&= \mathbb{E}\langle Y_s(a_1, \ldots, a_m), \alpha\rangle \langle Y_t(a_1, \ldots, a_m), \beta\rangle.
\end{aligned}$$

This proves the third property in Definition 8 and hence (iii).

The proof of (iv) is similar to the proof of [31, Theorem 2(v)] with minor modifications.

For the proof of $(v)(a)$, see [31, Theorem 2(ii)(2)].

We now proceed to prove $(v)(b)$ by induction. We have that for $a \in U^*$, $X_0 a = Y_0 a = \sum_{k=0}^{\infty} Z_{-k}((A^*)^k a)$ by assumption, therefore

$$\begin{aligned} X_1 a = \quad & AX_0 a + Z_1 a = A\sum_{k=0}^{\infty} Z_{-k}((A^*)^k a) + Z_1 a = \sum_{k=0}^{\infty} Z_{-k}((A^*)^{k+1} a) + Z_1 a \\ = \quad & \sum_{k=1}^{\infty} Z_{-(k-1)}((A^*)^k a) + Z_1 a = \sum_{k=0}^{\infty} Z_{1-k}((A^*)^k a) = Y_1 a. \end{aligned}$$

Suppose now that $X_{n-1} = Y_{n-1}$ for some $n \geq 2$, then we have $X_n a = AX_{n-1}a + Z_n a = AY_{n-1}a + Z_n a = Y_n a$.

For the proof of $(v)(c)$, see [31, Theorem 2(ii)(3)].

QED

3 Properties of solutions

3.1 Decomposability

Let $X : U^* \to L_0(\Omega, \mathcal{F}, \mathbb{P})$ be a cylindrical random variable in U. If μ is the cylindrical distribution of X, then by Remark 4, for $a \in U^*$, $\hat{\mu}(a) = \mathbb{E}(e^{iXa})$, where $\hat{\mu}$ denotes the characteristic functional of μ. Therefore $\mathbb{E}(e^{iXa}) = \int_U e^{i\langle x,a\rangle}\mu(dx)$.

We then have that for a bounded linear operator $A : U \to U$,

$$\begin{aligned} \mathbb{E}(e^{i(AX)a}) = \mathbb{E}(e^{iX(A^*a)}) \quad &= \quad \int_U e^{i\langle x,A^*a\rangle}\mu(dx) = \int_U e^{i\langle Ax,a\rangle}\mu(dx) \\ &= \quad \int_U e^{i\langle x,a\rangle}(A\mu)(dx), \end{aligned}$$

where $A\mu$ is defined by $(A\mu)(B) := \mu(A^{-1}(B))$ for every $B \in \mathcal{B}(U)$. Therefore, if μ is the cylindrical distribution of X, then $A\mu$ is the cylindrical distribution of AX. We note this as a Lemma:

Lemma 2. Let $X : U^* \to L_0(\Omega, \mathcal{F}, \mathbb{P})$ be a cylindrical random variable in U and let $A : U \to U$ be a bounded linear operator. If μ is the cylindrical distribution of X, then $A\mu$ is the cylindrical distribution of AX.

Definition 11. Let μ and ν be cylindrical probability measures on $\mathcal{Z}(U^*)$, then the convolution of μ and ν is the cylindrical probability measure defined by $\mu * \nu(B) := \int\limits_U \int\limits_U 1_B(x+y)\mu(dx)\nu(dy),\ B \in \mathcal{Z}(U^*)$.

Remark 5. By computing characteristic functionals one can easily check that if X and Y are cylindrical random variables which are weakly independent with cylindrical distributions μ and ν respectively, then the cylindrical distribution of $X+Y$ is $\mu * \nu$.

In analogy to Urbanik [32], we give the following definition:

Definition 12. Let U be a Banach space, $A : U \to U$ be a bounded linear operator on U and μ be a cylindrical measure on $\mathcal{Z}(U^*)$. We say that μ is A-decomposable if there exists a cylindrical measure ν such that $\mu = (A\mu) * \nu$.

Lemma 3. Let A be a bounded linear operator on U, X_0 a cylindrical random variable in U and $\{X_n\}_{n\geq 1}$ be a sequence of cylindrical random variables in U such that $X_n = AX_{n-1} + Z_n$, where $\{Z_n\}_{n\in\mathbb{Z}}$ is a sequence of weakly independent and identically distributed cylindrical random variables in U. Suppose that $\{Z_n\}_{n\geq 1}$ is weakly independent of X_0 and that there exists a cylindrical random variable X in U such that $\{X_n a\}$ converges to Xa in distribution for all a in U^* as n tends to infinity, then:

(i) If Z is a cylindrical random variable with the same distribution as Z_1 which is weakly independent of X, then $X \overset{d}{=} AX + Z$.

(ii) If μ denotes the cylindrical distribution of X, then μ is A-decomposable.

Proof. We show first that $X \overset{d}{=} AX + Z$ for any cylindrical random variable Z in U with the same cylindrical distribution as Z_1, which is weakly independent of X. Since Z_n is weakly independent of X_0 for all $n \geq 1$ and the sequence $\{Z_n\}_{n\in\mathbb{Z}}$ is a sequence of weakly independent cylindrical random variables, it follows that X_{n-1} is weakly independent of Z_n for all $n \geq 1$, i.e. $X_{n-1}a$ is independent of $Z_n b$ for all $a,\ b \in U^*$ and $n \geq 1$. Therefore

$$\begin{aligned} \mathbb{E}(e^{i\lambda X_n a}) &= \mathbb{E}(\exp\{i\lambda(X_{n-1}(A^*a) + Z_n a)\}) = \mathbb{E}(e^{i\lambda X_{n-1}(A^*a)})\mathbb{E}(e^{i\lambda Z_n a}) \\ &= \mathbb{E}(e^{i\lambda X_{n-1}(A^*a)})\mathbb{E}(e^{i\lambda Z_1 a}). \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{E}(e^{i\lambda X a}) &= \lim_{n\to\infty} \mathbb{E}(e^{i\lambda X_n a}) = \lim_{n\to\infty} \mathbb{E}(\exp\{i\lambda(X_{n-1}(A^*a) + Z_n a)\}) \\ &= \lim_{n\to\infty} \mathbb{E}(e^{i\lambda X_{n-1}(A^*a)})\mathbb{E}(e^{i\lambda Z_1 a}) = \mathbb{E}(e^{i\lambda X(A^*a)})\mathbb{E}(e^{i\lambda Z_1 a}), \end{aligned}$$

since $\{X_n a\}$ converges to Xa in distribution for all a in U^* and $Z_n a$ has the same distribution as $Z_1 a$ for all $n \in \mathbb{Z}$ and $a \in U^*$. If Z is a cylindrical random variable with the same distribution as Z_1 and is weakly independent of X, then $\mathbb{E}(e^{i\lambda Xa}) = \mathbb{E}(e^{i\lambda X(A^*a)})\mathbb{E}(e^{i\lambda Za})$ and hence X and $AX + Z$ have the same characteristic functional. Therefore $X \overset{d}{=} AX + Z$.

For the characteristic functionals $\hat{\mu}$ of the cylindrical distribution μ of X, $\hat{\nu}$ of the cylindrical distribution ν of Z and $a \in U^*$ chosen arbitrarily, we have by Lemma 2 that

$$\begin{aligned} \hat{\mu}(a) &= \mathbb{E}(e^{iXa}) = \mathbb{E}(e^{i((AX)a+Za)}) = \mathbb{E}(e^{i((AX)a)})\mathbb{E}(e^{i(Za)}) \\ &= \widehat{(A\mu)}(a)\hat{\nu}(a). \end{aligned}$$

Also since for cylindrical measures μ_1 and μ_2 on $\mathcal{Z}(U^*)$ we have $\widehat{\mu_1 * \mu_2} = \widehat{\mu_1}\widehat{\mu_2}$ (see Applebaum and Riedle [1, page 699]), we have $\widehat{(A\mu)}(a)\hat{\nu}(a) = \widehat{(A\mu) * \nu}(a)$. Therefore $\hat{\mu}(a) = \widehat{(A\mu) * \nu}(a)$ for all $a \in U^*$ and hence by [33, Chapter VI Proposition 3.2(a), page 392] , $\mu = (A\mu) * \nu$. QED

Lemma 4. Let $\{Z_n\}_{n\in\mathbb{Z}}$ be a sequence of identically distributed and weakly independent cylindrical random variables in U, $Z_n \in \mathfrak{C}_2$ for all n and A be a bounded linear operator on U. Further, let $\{X_n\}_{n\geq 0}$ be a sequence of cylindrical random variables in U such that $X_0 \in \mathfrak{C}_2$, $X_n = AX_{n-1} + Z_n$, $n \geq 1$ and X_0 is weakly independent of $\{Z_n\}_{n\geq 1}$. Let $\sum_{k=0}^{\infty} \|A^k\| < \infty$, and $X := \sum_{k=1}^{\infty} Z_k((A^*)^{k-1}(\cdot))$, where the series converges in $\mathfrak{C}_2$ then $\{X_n a\}$ converges in distribution to Xa for all $a \in U^*$.

Proof. Since $\{Z_k\}$ is a sequence of weakly independent and identically distributed cylindrical random variables, we have that for $a \in U^*$ and $n \geq 1$,

$$\sum_{k=1}^{n} Z_k((A^*)^{n-k}a) \overset{d}{=} \sum_{k=1}^{n} Z_k((A^*)^{k-1}a).$$

Since X_0 is weakly independent of $\{Z_n\}_{n\geq 1}$,

$$X_n a = X_0((A^*)^n a) + \sum_{k=1}^{n} Z_k((A^*)^{n-k}a) \overset{d}{=} X_0((A^*)^n a) + \sum_{k=1}^{n} Z_k((A^*)^{k-1}a).$$

Since $\sum_{k=0}^{\infty} \|A^k\| < \infty$, by Lemma 1 ($b$) ($i$), $\xi := \mathfrak{C}_2$-limit of $\sum_{k=1}^{\infty} Z_k((A^*)^{k-1}(\cdot))$ exists. By Lemma 1 (b) (ii), $\eta(a) := L^2$-limit of $\sum_{k=1}^{\infty} Z_k((A^*)^{k-1}a)$ exists for all

$a \in U^*$ and by Lemma 1 (b) (iii), $\beta(a) :=$ almost sure limit of $\sum_{k=1}^{\infty} Z_k((A^*)^{k-1}a)$ exists for all $a \in U^*$.

By the second part of [31, Remark 4, page 117], it follows that $\xi(a) = \eta(a)$ for all $a \in U^*$. It is clear that $\eta(a) = \beta(a)$ almost surely, for all $a \in U^*$. Therefore $\xi(a) = \eta(a) = \beta(a)$ almost surely, for all $a \in U^*$. Also, $X = \xi$. Since $X_0 \in \mathfrak{C}_2$, we have $\|X_0((A^*)^n a)\| \leq \|X_0\| \|(A^*)^n\| \|a\|$, hence since $\sum_{k=1}^{\infty} \|A^k\| < \infty$ we have that $\lim_{n\to\infty} \|X_0((A^*)^n a)\| = 0$ almost surely. Therefore the sequence $\{\psi_n(a)\}$ with $\psi_n(a) := X_0((A^*)^n a) + \sum_{k=1}^{n} Z_k((A^*)^{k-1}a)$ converges almost surely to Xa and hence since $\psi_n(a) \stackrel{d}{=} X_n a$ for all n and $a \in U^*$, it follows that $\{X_n a\}$ converges in distribution to Xa for all $a \in U^*$. QED

Putting Lemma 2, Lemma 3 and Lemma 4 together, we obtain the following Theorem, which is the main result of this section:

Theorem 2. Let μ_n denote the cylindrical distribution of Y_n as defined in Theorem 1 (i), where we assume that

$$\sum_{k=0}^{\infty} \|A^k\| < \infty$$

and $\{Z_k\}$ is a sequence of identically distributed and weakly independent cylindrical random variables in U, with $\{Z_n\} \subseteq \mathfrak{C}_2$. Then for all $n \in \mathbb{Z}$, μ_n is independent of n and moreover, μ_n is A-decomposable.

Proof. Let X_0 be any element of $\mathfrak{C}_2$ which is weakly independent of $\{Z_n\}_{n\in\mathbb{Z}}$, and define the sequence $\{X_n\}_{n\geq 1}$ by $X_n := AX_{n-1} + Z_n$. Since $\sum_{k=0}^{\infty} \|A^k\| < \infty$, we may use Lemma 4 to conclude that $\{X_n a\}$ converges to Xa in distribution as n tends to infinity for all $a \in U^*$, where $Xa := \sum_{k=1}^{\infty} Z_k((A^*)^{k-1}a)$. By Lemma 3 if μ is the cylindrical distribution of X, then μ is A-decomposable. However it is easy to see that $Y_n \stackrel{d}{=} X$ for all $n \in \mathbb{Z}$. This completes the proof. QED

3.2 Induced Processes

Definition 13. Let ξ be a random element in $(U, \mathcal{B}(U))$. We say that a cylindrical random variable X in U is induced by ξ if $X = \langle \xi, \cdot \rangle$, i.e. for all $a \in U^*$, $Xa = \langle \xi, a \rangle$ almost surely, where $\langle \xi, a \rangle(\omega) := \langle \xi(\omega), a \rangle$, $\omega \in \Omega$.

We note that while some cylindrical random variables may be induced by random elements, not every cylindrical random variable arises this way. An example of a situation where a cylindrical random variable is not induced by a random element is given in [33, page 95]. It is therefore of relevance to determine conditions under which a cylindrical random variable in U is induced by a U-valued random variable. We are interested in conditions under which Y_n is induced by a U-valued random variable.

For a U-valued random variable X and $a_1, \ldots, a_m \in U^*,\ m \geq 1$ let us define:

$$\langle X, (a_1, \ldots, a_m) \rangle \quad := \quad (\langle X, a_1 \rangle, \ldots, \langle X, a_m \rangle). \tag{19}$$

We note that if the cylindrical random variable Z is induced by X, then $\langle X,\ (a_1, \ldots, a_m) \rangle = Z(a_1, \ldots, a_m)$ almost surely.

In [31, Lemma 2, page 120], it was shown that if $\{Z_n\}_{n \in T},\ T = \mathbb{N}$ or $\mathbb{Z}$ is a sequence of weakly independent and identically distributed cylindrical random variables in U then for all $m \in \mathbb{N}$ and $a_1, \ldots, a_m \in U^*$, $\{Z_n(a_1, \ldots, a_m)\}_{n \in T}$ is a sequence of i.i.d. random vectors in $\mathbb{R}^m$. The converse of this statement is also true. We shall formulate this alongside other related statements which we shall need:

Lemma 5. Let $\{Z_n\}_{n \in T},\ T = \mathbb{N}$ or $\mathbb{Z}$ be a sequence of cylindrical random variables in U then

(a) $\{Z_n\}_{n \in T}$ is weakly independent and identically distributed if and only if for all $m \geq 1$ and $a_1, \ldots, a_m \in U^*$, $\{Z_n(a_1, \ldots, a_m)\}_{n \in T}$ is a sequence of i.i.d. random vectors in $\mathbb{R}^m$.

(b) Suppose that Z_n is induced by a U-valued random variable X_n for each $n \in T$ and that $\{X_n\}_{n \in T}$ is i.i.d., then for all $m \geq 1$ and $a_1, \ldots, a_m \in U^*$, $\{\langle X_n, (a_1, \ldots, a_m) \rangle\}_{n \in T}$ and $\{Z_n(a_1, \ldots, a_m)\}_{n \in T}$ are also i.i.d.

Proof. (a) For the proof that if $\{Z_n\}_{n \in T}$ is weakly independent and identically distributed, then for all $m \geq 1$ and $a_1, \ldots, a_m \in U^*$, $\{Z_n(a_1, \ldots, a_m)\}_{n \in T}$ is a sequence of i.i.d. random vectors in $\mathbb{R}^m$, see [31, Lemma 2 page 120]. The converse is immediate if we set $m = 1$.

The proof of the other assertion is obvious. □ QED

Theorem 3. Suppose that $\{\xi_n\}_{n \in \mathbb{Z}}$ is a sequence of i.i.d. U-valued random variables with $\mathbb{E}\|\xi_1\|^2 < \infty$. Let $\{Z_n\}_{n \in \mathbb{Z}}$ be a sequence of cylindrical random variables such that Z_n is induced by ξ_n. If $A : U \to U$ is a bounded linear operator satisfying $\sum_{k=0}^{\infty} \|A^k\| < \infty$, then $Y_n := \sum_{k=0}^{\infty} Z_{n-k}((A^*)^k(\cdot))$ is induced by a Borel random variable for each n. Further, $Y_n \in \mathfrak{C}_2$.

Proof. For each $\omega \in \Omega$, $a \in U^*$ and $m \geq 1$,

$$\begin{aligned}[\sum_{k=0}^{m} Z_{n-k}((A^*)^k a)](\omega) = & \sum_{k=0}^{m}(Z_{n-k}((A^*)^k a))(\omega) = \sum_{k=0}^{m}\langle \xi_{n-k}(\omega), (A^*)^k a\rangle \\ = & \sum_{k=0}^{m}\langle A^k \xi_{n-k}(\omega), a\rangle = \langle \sum_{k=0}^{m} A^k \xi_{n-k}(\omega), a\rangle,\end{aligned}$$

where the last equality is justified by the linearity of $\langle \cdot, a\rangle$.

By Lemma 1 (b) (iv), $\sum_{k=0}^{\infty} Z_{n-k}((A^*)^k a) := \lim_{m\to\infty} \sum_{k=0}^{m} Z_{n-k}((A^*)^k a)$ exists almost surely for all $a \in U^*$. Therefore for each $a \in U^*$ and all ω in a set of measure 1 depending on a,

$$[\sum_{k=0}^{\infty}(Z_{n-k}((A^*)^k a))](\omega) = \lim_{m\to\infty}\sum_{k=0}^{m}(Z_{n-k}((A^*)^k a))(\omega) = \lim_{m\to\infty}\langle \sum_{k=0}^{m} A^k \xi_{n-k}(\omega), a\rangle.$$

We will now show that under the condition $\sum_{k=0}^{\infty} \|A^k\| < \infty$, $\sum_{k=0}^{\infty} A^k \xi_{n-k} := \lim_{m\to\infty} \sum_{k=0}^{m} A^k \xi_{n-k}$ exists almost surely. Let $S_m := \sum_{k=0}^{m} A^k \xi_{n-k}$ and $\varepsilon > 0$ be chosen arbitrarily, then for $l \leq m$ and by Chebyshev's inequality,

$$\mathbb{P}(\|S_m - S_l\| > \varepsilon) \leq \frac{\mathbb{E}\|S_m - S_l\|}{\varepsilon} \leq \frac{\sum_{k=l+1}^{m} \|A^k\| \mathbb{E}\|\xi_{n-k}\|}{\varepsilon}.$$

Since $\{\xi_n\}_{n\in\mathbb{Z}}$ is a sequence identically distributed random variables, $\mathbb{E}\|\xi_{n-k}\| = \mathbb{E}\|\xi_1\| < \infty$ for all $k \in \mathbb{Z}$ as $\mathbb{E}\|\xi_1\|^2 < \infty$.

Since $\sum_{k=0}^{\infty} \|A^k\| < \infty$, $\{S_m\}$ is a Cauchy sequence in probability and hence converges in probability. We note that the sequence $\{A^k \xi_{n-k}\}_{k\geq 0}$ is a sequence of independent random elements of U. Since $\{S_m\}$ converges in probability it follows that it converges almost surely. Let $a \in U^*$ be chosen arbitrarily, then since a is continuous, we have that

$$\begin{aligned}\lim_{m\to\infty}\langle \sum_{k=0}^{m} A^k \xi_{n-k}(\omega), a\rangle = & \langle (\lim_{m\to\infty} \sum_{k=0}^{m} A^k \xi_{n-k}(\omega)), a\rangle = \langle (\sum_{k=0}^{\infty} A^k \xi_{n-k}(\omega)), a\rangle \\ = & \langle (\sum_{k=0}^{\infty} A^k \xi_{n-k})(\omega), a\rangle = \langle \sum_{k=0}^{\infty} A^k \xi_{n-k}, a\rangle(\omega),\end{aligned}$$

where the third equality is the definition above of $\sum_{k=0}^{\infty} A^k \xi_{n-k}$ as the almost sure limit of $\sum_{k=0}^{m} A^k \xi_{n-k}$. We therefore have that

$$[\sum_{k=0}^{\infty} Z_{n-k}((A^*)^k a)](\omega) = \langle \sum_{k=0}^{\infty} A^k \xi_{n-k}, a\rangle(\omega)$$

almost surely. By the second part of [31, Remark 4, page 117], we have that $[\sum_{k=0}^{\infty} Z_{n-k}((A^*)^k a)] = [\sum_{k=0}^{\infty} Z_{n-k}((A^*)^k(\cdot))](a)$ for all $a \in U^*$.
Therefore $[\sum_{k=0}^{\infty} Z_{n-k}((A^*)^k(\cdot))](a) = \langle \sum_{k=0}^{\infty} A^k \xi_{n-k}, a\rangle$ almost surely, for all $a \in U^*$ and hence $\{Y_n\}$ is induced by $\sum_{k=0}^{\infty} A^k \xi_{n-k}$.
We now show that $Y_n \in \mathfrak{C}_2$. We have that
$\|Y_n\| = \|\sum_{k=0}^{\infty} Z_{n-k}((A^*)^k(\cdot)\| \leq \sum_{k=0}^{\infty} \|Z_{n-k}((A^*)^k(\cdot))\|$. Also,

$$\begin{aligned} \|Z_{n-k}((A^*)^k(\cdot))\|^2 &= \sup\{\mathbb{E}|Z_{n-k}((A^*)^k a)|^2 : a \in U^*, \|a\| \leq 1\} \\ &= \sup\{\mathbb{E}|\langle \xi_{n-k}, (A^*)^k a\rangle|^2 : a \in U^*, \|a\| \leq 1\} \\ &\leq \sup\{\mathbb{E}\|\xi_{n-k}\|^2 \|(A^*)^k\|^2_{\mathcal{L}(U^*)} \|a\|^2_{U^*} : a \in U^*, \|a\| \leq 1\} \\ &\leq \|(A^*)^k\|^2_{\mathcal{L}(U^*)} \mathbb{E}\|\xi_1\|^2. \end{aligned}$$

Therefore $\|Y_n\| \leq (\mathbb{E}\|\xi_1\|^2)^{\frac{1}{2}} \sum_{k=0}^{\infty} \|(A^*)^k\|_{\mathcal{L}(U^*)} < \infty$ which completes the proof.

QED

The pre-requsite material required for the rest of this section is given in section 5.

Lemma 6. Let $\{X_n\}_{n\geq 0}$ be a sequence of weakly independent cylindrical random variables in U. Suppose that for each n there exists a Radon measure μ_n on $\mathcal{B}(U)$ such that $\mathbb{E}(e^{iX_n a}) = \hat{\mu}_n(a)$ for all $a \in U^*$. For each $m \geq 1$ define $W_m := \sum_{n=0}^{m} X_n$, then for each $m \geq 1$ there exists a Radon measure ν_m such that $\mathbb{E}(e^{iW_m a}) = \widehat{\nu_m}(a)$ for all $a \in U^*$.

Proof. Since $\{X_n\}_{n\geq 0}$ is a sequence of weakly independent cylindrical random variables in U, it follows that for $a \in U^*$,

$$\mathbb{E}(e^{iW_m a}) = \mathbb{E}(e^{i\sum_{k=0}^{m} X_k a}) = \mathbb{E}(\prod_{k=0}^{m} e^{iX_k a}) = \prod_{k=0}^{m} \mathbb{E}(e^{iX_k a}) = \prod_{k=0}^{m} \widehat{\mu_k}(a) = \widehat{\nu_m}(a)$$

where $\nu_m := \mu_0 * \cdots * \mu_m$, i.e. the convolution of $\mu_0, \ldots, \mu_m$. Since μ_k is a Radon measure for each $k = 0, \ldots, m$, it follows from Lemma 8 (b) that ν_m is a Radon measure for each $m \geq 1$. $\boxed{QED}$

Theorem 4. Let U be a Banach space and $\{Z_n\}_{n\in\mathbb{Z}}$ be sequence of weakly independent and identically distributed cylindrical random variables in U and $A : U \to U$ be a bounded and injective linear operator such that $\sum_{k=0}^{\infty} \|A^k\| < \infty$. Further assume that there exists a Radon measure ν such that $\mathbb{E}(e^{iZ_1 a}) = \widehat{\nu}(a)$ for all $a \in U^*$ and define $\lambda_m := \nu * (A\nu) * \cdots * (A^m\nu)$. For $n \in \mathbb{Z}$, and $a \in U^*$ define $Y_n a := \sum_{k=0}^{\infty} Z_{n-k}((A^*)^k a)$. If $\{\lambda_m\}$ is weakly relatively compact in $\mathcal{M}_t(U)$, then Y_n is induced by a U-valued random variable.

Proof. We will show that under the assumptions of the Theorem, there exists a Radon probability measure μ on U such that $\mathbb{E}(e^{iY_n a}) = \hat{\mu}(a)$ for all $a \in U^*$. The assertion will then follow from Theorem 6. For $n \in \mathbb{Z},\ l, m \in \mathbb{N}$ with $m \geq l \geq 0$ and $a \in U^*$, define $\eta_{nlm} a := \sum_{k=l}^{m} Z_{n-k}((A^*)^k a)$, then for all $a \in U^*$, $\mathbb{E}(e^{i\eta_{nlm} a}) = \prod_{k=l}^{m} \mathbb{E}(e^{iZ_{n-k}(A^*)^k a}) = \prod_{k=l}^{m} \mathbb{E}(e^{iZ_1(A^*)^k a}) = \prod_{k=l}^{m} \widehat{\nu}((A^*)^k a) = \prod_{k=l}^{m} \int_U e^{i\langle x\ ,\ (A^*)^k a\rangle} d\nu = \prod_{k=l}^{m} \int_U e^{i\langle A^k x\ ,\ a\rangle} d\nu = \prod_{k=l}^{m} \int_U e^{i\langle x\ ,\ a\rangle} d(A^k\nu)$
$= \prod_{k=l}^{m} \widehat{(A^k\nu)}(a) = \widehat{\lambda_{lm}}(a)$, where we define $\lambda_{lm} := (A^l\nu) * \cdots * (A^m\nu)$. Since ν is a Radon measure, it follows from Lemma 8 (a)-(b) that λ_{lm} is a Radon measure for all $0 \leq l \leq m$. From Lemma 1 (b) (ii) and (a) (ii) it follows that $\hat{\mu}(a) := \lim_{m\to\infty} \widehat{\lambda_{0m}}(a)$ exists for all $a \in U^*$. We note that $\lambda_m = \lambda_{0m}$ for all $m \geq 0$ and that by assumption $\{\lambda_m\}$ is weakly relatively compact. Then it follows from Theorem 7 that $\{\lambda_{0m}\}$ converges weakly to a Radon measure μ on $\mathcal{B}(U)$ with characteristic functional $\hat{\mu}$. Obviously Y_n and the Radon measure have the same characteristic functional $\hat{\mu}$. $\boxed{QED}$

3.3 Markov property of a solution process

Definition 14. A cylindrical process $\{X_n\}_{n\geq 0}$ in U is said to be adapted to a filtration $\{\mathcal{F}_n\}_{n\geq 0}$ if the process $\{X_n a\}_{n\geq 0}$ is adapted to $\{\mathcal{F}_n\}_{n\geq 0}$ for all $a \in U^*$.

Definition 15. A cylindrical process $\{X_n\}_{n\geq 0}$ in U is said to be a cylindrincal Markov process relative to a filtration $\{\mathcal{F}_n\}_{n\geq 0}$ if it is adapted to $\{\mathcal{F}_n\}_{n\geq 0}$ and for all $m \in \mathbb{N}$, $a_1, \ldots, a_m \in U^*$, $\{X_n(a_1, \ldots, a_m)\}_{n\geq 0}$ is an m-dimensional Markov process.

We need the following standard observation:

Lemma 7. Let $(E_i, \mathcal{E}_i),\ i = 1, 2$ be measurable spaces, $f : E_1 \times E_2 \to \mathbb{R}$ be a bounded $\mathcal{E}_1 \otimes \mathcal{E}_2$-measurable function and $\xi,\ \eta$ be random elements of E_1 and E_2 respectively defined on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$. Suppose that $\mathcal{F} \subset \mathcal{G}$ and that ξ is $\mathcal{F}$ measurable, then $\mathbb{E}(f(\xi, \eta)|\mathcal{F}) = \mathbb{E}(f(x, \eta)|\mathcal{F})|_{x=\xi}$-almost surely.

Proof. See [12, Chapter 1 § 2 Lemma 1 page 38]. $\boxed{QED}$

We have the following theorem:

Theorem 5. Let $\{Z_n\}_{n\in\mathbb{Z}}$ be a sequence of weakly independent and identically distributed cylindrical random variables in U, X_0 a cylindrical random variable in U which is weakly independent of $\{Z_n\}_{n\geq 1}$. Let $\{\mathcal{F}_n\}_{n\geq 0}$ be the filtration defined by

$$\mathcal{F}_n := \mathcal{G}_n \vee \mathcal{H}$$

where

$$\mathcal{G}_0 := \emptyset,\ \mathcal{G}_n := \sigma(\{Z_m a : a \in U^*, m = 1, \ldots, n\}),\ n \geq 1,$$

and

$$\mathcal{H} := \sigma(\{X_0(a) : a \in U^*\}).$$

Define $X_n = AX_{n-1} + Z_n,\ n \geq 1$ then $\mathbb{E}(f(X_n(a_1, \ldots, a_m))|X_{n-1}(a_1, \ldots, a_m))$ is a version of $\mathbb{E}(f(X_n(a_1, \ldots, a_m))|X_{n-1}(A^*a_1, \ldots, A^*a_m))$ for all $n,\ m \in \mathbb{N}$, bounded and measurable function $f : \mathbb{R}^m \to \mathbb{R}$ and $a_1, \ldots, a_m \in U^*$ if and only if $\{X_n\}_{n\geq 0}$ is a cylindrical Markov process relative to $\{\mathcal{F}_n\}_{n\geq 0}$.

Proof. We note that for each $a \in U^*$,

$$X_n a = X_0((A^*)^n a) + Z_1((A^*)^{n-1}a) + \cdots + Z_{n-1}(A^*a) + Z_n a.$$

and hence

$$X_{n-1}a = X_0((A^*)^{n-1}a) + Z_1((A^*)^{n-2}a) + \cdots + Z_{n-1}a$$

from which we have that

$$X_{n-1}(A^*a) = X_0((A^*)^n a) + Z_1((A^*)^{n-1}a) + \cdots + Z_{n-1}(A^*a).$$

One can check that $\{X_n\}$ is adapted to $\{\mathcal{F}_n\}$ hence for each $n \geq 0$ and $a \in U^*$, $X_n(A^*a)$ is $\mathcal{F}_n$ measurable. Let $m \geq 1$ and $f : \mathbb{R}^m \to \mathbb{R}$ be a bounded and measurable function, then we have that

$$\begin{aligned}
&\mathbb{E}(f(X_n(a_1,\ldots,a_m))|\mathcal{F}_{n-1}) \\
=\ &\mathbb{E}(f((AX_{n-1})(a_1,\ldots,a_m) + Z_n(a_1,\ldots,a_m))|\mathcal{F}_{n-1}) \\
=\ &\mathbb{E}(f(X_{n-1}(A^*a_1,\ldots,A^*a_m) + Z_n(a_1,\ldots,a_m))|\mathcal{F}_{n-1})
\end{aligned}$$

Since for each $a \in U^*$, $X_{n-1}(A^*a)$ is measurable relative to $\mathcal{F}_{n-1}$, we have that $X_{n-1}(A^*a_1,\ldots,A^*a_m)$ is measurable relative to $\mathcal{F}_{n-1}$. In addition, $Z_n(a_1,\ldots,a_m)$ is independent of $\mathcal{F}_{n-1}$. From Lemma 7, it follows that

$$\begin{aligned}
&\mathbb{E}(f(X_n(a_1,\ldots,a_m))|\mathcal{F}_{n-1}) \\
=\ &\mathbb{E}(f(X_{n-1}(A^*a_1,\ldots,A^*a_m) + Z_n(a_1,\ldots,a_m))|\mathcal{F}_{n-1}) \\
=\ &\mathbb{E}(f(x + Z_n(a_1,\ldots,a_m)))|_{x=X_{n-1}(A^*a_1,\ldots,A^*a_m)}.
\end{aligned}$$

Since $X_{n-1}(A^*a_1,\ldots,A^*a_m)$ is measurable relative to $\sigma(X_{n-1}(A^*a_1,\ldots,A^*a_m))$, and $Z_n(a_1,\ldots,a_m)$ is independent of $\sigma(X_{n-1}(A^*a_1,\ldots,A^*a_m))$, the same argument shows that

$$\begin{aligned}
&\mathbb{E}(f(X_n(a_1,\ldots,a_m))|X_{n-1}(A^*a_1,\ldots,A^*a_m)) \\
=\ &\mathbb{E}(f(x + Z_n(a_1,\ldots,a_m)))|_{x=X_{n-1}(A^*a_1,\ldots,A^*a_m)}.
\end{aligned}$$

Thefore

$$\mathbb{E}(f(X_n(a_1,\ldots,a_m))|\mathcal{F}_{n-1}) = \mathbb{E}(f(X_n(a_1,\ldots,a_m))|X_{n-1}(A^*a_1,\ldots,A^*a_m)).$$

Hence $\{X_n\}_{n\geq 0}$ is a cylindrical Markov process relative to $\{\mathcal{F}_n\}$ if and only if $\mathbb{E}(f(X_n(a_1,\ldots,a_m))|X_{n-1}(a_1,\ldots,a_m))$ is a version of $\mathbb{E}(f(X_n(a_1,\ldots,a_m))|X_{n-1}(A^*a_1,\ldots,A^*a_m))$ for all $n \geq 0$, $m \in \mathbb{N}$, bounded and measurable function $f : \mathbb{R}^m \to \mathbb{R}$ and $a_1,\ldots,a_m \in U^*$. QED

From Theorem, 5 we now obtain the following Corollary:

Corollary 1. Suppose that the assumptions of Theorem 5 hold and that A is the identity operator on U, then $\{X_n\}_{n\geq 0}$ is a cylindrical Markov process relative to $\{\mathcal{F}_n\}_{n\geq 0}$.

4 Conclusion

We have considered the cylindrical difference equation $X_n = AX_{n-1} + Z_n$, $n \in \mathbb{Z}$ in a separable Banach space U, where $\{Z_n\}$ is a sequence of weakly independent and identically distributed cylindrical random variables, $Z_n \in \mathfrak{C}_2$ for all n, and A is a bounded linear operator on U. Assuming only that the the operator A satisfies a summability condition, we show that the difference equation has a unique cylindrical weakly stationary solution $\{Y_n\}_{n\in\mathbb{Z}}$. In addition to studying the important question as to when Y_n is induced by a U-valued random variable, we show that several properties similar to those which hold for autoregressive processes in Euclidean space also hold for $\{Y_n\}$.

5 Appendix

Definition 16. A finite Borel measure μ on a Hausdorff topological space X is said to be a Radon measure if for each $B \in \mathcal{B}(X)$, $\mu(B) = sup\{\mu(K) : K \subset B,\ K\ compact\}$.

$M_t(X)$ will denote the set of Radon probability measures on the Hausdorff topological space X.

Theorem 6. Let U be a metric space and let Γ be a vector space of real valued continuous functions defined on U that separate the points of U. Let

$$X : \Gamma \to L_0(\Omega, \mathcal{F}, \mathbb{P})$$

be a linear operator. Then the following statements are equivalent:

(i) X is induced by a Borel measurable, separably valued random variable $\xi : \Omega \to U$ with a Radon probability distribution $\mathbb{P}_\xi$.

(ii) There exists a Radon probability measure μ on U such that $\varphi_X(a) = \hat{\mu}(a)$ for all $a \in \Gamma$.

Proof. See [33, Theorem 2.5, Chapter IV, page 216]. QED

Theorem 7. Let X be a metric space (or a completely regular Hausdorff topological space) and let Γ be a separating vector subspace of $C(X)$ and let $\chi : \Gamma \to \mathbb{C}$ be a functional. If $\{\mu_\alpha\}$ is a weakly relatively compact in $M_t(X)$ net of Radon probability measures on X and $\lim_\alpha \hat{\mu}_\alpha(a) = \chi(a)$ for each $a \in \Gamma$, then μ_α converges weakly to a Radon probability measure, the characteristic function thereof coincides on Γ with χ.

Proof. See [33, Theorem 3.1, Chapter IV, page 224]. QED

Lemma 8. Let U be a Banach space. Let μ, ν be finite measures on $\mathcal{B}(U)$.

(a) If $A : U \to U$ is a bounded and injective linear operator and μ is a Radon measure, then $A\mu$ is a Radon measure.

(b) If μ and ν are Radon measures then the convolution $\mu * \nu$ of μ and ν is a Radon measure.

Proof. Since $(A\mu)(\cdot) = \mu(A^{-1}(\cdot))$ and μ is finite, it follows that $A\mu$ is finite. Further, if $B \in \mathcal{B}(U)$, then

$$\begin{aligned}(A\mu)(B) &= \mu(A^{-1}B)\\ &= \sup\{\mu(K) : K \textit{ compact},\ K \subseteq A^{-1}B\}\\ &= \sup\{\mu(K) : K \textit{ compact},\ AK \subseteq B\}\\ &= \sup\{\mu(A^{-1}C) : A^{-1}C \textit{ compact},\ C \subseteq B\}\\ &= \sup\{\mu(A^{-1}C) : C \textit{ compact},\ C \subseteq B\}.\end{aligned}$$

The second to the last equality is justified by the fact that A is injective, while the last equality is justified by the open mapping theorem and the fact that A is injective and bounded.

For the proof of the second assertion see [33, Chapter I Proposition 4.4, page 64]. QED

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