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Intrinsically quasi-isometric sections in metric spaces

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Abstract. This note contributes to the study of large-scale geometry. Specifically, we introduce the concept of intrinsically quasi-isometric sections in metric spaces and investigate their properties. In particular, we examine their Ahlfors-David regularity at large scales. Building on Cheeger's theory, we define appropriate sets that enable the determination of convexity and establish whether these sections form a vector space over $\mathbb{R}$ or $\mathbb{C}$. Furthermore, inspired by Cheeger's approach, we propose an equivalence relation for this class of sections. Throughout the paper, we employ fundamental mathematical tools.

Keywords: Large scale geometry, Quasi-isometric graphs, vector space, Ahlfors-David regularity, Metric spaces

MSC 2020 classification: primary 26A16, secondary 51F30, 46B04

1 Introduction

This paper contributes to the field of Large Scale Geometry, also known as coarse geometry. Large Scale Geometry focuses on the study of geometric objects from a global perspective, examining their behavior at large distances. It plays a crucial role in geometric group theory, algebraic K-theory, non-commutative geometry, and other related areas of analysis. The reader can see [21, 6, 22].

In particular, we focus on the concept of quasi-isometric graphs in metric spaces. Quasi-isometric maps [17, 20, 24] can be seen as large-scale analogues of biLipschitz maps, the latter having been studied by Le Donne and the author in [14]. More specifically, in [14], Le Donne and the author introduced an alternative notion of Lipschitz graphs, based on two simple observations:

(1) Franchi, Serapioni and Serra Cassano [16, 15] introduced and studied the class of intrinsically Lipschitz maps in subRiemannian Carnot groups in order to establish a good notion of rectifiability sets in the context of subRiemannian Carnot groups [1, 5, 7] after the negative result in [2].

(2) we consider graphs instead of maps.

In our context we consider a section φ of π (i.e., $\pi \circ \varphi = id$) such that $\pi : X \to Y$ produces a foliation for X, i.e., $X = \coprod \pi^{-1}(y)$ and the Lipschitz property of φ consists to ask that the distance between two points is comparable with the distance between a point and a fiber. Following this idea, it is natural to explore related notions, such as intrinsically Hölder [10] and quasi-symmetric [9] sections, where the right-hand side term involves the distance to a fiber rather than to a single point. In this paper, we introduce the concept of intrinsically quasi-isometric sections. Specifically, we consider a metric space X, a topological space Y, and a quotient map $\pi : X \to Y$, which is assumed to be continuous, open, and surjective. The author also studies the link between this notion and Hopf-Lax semigroup in [13], [12].

The standard example for us is when X is a metric Lie group G (meaning that the Lie group G is equipped with a left-invariant distance that induces the manifold topology), for example a subRiemannian Carnot group, and Y is the space of left cosets G/H, where $H < G$ is a closed subgroup and $\pi : G \to G/H$ is the projection modulo H, $g \mapsto gH$.

Definition 1. We say that a map $\varphi : Y \to X$ is an *intrinsically* (L, M)-*roughly quasi-isometric section of* π or, simply, intrinsically (L, M)-quasi-isometric section of π, with $L \geq 1$ and $M \geq 0$, if

$$\pi \circ \varphi = \mathrm{id}_Y, \tag{1}$$

and

$$d(\varphi(y_1), \varphi(y_2)) \leq L d(\varphi(y_1), \pi^{-1}(y_2)) + M, \quad \text{for all } y_1, y_2 \in Y. \tag{2}$$

Here d denotes the distance on X, and, as usual, for a subset $A \subset X$ and a point $x \in X$, we have $d(x, A) := \inf\{d(x, a) : a \in A\}$.

If $L = 1$, then φ is called an intrinsically (L, M)-roughly isometric; on the other hand if $M = 0$ the intrinsically (L, M)-roughly quasi-isometric sections are intrinsically L-Lipschitz sections studied in [14]. Moreover, we underline that, when $M = 0$ and π is a Lipschitz quotient or submetry [4, 3], the results trivialize, since in this case being intrinsically Lipschitz is equivalent to biLipschitz embedding, see Proposition 2.4 in [14].

It is easy to see that an intrinsically (L, M)-quasi-isometric section need not be continuous. Moreover, by the simply fact that $\varphi(y) \in \pi^{-1}(y)$, we have that

$$\frac{1}{L} d(\varphi(y_1), \pi^{-1}(y_2)) - M \leq d(\varphi(y_1), \pi^{-1}(y_2)) \leq d(\varphi(y_1), \varphi(y_2)),$$

for any $y_1, y_2 \in Y$ and so we get the left part of classical quasi-isometric definition is trivial, i.e.,

$$\frac{1}{L} d(\varphi(y_1), \pi^{-1}(y_2)) - M \leq d(\varphi(y_1), \varphi(y_2)) \leq L d(\varphi(y_1), \pi^{-1}(y_2)) + M,$$

for any $y_1, y_2 \in Y$.

The main result of this paper is the following.

Theorem 1 (Ahlfors-David regularity). *Let $\pi : X \to Y$ be a quotient map between a metric space X and a topological space Y such that there is a measure μ on Y such that for every $r_0 > 0$ and every $x, x' \in X$ with $\pi(x) = \pi(x')$ there is $C > 0$ such that*

$$\mu(\pi(B(x,r))) \leq C\mu(\pi(B(x',r))), \tag{3}$$

for every $r > r_0$.

We also assume that $\varphi : Y \to X$ is an intrinsically (L, M)-quasi-isometric section of π such that $\varphi(Y)$ is, for large scale, Q-Ahlfors-David regular with respect to the measure $\varphi_*\mu$, with $Q \in (0, \infty)$.

Then, for every intrinsically (L, M)-quasi-isometric section $\psi : Y \to X$, the set $\psi(Y)$ is such that

$$(r - M)^Q \lesssim \psi_*\mu\big(B(\psi(y), r) \cap \psi(Y)\big) \lesssim (r + M)^Q, \tag{4}$$

for every $r > r_0$, with $r_0 \geq M$.

Namely, in Theorem 1 Ahlfors-David Q-regularity means that the measure $\varphi_*\mu$ is such that for each point $x \in \varphi(Y)$ there exist $r_0 > 0$ and $C > 0$ so that

$$C^{-1}r^Q \leq \varphi_*\mu\big(B(x,r) \cap \varphi(Y)\big) \leq Cr^Q, \qquad \text{for all } r > r_0. \tag{5}$$

Finally, inspired by the seminal work of Cheeger [8] (see also [18, 19]) we give an equivalent property of quasi-isometric sections which we will be the basic point in order to obtain the following results:

(1) Proposition 4: a suitable set of this class of sections is convex one.

(2) Theorem 2 and 3: a suitable set of this class of sections is a vector space over $\mathbb{R}$ or $\mathbb{C}$.

(3) Theorem 4 gives an equivalence relation of this class of sections.

Regarding being vector spaces over $\mathbb{R}$ or $\mathbb{C}$, we present two different approaches: in Theorem 2 we used to be quasi-isometric section with respect to another section; on the other hand, in Theorem 3 we used the finite distance between two points belong to the same fiber.

Proposition 1. *Let X be a metric space, Y a topological space, $\pi : X \to Y$ a quotient map, $L \geq 1$ and $M \geq 0$. Assume that every point $x \in X$ is contained in the image of an intrinsic (L, M)-quasi-isometric section ψ_x for π. Then for every section $\varphi : Y \to X$ of π the following are equivalent:*

(1) for all $x \in \varphi(Y)$ the section φ is intrinsically (L_1, M_1)-quasi-isometric with respect to ψ_x at x;

(2) the section φ is intrinsically (L_2, M_2)-quasi-isometric.

Moreover, if ψ is intrinsically Lipschitz (i.e. $M = 0$), then $M_1 = M_2$.

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2 Intrinsically quasi-isometric sections

Definition 2 (Intrinsic quasi-isometric section)**.** Let (X, d) be a metric space and let Y be a topological space. We say that a map $\varphi : Y \to X$ is a *section* of a quotient map $\pi : X \to Y$ if

$$\pi \circ \varphi = \mathrm{id}_Y.$$

Moreover, we say that φ is an *intrinsically* (L, M)*quasi-isometric section* with constants $L \geq 1$ and $M \geq 0$ if in addition

$$d(\varphi(y_1), \varphi(y_2)) \leq Ld(\varphi(y_1), \pi^{-1}(y_2)) + M, \quad \text{for all } y_1, y_2 \in Y.$$

Equivalently, we are requesting that

$$d(x_1, x_2) \leq Ld(x_1, \pi^{-1}(\pi(x_2))) + M, \quad \text{for all } x_1, x_2 \in \varphi(Y).$$

We further rephrase the definition as saying that $\varphi(Y)$, which we call the *graph* of φ, avoids some particular sets (which depend on L, M and φ itself):

Proposition 2. *Let $\pi : X \to Y$ be a quotient map between a metric space and a topological space, $\varphi : Y \to X$ be a section of π, $L \geq 1$ and $M \geq 0$. Then φ is intrinsically (L, M)-quasi-isometric if and only if*

$$\varphi(Y) \cap R_{x,L} = \emptyset, \quad \textit{for all } x \in \varphi(Y),$$

where

$$R_{x,L} := \left\{x' \in X \mid Ld(x', \pi^{-1}(\pi(x))) + M < d(x', x)\right\}.$$

Proposition 2 is a triviality, still its purpose is to stress the analogy with the intrinsically Lipschitz sections theory introduced in [14] when $M = 0$ and, in particular, the sets $R_{x,L}$ are the intrinsic cones in the sense of Franchi, Serapioni and Serra Cassano considered in subRiemannian Carnot groups when $M = 0$. The reader can see [11] for a suitable notion of intrinsic cones in metric groups.

However, we recall some examples of linear sections and intrinsically quasi-isometric sections.

(1) Let the general linear group $X = GL(n,\mathbb{R})$ or $X = GL(n,\mathbb{C})$ of degree n which is the set of $n \times n$ invertible matrices, together with the operation of ordinary matrix multiplication. We consider $Y = \mathbb{R}^* = GL(n,\mathbb{R})/SL(n,\mathbb{R})$ or $Y = \mathbb{C}^* = GL(n,\mathbb{C})/SL(n,\mathbb{C})$ where the special linear group $SL(n,\mathbb{R})$ (or $SL(n,\mathbb{C})$) is the subgroup of $GL(n,\mathbb{R})$ (or $GL(n,\mathbb{C})$) consisting of matrices with determinant of 1. Here the linear map $\pi = det : GL(n,\mathbb{R}) \to \mathbb{R}^*$ is a surjective homomorphism where $Ker(\pi) = SL(n,\mathbb{R})$.

(2) Let $X = GL(n,\mathbb{R})$ as above and $Y = GL(n,\mathbb{R})/O(n,\mathbb{R})$ where $O(n,\mathbb{R})$ is the orthogonal group in dimension n. Recall that Y is diffeomorphic to the space of upper-triangular matrices with positive entries on the diagonal, the natural map $\pi : X \to Y$ is linear.

(3) Let $X = \mathbb{R}^2, Y = \mathbb{R}$ endowed with the Euclidean distance and $\pi : \mathbb{R}^2 \to \mathbb{R}$ defined as $\pi((x_1, x_2)) := x_1 + x_2$ for any $(x_1, x_2) \in \mathbb{R}^2$. An easy example of quasi-isometric sections of π is the following one: let $\varphi : \mathbb{R} \to \mathbb{R}^2$ given by
$$\varphi(y) = (by + af(y), (1-b)y - af(y)), \quad \forall y \in \mathbb{R},$$
where $a, b \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ might not be a continuous map, but it will be bounded by a constant M.

(4) Let $X = \mathbb{R}^{2\kappa}, Y = \mathbb{R}$ endowed with the Euclidean distance and $\pi : \mathbb{R}^{2\kappa} \to \mathbb{R}$ defined as $\pi((x_1, \dots, x_{2\kappa})) := x_1 + \dots + x_{2\kappa}$ for any $(x_1, \dots, x_{2\kappa}) \in \mathbb{R}^{2\kappa}$. An easy example of quasi-isometric sections of π is the following one: let $\varphi : \mathbb{R} \to \mathbb{R}^{2\kappa}$ be given by
$$\varphi(y) = (y + a_1 f_1(y), -a_1 f_1(y), a_2 f_2(y), -a_2 f_2(y), \dots, a_\kappa f_\kappa(y), -a_\kappa f_\kappa(y)),$$
for all $y \in \mathbb{R}$, where $a_i \in \mathbb{R}$ and $f_i : \mathbb{R} \to \mathbb{R}$ might not be continuous maps, but they will be bounded by a constant M for any $i = 1, \dots, \kappa$.

(5) Regarding examples of intrinsic quasi-isometric sections in the contex of Carnot groups the reader can see [23, Example 4.58].

2.1 Intrinsic quasi-isometric with respect to families of sections

In this section we continue to fix a quotient map $\pi : X \to Y$ between a metric space X and a topological space Y.

Definition 3 (Intrinsic quasi-isometric with respect to a section)**.** Given sections $\varphi, \psi : Y \to X$ of π. We say that φ is *intrinsically* (L, M)*-quasi-isometric with respect to* ψ *at point* $\hat{x}$, with $L \geq 1, M \geq 0$ and $\hat{x} \in X$, if

(1) $\hat{x} \in \psi(Y) \cap \varphi(Y)$;

(2) $\varphi(Y) \cap C^{\psi}_{\hat{x},L} = \emptyset$,

where
$$C^{\psi}_{\hat{x},L} := \{x \in X \,:\, d(x, \psi(\pi(x))) > Ld(\hat{x}, \psi(\pi(x))) + M\}.$$

Remark 1. Definition 3 can be rephrased as follows. A section φ is intrinsically (L, M)-quasi-isometric with respect to ψ at point $\hat{x}$ if and only if there is $\hat{y} \in Y$ such that $\hat{x} = \varphi(\hat{y}) = \psi(\hat{y})$ and

$$d(x, \psi(\pi(x))) \leq Ld(\hat{x}, \psi(\pi(x))) + M, \quad \forall x \in \varphi(Y),$$

which equivalently means

$$d(\varphi(y), \psi(y)) \leq Ld(\psi(\hat{y}), \psi(y)) + M, \qquad \forall y \in Y. \tag{6}$$

The proof of Proposition 1 is an immediate consequence of the following result.

Proposition 3. *Let X be a metric space, Y a topological space, and $\pi : X \to Y$ a quotient map. Let $L \geq 1, M \geq 0$ and $y_0 \in Y$. Assume $\varphi_0 : Y \to X$ is an intrinsically (L, M)-quasi-isometric section of π. Let $\varphi : Y \to X$ be a section of π such that $x_0 := \varphi(y_0) = \varphi_0(y_0)$. Then the following are equivalent:*

(1) For some $L_1 \geq 1$ and $M_1 \geq 0$, φ is intrinsically (L_1, M_1)-quasi-isometric with respect to φ_0 at x_0;

(2) For some $L_2 \geq 1$ and $M_2 \geq 0$, φ satisfies

$$d(x_0, \varphi(y)) \leq L_2 d(x_0, \pi^{-1}(y)) + M_2, \quad \forall y \in Y. \tag{7}$$

Moreover, the constants L_1 and L_2 are quantitatively related in terms of L and M_1 and M_2 are quantitatively related in terms of M.

Proof. (1) $\Rightarrow$ (2). For every $y \in Y$, it follows that

$$\begin{aligned} d(\varphi(y), x_0) &\leq d(\varphi(y), \varphi_0(y)) + d(\varphi_0(y), x_0) \\ &\leq (L_1 + 1) d(\varphi_0(y), x_0) + M_1 \\ &\leq L(L_1 + 1) d(x_0, \pi^{-1}(y)) + M_1 + M, \end{aligned}$$

where in the first inequality we used the triangle inequality, and in the second one the intrinsic quasi-isometric property of φ. Then, in the third inequality we used the intrinsic quasi-isometric property of φ_0.

(2) $\Rightarrow$ (1). For every $y \in Y$, we have that

$$\begin{aligned} d(\varphi(y), \varphi_0(y)) &\leq d(\varphi(y), x_0) + d(x_0, \varphi_0(y)) \\ &\leq (L_2+1) d(\varphi_0(y), x_0) + M_2, \end{aligned}$$

where in the first equality we used the triangle inequality, and in the second one we used (7) and the fact that $\varphi_0(y)$ belongs to the fiber $\pi^{-1}(y)$. QED

Remark 2. Notice that if φ_0 in Proposition 3 is L-intrinsically Lipschitz (i.e., $M=0$), then $M_1 = M_2$.

2.2 Convex set

We show that a class of intrinsically quasi-isometric sections with respect to another is a convex set. We begin by considering the following set.

Definition 4 (Intrinsic quasi-isometric set with respect to ψ)**.** Let $\psi : Y \to X$ a section of π. We define the set of all intrinsically quasi-isometric sections of π with respect to ψ at point $\hat{x}$ as

$$\begin{aligned} I_{\psi,\hat{x}} := \{ \varphi : Y \to X \text{ a section of } \pi \, : \, \varphi \text{ is intrinsically } (L,M)\text{-quasi-isometric} \\ \text{w.r.t. } \psi \text{ at point } \hat{x} \text{ for some } L \geq 1 \text{ and } M \geq 0 \}. \end{aligned}$$

Proposition 4. *Let $\pi : X \to Y$ be a linear and quotient map between normed spaces X and Y. Assume also that $\psi : Y \to X$ a section of π and $\hat{x} \in \psi(Y)$. Then, the set $I_{\psi,\hat{x}}$ is a convex set.*

Proof. Let $\varphi, \eta \in I_{\psi,\hat{x}}$ and let $t \in [0,1]$. We want to show that

$$w := t\varphi + (1-t)\eta \in I_{\psi,\hat{x}}.$$

Notice that by linearity of π it holds

$$\pi(w(y)) = \pi(t\varphi(y) + (1-t)\eta(y)) = t\pi(\varphi(y)) + (1-t)\pi(\eta(y)) = y,$$

for any $y \in Y$, i.e., w is a section of π. Moreover, $w(\bar{y}) = \varphi(\bar{y}) = \eta(\bar{y}) = \hat{x}$ for some $\bar{y} \in Y$ and for every $y \in Y$ we have

$$\|w(y) - \psi(y)\| = \|t(\varphi(y) - \psi(y)) + (1-t)(\eta(y) - \psi(y))\|,$$

and so

$$\|w(y) - \psi(y)\| \leq t\|\varphi(y) - \psi(y)\| + (1-t)\|\eta(y) - \psi(y)\|.$$

Hence,

$$\begin{aligned} d(w(y), \psi(y)) &\leq tL_\varphi d(\psi(\bar{y}), \psi(y)) + M_\varphi + (1-t)L_\eta d(\psi(\bar{y}), \psi(y)) + M_\eta \\ &= [t(L_\varphi - L_\eta) + L_\eta] d(\psi(\bar{y}), \psi(y)) + (M_\varphi + M_\eta), \end{aligned}$$

for every $y \in Y$, as desired. QED

2.3 Vector space: version 1

In this section we show that a 'large' class of intrinsically quasi-isometric sections is a vector space over $\mathbb{R}$ or $\mathbb{C}$ defined as follows. Notice that it is not possible to obtain that $I_{\psi,\hat{x}}$ is a vector space for $\mathbb{R}$ since the simply observation that if $\psi(\hat{y}) = \hat{x}$ then $\psi(\hat{y})+\psi(\hat{y}) \neq \hat{x}$. On the other hand, we have the following result.

Theorem 2. *Let $\pi : X \to Y$ is a linear and quotient map between normed spaces X and Y. Assume also that $\psi : Y \to X$ is a section of π and $\{\lambda\hat{x} : \lambda \in \mathbb{R}^+\} \subset X$ with $\hat{x} \in \psi(Y)$.*

Then, for any $\alpha \in (0, 1]$, the set $\bigcup_{\lambda\in\mathbb{R}^+} I_{\lambda\psi,\lambda\hat{x}} \cup \{0\}$ is a vector space over $\mathbb{R}$ or $\mathbb{C}$.

Proof. Let $\varphi, \eta \in \bigcup_{\lambda\in\mathbb{R}^+} I_{\lambda\psi,\lambda\hat{x}}$ and $\beta \in \mathbb{R} - \{0\}$. We want to show that

(1) $w := \varphi + \eta \in \bigcup_{\lambda\in\mathbb{R}^+} I_{\lambda\psi,\lambda\hat{x}}$.

(2) $\beta\varphi \in \bigcup_{\lambda\in\mathbb{R}^+} I_{\lambda\psi,\lambda\hat{x}}$.

(1). If $\varphi \in I_{\delta_1\psi,\delta_1\hat{x}}$ and $\eta \in I_{\delta_2\psi,\delta_2\hat{x}}$ for some $\delta_1, \delta_2 \in \mathbb{R}^+$ it holds

$$w \in I_{(\delta_1+\delta_2)\psi,(\delta_1+\delta_2)\hat{x}}.$$

To simplify, we choose $\varphi, \eta \in I_{\psi,\hat{x}}$ and so it remains to prove

$$w \in I_{2\psi,2\hat{x}}.$$

By linear property of π, w is a section of $1/2\pi$. On the other hand, if $\psi(\bar{y}) = \hat{x}$, then $w(\bar{y}) = \varphi(\bar{y}) + \eta(\bar{y}) = 2\psi(\bar{y}) \in X$. Moreover, using (6), we deduce

$$\begin{aligned}
\|w(y) - 2\psi(y)\| &= \|\varphi(y) + \eta(y) - 2\psi(y)\| \\
&\leq \|\varphi(y) - \psi(y)\| + \|\eta(y) - \psi(y)\| \\
&\leq 2L\|\psi(\bar{y}) - \psi(y)\| + 2M \\
&= L\|2\psi(\bar{y}) - 2\psi(y)\| + 2M,
\end{aligned}$$

for any $y \in Y$, as desired.

(2). If $\varphi \in I_{\delta_1\psi,\delta_1\hat{x}}$ then in a similar way to the point (1) it is possible to deduce that $\beta\varphi \in I_{\beta\delta_1\psi,\beta\delta_1\hat{x}}$. *QED*

2.4 Vector space: version 2

In this section we present another vector space over $\mathbb{R}$ or $\mathbb{C}$ for a suitable class of quasi-isometric sections of a linear map. Here we do not ask the condition given by Definition 4 but we want that the distance between two points belong to the same fiber is finite.

Theorem 3. *Let $\pi : X \to Y$ is a linear and quotient map between normed spaces X and Y. Assume also that the distance between two points of the same fiber is bounded by $\ell < \infty$. Then, the set of all quasi-isometric sections of π joint with the zero map is a vector space over $\mathbb{R}$ or $\mathbb{C}$.*

Proof. Let $\lambda \in \mathbb{R} - \{0\}$ and φ, ψ be two quasi-isometric sections of π with constants (L_φ, M_φ) and (L_ψ, M_ψ), respectively. We want to show that

(1) $\varphi + \psi$ is a quasi-isometric section of π.

(2) $\lambda\varphi$ is a quasi-isometric section of π.

(1). The fact that $\varphi + \psi$ is a section of $1/2\pi$ follows from the linearity of π. Moreover, for any $y_1, y_2 \in Y$ we consider $a \in 1/2\pi^{-1}(y_2)$ such that $d(\varphi(y_1) + \psi(y_1), 1/2\pi^{-1}(y_2)) = d(\varphi(y_1) + \psi(y_1), a)$ and so

$$\frac{1}{2}a \in \pi^{-1}(y_2).$$

As a consequence, by triangle inequality,

$$\begin{aligned}
\|\varphi(y_1) + \psi(y_1) - \varphi(y_2) - \psi(y_2)\| &\leq \|\varphi(y_1) + \psi(y_1) - a\| + \|a - \varphi(y_2) - \psi(y_2)\| \\
&\leq d(\varphi(y_1) + \psi(y_1), a) + \|1/2a - \varphi(y_2)\| + \\
&\quad + \|1/2a - \psi(y_2)\| \\
&\leq d(\varphi(y_1) + \psi(y_1), 1/2\pi^{-1}(y_2)) + 2\ell,
\end{aligned}$$

where in the last inequality we used that the distance between two points in the same fiber is bounded by ℓ.

(2). The fact that $\lambda\varphi$ is a section of $1/\lambda\pi$ is trivial using the linearity of π. On the other hand, for any $y_1, y_2 \in Y$

$$\begin{aligned}
\|\lambda\varphi(y_1) - \lambda\varphi(y_2)\| &\leq L_\varphi|\lambda| d(\varphi(y_1), \pi^{-1}(y_2)) + M_\varphi \\
&= L_\varphi d(\lambda\varphi(y_1), (1/\lambda\pi)^{-1}(y_2)) + M_\varphi,
\end{aligned}$$

i.e., the thesis holds. This fact follows by these observations:

(1) if $d(\varphi(y_1), \pi^{-1}(y_2)) = d(\varphi(y_1), a)$ then $|\lambda| d(\varphi(y_1), \pi^{-1}(y_2)) = \|\lambda\varphi(y_1) - \lambda a\|$.

(2) $\lambda a \in \pi^{-1}(\lambda y)$.

(3) $\pi^{-1}(\lambda y) = (1/\lambda\pi)^{-1}(y)$.

The second point is true because using the linearity of π we have that $\pi(\lambda a) = \lambda\pi(a) = \lambda y$. Finally, the third point holds since

$$\pi^{-1}(\lambda y) = \{x \in X \,:\, \pi(x) = \lambda y\} = \{x \in X \,:\, 1/\lambda\pi(x) = y\} = (1/\lambda\pi)^{-1}(y),$$

as desired.

QED

We stress that in the first point of the last proof we do not use the quasi-isometric hypothesis of φ and ψ and so we have the following result.

Proposition 5. *Let $\pi : X \to Y$ is a linear and quotient map between normed spaces X and Y. Assume also that the distance between two points of the same fiber is bounded by $\ell < \infty$. Then, the sum of two sections of π is a $(1, 2\ell)$-quasi isometric section of $1/n\pi$.*

3 An equivalence relation

In this section X is a metric space, Y a topological space and $\pi : X \to Y$ a quotient map (we do *not* ask that π is a linear map). We stress that Definition 3 does not induce an equivalence relation, because of lack of symmetry in the right-hand side of (6). As a consequence we must ask a stronger condition in order to obtain an equivalence relation.

Definition 5 (Intrinsic quasi-isometric with respect to a section in strong sense)**.** Given sections $\varphi, \psi : Y \to X$ of π. We say that φ is *intrinsically* (L, M)-*quasi-isometric with respect to* ψ *at point* $\hat{x}$ *in strong sense*, with $L \geq 1, M \geq 0$ and $\hat{x} \in X$, if

(1) $\hat{x} \in \psi(Y) \cap \varphi(Y)$;

(2) it holds

$$d(\varphi(y), \psi(y)) \leq L \min\{d(\psi(\hat{y}), \psi(y)), d(\psi(\hat{y}), \varphi(y))\} + M, \qquad \forall y \in Y. \tag{8}$$

At this point, we are able to give the main theorem.

Theorem 4. *Let $\pi : X \to Y$ be a quotient map from a metric space X to a topological space Y. Assume also that $\psi : Y \to X$ is a section of π and $\hat{x} \in X$. Then, being intrinsically quasi-isometric with respect to ψ at point $\hat{x}$ in strong sense induces an equivalence relation. We will write the class of equivalence of ψ at point $\hat{x}$ as*

$$:= \{\varphi : Y \to X \text{ a section of } \pi : \varphi \text{ is intrinsically } (L, M)\text{-quasi-isometric with respect to } \psi \text{ at point } \hat{x} \text{ in strong sense, for some } L \geq 1, M \geq 0\}.$$

An interesting observation is that, considering $I_{\psi,\hat{x}}$, the intrinsic constants L and M can be change but it is fundamental that the point $\hat{x}$ is a common one for the every section.

Proof. We need to show:

(1) reflexive property;

(2) symmetric property;

(3) transitive property;

(1). It is trivial that $\varphi \backsim \varphi$.
(2). If $\varphi \backsim \psi$, then $\psi \backsim \varphi$. This follows from (8).
(3). We know that $\varphi \backsim \psi$ and $\psi \backsim \eta$. Hence, $\hat{x} = \varphi(\hat{y}) = \psi(\hat{y}) = \eta(\hat{y})$. Moreover, by (8), it holds

$$\begin{aligned} d(\varphi(y), \psi(y)) &\leq L_1 \min\{d(\psi(\hat{y}), \psi(y)), d(\psi(\hat{y}), \varphi(y))\} + M_\varphi, \\ d(\psi(y), \eta(y)) &\leq L_2 \min\{d(\eta(\hat{y}), \eta(y)), d(\eta(\hat{y}), \psi(y))\} + M_\eta, \end{aligned}$$

for any $y \in Y$ and, consequently, if $\tilde{L} = \min\{L_1, L_2\}$, then

$$\begin{aligned} d(\varphi(y), \psi(y)) &\leq d(\varphi(y), \psi(y)) + d(\psi(y), \eta(y)) \\ &\leq \tilde{L} \min\{d(\eta(\hat{y}), \eta(y)), d(\psi(\hat{y}), \varphi(y))\} + M_\varphi + M_\eta, \\ &= \tilde{L} \min\{d(\eta(\hat{y}), \eta(y)), d(\eta(\hat{y}), \varphi(y))\} + M_\varphi + M_\eta, \end{aligned} \tag{9}$$

for any $y \in Y$. This means that $\varphi \backsim \eta$, as desired. QED

4 Proof of Ahlfors-David regularity

We need to a preliminary result.

Lemma 1. *Let X be a metric space, Y a topological space, and $\pi : X \to Y$ a quotient map. If $\varphi : Y \to X$ is an intrinsically (L, M)-quasi-isometric section of π with $L \geq 1$ and $M \geq 0$, then it holds*

$$\pi\left(B\left(p, \frac{r}{L}\right)\right) \subset \pi(B(p, r+M) \cap \varphi(Y)) \subset \pi(B(p, r+M)), \quad \forall p \in \varphi(Y), \forall r > 0. \tag{10}$$

Proof. Regarding the first inclusion, fix $p \in \varphi(Y), r > 0$ and $q \in B(p, \frac{r}{L})$. We need to show that $\pi(q) \in \pi(\varphi(Y) \cap B(p, r+M))$. Actually, it is enough to prove that

$$\varphi(\pi(q)) \in B(p, r+M), \tag{11}$$

because if we take $g := \varphi(\pi(q))$, then $g \in \varphi(Y)$ and

$$\pi(g) = \pi(\varphi(\pi(q))) = \pi(q) \in \pi(\varphi(Y) \cap B(p, r+M)).$$

Hence using the intrinsic quasi-isometric property of φ and the fact that q and g belong to the same fiber because $\pi(g) = \pi(q)$, we have that for any $p, q, g \in \varphi(Y)$ with $g = \varphi(\pi(q))$,

$$d(p,g) \leq Ld(p, \pi^{-1}(\pi(g))) + M = Ld(p, \pi^{-1}(\pi(q))) + M \leq Ld(p,q) + M < r + M, \tag{12}$$

i.e., (11) holds, as desired. Finally, the second inclusion in (10) follows immediately noting that $\pi(\varphi(Y)) = Y$ because φ is a section and the proof is complete. QED

Now we are able to prove Theorem 1.

Proof of Theorem 1. Let φ and ψ intrinsically (L, M)-quasi-isometric sections, with $L \geq 1$ and $M \geq 0$. Fix $y \in Y$. By Ahlfors-David regularity of $\varphi(y)$, we know that there are $c_1, c_2, r_0 > 0$ such that

$$c_1 r^Q \leq \varphi_* \mu\big(B(\varphi(y), r) \cap \varphi(Y)\big) \leq c_2 r^Q, \tag{13}$$

for all $r > r_0$. We would like to show that there is $c_3, c_4 > 0$ such that

$$c_4 (r-M)^Q \leq \psi_* \mu\big(B(\psi(y), r) \cap \psi(Y)\big) \leq c_4 (r+M)^Q, \tag{14}$$

for every $r > r_0$. We begin noticing that, by symmetry and (3)

$$C^{-1} \mu(\pi(B(\psi(y), r))) \leq \mu(\pi(B(\varphi(y), r))) \leq C \mu(\pi(B(\psi(y), r))). \tag{15}$$

Moreover,

$$\begin{aligned} \psi_* \mu\big(B(\psi(y), r) \cap \psi(Y)\big) &= \mu(\psi^{-1}\big(B(\psi(y), r) \cap \psi(Y)\big)) \\ &= \mu(\pi\big(B(\psi(y), r) \cap \psi(Y)\big)), \end{aligned} \tag{16}$$

and, consequently,

$$\begin{aligned} \psi_* \mu\big(B(\psi(y), r) \cap \psi(Y)\big) &\geq \mu(\pi(B(\psi(y), (r-M)/L))) \\ &\geq C^{-1} \mu(\pi(B(\varphi(y), (r-M)/L))) \\ &\geq C^{-1} \mu(\pi(B(\varphi(y), (r-M)/L) \cap \varphi(Y))) \\ &= C^{-1} \varphi_* \mu\big(B(\varphi(y), (r-M)/L) \cap \varphi(Y)\big) \\ &\geq c_1 C^{-1} L^{-Q} (r-M)^Q \end{aligned}$$

where in the first inequality we used the first inclusion of (10) with ψ in place of φ, and in the second one we used (15). In the third inequality we used the second inclusion of (10) and in the fourth one we used (16) with φ in place of ψ. Moreover, in a similar way we have that

$$\begin{aligned}\psi_*\mu\big(B(\psi(y),r)\cap\psi(Y)\big) &\leq \mu(\pi(B(\psi(y),r))) \leq C\mu(\pi(B(\varphi(y),r)))\\ &\leq C\mu(\pi(B(\varphi(y),L(r+M))\cap\varphi(Y)))\\ &= C\varphi_*\mu\big(B(\varphi(y),L(r+M))\cap\varphi(Y)\big)\\ &\leq c_2CL^Q(r+M)^Q.\end{aligned}$$

Hence, putting together the last two inequalities we have that (14) holds with $c_3 = c_1C^{-1}L^{-Q}$ and $c_4 = c_2CL^Q$. QED

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