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A note on Hilbert functions of projective homogeneous varieties

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Abstract. We generalize a theorem of Nanduri expressing the Hilbert function for Schubert varieties in minuscule homogeneous spaces in terms of the combinatorics of the corresponding Bruhat graph.

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1 Introduction

Let X a complex projective variety together with an embedding $X \hookrightarrow \mathbb{P}(V)$. Two valuable tools to study the geometry of X are provided by the Hilbert function HF_X and the Hilbert series HS_X, which encode important information such as the dimension and the degree of X.
A particularly interesting case to explore is given by rational homogeneous varieties G/P, i.e. quotients of a semisimple, simply connected algebraic group G by a parabolic subgroup P and their equivariant embeddings. The theorem of Borel and Remmert identifies them as the building blocks of projective complex varieties admitting a transitive action by their automorphism groups. Further to this their relevance lies in the fact that their geometry is dictated by the representation theory of G and thus one can use powerful representation theoretic methods to address it. The wide class of rational homogeneous varieties comprises notably projective spaces and Grassmannians.
Hodge began the study of what we now call Hilbert functions in the case of Grassmannians $Gr(k,n)$ and their Schubert varieties in the 1940's. He computed a basis of the homogeneous coordinate ring of $Gr(k,n)$ in terms of the coordinates provided by their Plücker embedding into $\mathbb{P}(\wedge^k\mathbb{C}^n)$. This basis has been used by Hodge to construct the first example of *standard monomial theory* which can be applied to minuscule partial flag varieties. In a series of papers [Ses78, LS78, LMS79a, LMS79b, LS86] Seshadri and collaborators studied in detailed the geometry of partial flag varieties and provided the background for the subsequent realization of Hodge algebras. De Concini and Lakshmibai gave a first generalization to allow for treatment of classical types for all fundamental weights [DeCL81]. The case of arbitrary partial flag varieties and their Schubert varieties for symmetrizable Kac–Moody Lie algebras has been made possible by Littelmann's Lakshmibai-Seshadri (LS) path model [Lit94, Lit95, Lit96, Lit97, Lit98] proving a conjecture of Lakshmibai. This general case needs a further generalization of Hodge algebras to LS algebras (see e.g. [Chi00]).

Beside the results of [Hod42], [Hod43] and [HP52] in the classical case of the Plücker embedding of the Grassmannian, the Hilbert series has been studied in [Br19, Ch00, Mu03, Nan14, SSW17, Stu96]. Gross and Wallach computed the Hilbert polynomial and series for an arbitrary projective

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homogeneous variety in [GW11]. Far less is known about the Hilbert series of Schubert varieties in arbitrary G/P.
Recently Chirivì, Fan and Littelmann introduced the notion of *Seshadri stratification* [CFL22, CFL23, CFL24]. One of their goals was to find a geometrical interpretation of LS-paths. When applied to homogeneous spaces Seshadri stratifications yield the Bruhat stratification, however their scope is much wider. In fact, Chirivì, Fan and Littelmann show that all projective varieties which are smooth in codimension one admit a Seshadri stratification. One of the outcomes of their theory is a formula for the Hilbert series of an embedded projective variety admitting a Seshadri stratification in terms of what they call Newton–Okounkov simplicial complex.

As remarked above among rational homogeneous varieties the simplest to study are the so-called *minuscule* homogeneous varieties, because they share several geometric properties with Grassmannians (and are therefore sometimes referred to as generalized Grassmannians). Here we give a combinatorial description of the Hilbert function of Schubert varieties in a minuscule G/P, which generalizes a theorem of Nanduri for Grassmannians [Nan14], relying on results by Chirivì [Chi00].

The plan of the paper is the following: In Section 2 we fix notations and recall some basic facts on homogeneous varieties. In Section 3 we review known results about the computation of the Hilbert polynomial and the Hilbert series of projective homogeneous varieties. Section 4 is devoted to minuscule homogeneous varieties. We compute their Hilbert coefficients in terms of maximal chains of their Bruhat posets. Lastly, in Section 5, we focus on the results of [Br19] and [Nan14] known in the case of Grassmannians and their Schubert varieties and outline possible further developments.

2 Preliminaries about projective homogeneous varieties

2.1 Notations and conventions

We will work over the complex numbers. Fix G a semisimple, simply connected algebraic group, T a maximal torus, and B a Borel subgroup which contains T. The data $G \supset B \supset T$ identify a root system R as well as the sets of positive roots R^+ and of simple roots $S = \{\alpha_1, \dots, \alpha_l\}$, where $l = \operatorname{rank} G$. We denote by $\{\omega_1, \dots, \omega_l\}$ the corresponding fundamental weights.
Let $W = N_G(T)/C_G(T)$ be the Weyl group associated to G. Then W is generated by the reflections s_{α_i} corresponding to simple roots α_i. Any $w \in W$ can thus be written (non-uniquely) as a product of simple reflections. The length $\ell(w)$ of w is the minimum of the lengths of such an expression for w. If $X(T)$ is the character group of T, then there is a W-invariant inner product $(\, , \,)$ on the vector space $X(T) \otimes \mathbb{R}$.
Additionally we fix a parabolic subgroup P containing B. The set of parabolic subgroups is in one-to-one correspondence with subsets S_P of S. Among them the maximal parabolic subgroups P_i correspond to subsets $S \setminus \{\alpha_i\}$, while for the Borel subgroup we have $S_B = \emptyset$.
We denote by W_P the Weyl group of P, i.e. the subgroup of W which is generated by simple reflections s_{α_i} for $\alpha_i \in S_P$. Every coset $wW_P \in W/W_P$ admits a unique representative of minimal length. We denote by W^P the set of minimal representatives.

2.2 Projective homogeneous varieties

For any parabolic subgroup P of G we refer to the projective homogeneous variety G/P as a generalized flag variety. Indeed, if we take $G = SL(n)$, $P = B$ the Borel subgroup of upper triangular matrices, then $G/B = \mathcal{F}(1, \dots, n)$ is the classical flag variety. There is an induced action of T on P which has only finitely many fixed points $e_w = wW_P$ for $w \in W$. The B-orbit of such a T-fixed point $C_P(w) := B \cdot e_w$ is called *Schubert* (or *Bruhat*) cell associated to $w \in W^P$. It is a locally

closed subset of G/P isomorphic to the affine space $\mathbb{C}^{\ell(w)}$. The *Schubert variety* $X_P(w)$ is the Zariski closure of $C_P(w)$ endowed with the canonical reduced scheme structure. Hence Schubert varieties are indexed by elements of W^P. The set of Schubert varieties in G/P is a poset with respect to the partial order given by inclusion. Thus W^P can also be endowed with the partial order defined by $\tau \leq \sigma \iff X_P(\tau) \subseteq X_P(\sigma)$ (called *Bruhat–Chevalley order*).
Schubert cells provide for an affine stratification of the projective variety G/P. In fact, G/P and its Schubert varieties admit *Bruhat decompositions* as disjoint union of Schubert cells:

$$G/P = \bigsqcup_{w \in W^P} C_P(w); \qquad X_P(w) = \bigsqcup_{\tau \leq w} C_P(\tau), \qquad w \in W^P.$$

We will drop the subscript when the parabolic subgroup P is implied, denoting $X_P(w)$ simply by $X(w)$.
Let P be the maximal parabolic group associated to the fundamental weight ω. Then the Picard group $\operatorname{Pic} G/P$ is isomorphic to $\mathbb{Z}$ with ample generator denoted by L. The line bundle L gives rise to a projective embedding

$$X(w) \hookrightarrow G/P \hookrightarrow \mathbb{P}(H^0(G/P, L)).$$

Let $\mathfrak{g}$ be the complex simple Lie algebra of G. The choice of a simple root defines a **Z**-grading on $\mathfrak{g}$

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i, \text{ such that } [\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j} \text{ and } \mathfrak{g}_{-i} \simeq \mathfrak{g}_i^*.$$

A given representation $V(\omega)$ will also decompose accordingly

$$V(\omega) = \bigoplus_{i \in \mathbb{N}} V_i, \text{ where } \mathfrak{g}_i V_j \subseteq V_{i+j}.$$

Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. Then R is said to be *Cohen-Macaulay* if, for all $i > \dim R$, $\operatorname{Ext}^i_R(k, R)$ vanishes. If additionally R satisfies $\operatorname{Ext}^d_R(k, R) = R$, where $d = \dim R$, then R is called *Gorenstein*.

Definition 2.1. We say that a variety X is *Cohen-Macaulay at* $x \in X$ (respectively *Gorenstein at* $x \in X$) if $\mathcal{O}_{X,x}$ is Cohen-Macaulay (respectively Gorenstein), and that X is *Cohen-Macaulay* (respectively *Gorenstein*) if it is Cohen-Macaulay (respectively Gorenstein) for any point $x \in X$.

Let A be a standard graded finitely generated $\mathbb{C}$-algebra, and $(\Pi, \leq)$ a partially ordered set (poset) such that there exists an injective map $\Pi \hookrightarrow A$, $\theta \mapsto x_\theta$, whose image generates A as an algebra.

Definition 2.2. An *algebra with straightening laws* (ASL algebra) over a poset Π is a standard graded algebra A satisfying the following:

(i) the monomials $x_{\theta_1} \dots x_{\theta_r}$ with $\theta_1 \leq \dots \leq \theta_r$ form a basis of A as a vector space (*standard monomials*);

(ii) if $\alpha, \beta \in \Pi$ are incomparable, then $x_\alpha x_\beta = \sum_\xi c_\xi x_\xi$, $a_\xi \neq 0$, x_ξ standard monomial such that every x_ξ contains a factor x_μ, $\mu \in \Pi$, satisfying $\mu \leq \alpha$, $\mu \leq \beta$ (*straightening relation*).

We denote by $A(\omega)$ the coordinate ring of the Schubert variety $X(\omega)$. Ramanan and Ramanathan showed that $X(\omega)$ is projectively normal and Ramanan proved that $A(\omega)$ is Cohen–Macaulay.

3 Hilbert polynomials and Hilbert series of projective homogeneous varieties

Having reviewed basic facts about the geometry of projective homogeneous varieties and their Schubert varieties we proceed to recount different methods of computing their Hilbert polynomials and Hilbert series. Let ρ be half the sum of the positive roots and $\alpha^\vee$ the coroot corresponding to α. Set $c_\lambda(\alpha) := \frac{(\lambda,\alpha^\vee)}{(\rho,\alpha^\vee)} \in \mathbb{Q}^+$. Then Gross and Wallach prove that

Theorem 3.1 ([GW11])**.** The Hilbert function of $X \hookrightarrow \mathbb{P}(V_\lambda)$ decomposes as

$$\mathrm{HF}(t) = \prod_\alpha (1 + c_\lambda(\alpha)t),$$

where the product runs over all $\alpha \in R^+$ such that $(\lambda, \alpha^\vee) \neq 0$.

As a consequence one recovers the result of Borel and Hirzebruch on the degree of X:

$$\deg X = d! \prod_\alpha c_\lambda(\alpha),$$

where d, the number of positive roots satisfying $(\lambda, \alpha^\vee) \neq 0$, equals $\dim X$.

Definition 3.2. An ideal is called *Hilbertian* if for all nonnegative integers its Hilbert polynomial coincides with its Hilbert function.

Hilbertian ideals were introduced by Abhyankar in [Ab84] (see also [AK89]). It follows from the Theorem of Gross and Wallach (3.1) that all partial flag varieties G/P are Hilbertian and that the coefficients of the Hilbert polynomial form a strictly log-concave and therefore unimodal sequence (as shown in [Joh23]). Gross and Wallach also compute explicitly the Hilbert series of Segre and Veronese varieties as well as Grassmannian and flag varieties.

Remark 3.3. Abhyankar showed that the ideal of $r \times r$ minors of a generic $p \times q$ matrix is Hilbertian in [Ab84], thus addressing the case of Schubert varieties in classical flag varieties. This result was reproved by Stelzer and Yong in [SY25] who generalized it to matrix Schubert varieties. It would be interesting to see if the Hilbertian property holds for Schubert varieties in other homogeneous varieties G/P.

Based on the results of [GW11] Johnson gave an explicit formula for the Hilbert series of partial flag varieties in [Joh23]:

Theorem 3.4 ([Joh23])**.** If $g(x) = a_0 + a_1 x + \ldots + a_d x^d$ ($d = \dim G/P_k$) is the numerator polynomial in the Hilbert series of G/P_k

$$\mathrm{HS}(x) = \frac{g(x)}{(1-x)^{d+1}},$$

then $a_0 = 1$, and

$$a_i = \sum_{j=0}^{i} (-1)^j \binom{d+1}{j} D_{i-j}, \qquad 1 \leq i \leq d,$$

where $D_j := \dim V(G, j\omega_k)$ for $j = 0, \ldots, d$. In particular, Johnson shows that the linear coefficient of $g(x)$ measures the codimension of $G/P_k \subseteq \mathbb{P}(V_k)$:

Corollary 3.5 ([Joh23])**.**

$$a_1 = \dim V(G, \omega_k) - (\dim(G/P_k) + 1)\,.$$

Applying the above corollary to the case of $G = SL(n+1)$ he derives that $a_1 = \binom{n+1}{k} - (k(n-k+1))$ is non-negative. Looking at Table 3.6 we can immediately generalize this to :

Corollary 3.6. The linear coefficient of the numerator of the Hilbert series (3.4) of a minuscule G/P_k for G simply-laced is given by

G	P_k	a_1
D_n	ω_1	1
	ω_n, ω_{n-1}	$2^{n-1} - \frac{n(n-1)}{2} - 1$
E_6	ω_1, ω_6	10
E_7	ω_7	28

4 Minuscule representations of homogeneous spaces

Minuscule homogeneous varieties behave similarly to Grassmannians and are thus the most natural examples to study.

Definition 4.1. A fundamental weight ω is called *minuscule* if $\langle \omega, \beta \rangle \leq 1$ for all $\beta \in R^+$. Equivalently, W acts transitively on the set of its weights.

For a minuscule representation $V(\omega, G)$ all weights are extremal. In particular, all weight spaces are one-dimensional and the weights are in bijection with the elements of W^P. Accordingly, if P_k is the maximal parabolic subgroup associated to a minuscule weight ω_k, the corresponding G/P_k is said to be a *minuscule homogeneous space*. Minuscule representations have been classified (see [Bou], Ch. VIII, §7.3):

G	P_k	$\dim V(G, \omega_k)$
A_n	ω_k	$\binom{n+1}{k}$
D_n	ω_1	$2n$
	ω_n, ω_{n-1}	2^{n-1}
C_n	ω_1	$2n$
E_6	ω_1, ω_6	27
E_7	ω_7	56

In fact, we can restrict our study of minuscule homogeneous varieties and Schubert varieties therein to the simply-laced cases because of the following isomorphisms:

$$G(B_n)/P_n \simeq G(D_{n+1})/P_{n+1}, \qquad G(C_n)/P_1 \simeq G(A_{2n-1})/P_1.$$

The associated Schubert varieties are in one to one correspondence under these isomorphims.

In the A_n case all fundamental weights $\omega_1, \ldots, \omega_n$ are minuscule. The corresponding homogeneous spaces are Grassmannians $Gr(k,n)$, whose embedding into $P(V(\omega_k), A_n) \simeq P(\bigwedge^k V(\omega_1), A_n)$ is given by maximal minors of an $k \times n$ matrix.
For D_n there are three minuscule weights ω_1, ω_{n-1} and ω_n, which correspond to the natural and the two (isomorphic) half-spinor representations respectively. The associated homogeneous spaces are quadrics Q_n and one of the two connected components of the orthogonal Grassmannian $OGr(n, 2n)$.
E_6 and E_7 admit respectively two and one minuscule representation each, while E_8 has no minuscule representations. The homogeneous varieties $G(E_6)/P_1$ and $G(E_6)/P_6$ corresponding to the two fundamental representations are isomorphic as embedded projective varieties, but not equivariantly.

Let V_λ be one of the minuscule representation for G, v_λ a highest weight vector for V_λ as above. We denote the ring of coordinates of V_λ by $\mathrm{Sym}(V_\lambda^*)$. Then $\mathfrak{g}$ acts on $\mathrm{Sym}(V_\lambda^*)$. In particular, $\mathrm{Sym}^2(V_\lambda^*)$

decomposes as the direct sum $\mathfrak{U}(\mathfrak{g}) \cdot v_\lambda \oplus I$ for a unique $\mathfrak{g}$-submodule I. Konstant's Theorem states that the homogeneous coordinate ring of $G/P \subset \mathbb{P}(V_\lambda)$ is the quotient of $\mathrm{Sym}(V_\lambda^*)$ by the ideal generated by I. The algebra $A(\omega)$ is an ASL with respect to the embedding $\Pi(\omega) \hookrightarrow A(\omega)$, $\theta \mapsto x_\theta$, where x_θ is a vector of weight $-\theta(\omega_k)$, defined up to a non-zero scalar. We can identify the ideal I for minuscule homogeneous spaces with the ideal generated by the straightening relations of the non-standard monomials in x_θ's. We call the x_θ's generalized Plücker coordinates and the corresponding relations *generalized Plücker equations.* The generalized Plücker relations for orthogonal Grassmannians and for the Freudenthal variety were adressed by Chirivì and Maffei [CM13] and Chirivì, Littelmann, and Maffei in [CLM09] respectively.

Definition 4.2. An *LS path* $\pi = (\underline{\sigma} : \sigma_p > \sigma_{p-1} > \ldots > \sigma_1; \underline{a} : 0 < a_p < a_{p-1} < \ldots < a_1 = m)$ of shape λ and degree $m \geq 1$ is a pair of sequences linearly ordered elements $\sigma_i \in W$ and $a_i \in \mathbb{Q}$ such that there exists an (a_i, λ)-chain joining σ_{i-1} and σ_i for all i.

Let $L_r(\mathfrak{C})$ denote the set of LS paths of degree r with support in a complete chain $\mathfrak{C}$. We denote its cardinality by $h_r = |L_r(\mathfrak{C})|$. From the previous paragaraph it is clear that h_r is the dimension of $[A(\omega)]_r$, i.e. the r-th Hilbert coefficient of $A(\omega) = R(X(\omega))$. Chirivì gives the following expression for the numbers h_r:

Proposition 4.3 (Prop. 4 [Chi00])**.**

$$h_r = \sum_{\substack{i = 0, \ldots, d \\ \ell \in L_i'(\mathfrak{C})}} \binom{r - i + s_l - 1}{s_l - 1}$$

Using the inclusion-exclusion principle he computes the Hilbert coefficients for $A(\omega)$ over $\Pi(\omega)$. Taking all maximal chains $\mathfrak{C} = \{C_1, \ldots, C_t\}$ in $\Pi(\omega)$ one gets [Chi00]:

$$h_r(A(\omega)) = \sum_{i \geq 0; 1 \leq l_0 < \ldots < l_i \leq t} (-1)^i h_r(C_{l_0} \cap \ldots \cap C_{l_i}).$$

He then applies Proposition 4.3 in order to generalize Seshadri's computation of the degree of the Grassmannian (see also [CFL24]):

Proposition 4.4 ([Chi00], Theorem 3.6)**.** The degree of the embedded Schubert variety $X(\tau) \subseteq P(V(\lambda)_\tau)$ is

$$\deg X = \sum_{\mathfrak{C}} \prod_{j=1}^{s} b_{j,\mathfrak{C}},$$

where the sum runs over all maximal chains $\mathfrak{C}$ in $A(\tau)$ and the product over all bonds of a maximal chain $\mathfrak{C}$.

Remark 4.5. If ω_k is a minuscule weight of a semisimple Lie algebra the set of LS paths reduces to

$$B(\omega_k) = \left\{ (\sigma; 0, 1) \,|\, \sigma \in W^{P_k} \right\}.$$

In this case the weight spaces $V(\omega)_\mu$ are at most one-dimensional. In particular, $V(\omega)_\mu \neq 0$ if and only if $\mu = \sigma(\omega)$ for some $\omega \in W^{P_k}$. Hence in the minuscule case the number of maximal chains in $\Pi(\omega)$ equals the degree of $X(\omega)$, while the length of a (hence any) maximal chain is equal to the dimension of $X(\omega)$ minus 1.

Theorem 4.6. The r-th Hilbert coefficient $h_r(R[X(\omega)])$ of $R[X(\omega)]$ is given by

$$h_r(R[X(\omega)]) = \sum_{j=2}^{t} |M_j(r)|,$$

where

$$M_j(i) = \{(i_1, \dots i_t) \,|\, 1 \le i_1 < \dots < i_t \le s,\ |C_{i_1} \cap \dots \cap C_{i_t}| = |C_1| - r\}\,.$$

Proof. Let $X = G/P$ be a minuscule variety, where P is the parabolic subgroup associated to the fundamental weight ω. By Remark 4.5 for a minuscule representation $V(G, \omega)$ only the LS paths $(\sigma; 0, 1)$ with $\sigma \in W^{P_k}$ contribute to the Hilbert coefficients h_r, i.e. $L_i'(\mathfrak{C}) = \emptyset$ for all i. The underlying paths are precisely the maximal chains in the Bruhat poset. Thus applying Proposition 4.3 we get the result. QED

Remark 4.7. In the case of Grassmannian Theorem 4.6 has been proved by Nanduri in [Nan14] using methods from commutative algebra.

Example 4.8. The homogeneous variety $Y = G(D_5)/P_5$ is (one of the two connected components of) the orthogonal Grassmannian $OGr(5, 10)$. Y can be embedded into $\mathbb{P}(V(\omega_5, D_5))$, where $V(\omega_5, D_5)$ is one of the two 16-dimensional half-spinor representations of D_5. Using the grading with respect to the root α_5 the representation decomposes as:

$$V(\omega_5, D_5)_{\downarrow \alpha_5} = V(0, A_4) \oplus V(\omega_2, A_4) \oplus V(\omega_4, A_4) = \mathbb{C} \oplus \wedge^2\mathbb{C}^5 \oplus \wedge^4\mathbb{C}^5.$$

For this reason we denote the coordinates in $\mathbb{P}(V(\omega_5, D_5))$ by $\{y_0, y_{ij}, y_{klmn}\}$, where $1 \le i < j \le 5, 1 \le k < l < m < n \le 5$. Since

$$\mathrm{Sym}^2(V(\omega_5, D_5)) = V(\omega_1, D_5) + V(2\omega_5, D_5),$$

with $\dim V(\omega_1, D_5) = 10$, the ideal of Y in $\mathbb{C}[y_0, y_{ij}, y_{klmn}]$ is generated by 10 quadratic relations:

$$\begin{aligned} I(Y) = \\ &(-y_{15}y_{1234} + y_{14}y_{1235} - y_{13}y_{1245} + y_{12}y_{1345}, -y_{25}y_{1234} + y_{24}y_{1235} - y_{23}y_{1245} + y_{12}y_{2345},\\ &-y_{35}y_{1234} + y_{34}y_{1235} - y_{23}y_{1345} + y_{13}y_{2345}, -y_{45}y_{1234} + y_{34}y_{1245} - y_{24}y_{1345} + y_{14}y_{2345},\\ &-y_{45}y_{1235} + y_{35}y_{1245} - y_{25}y_{1345} + y_{15}y_{2345}, y_{25}y_{34} - y_{24}y_{35} + y_{23}y_{45} - y_0y_{2345},\\ &y_{15}y_{34} - y_{14}y_{35} + y_{13}y_{45} - y_0y_{1345}, y_{15}y_{24} - y_{14}y_{25} + y_{12}y_{45} - y_0y_{1245},\\ &y_{15}y_{23} - y_{13}y_{25} + y_{12}y_{35} - y_0y_{1235}, y_{14}y_{23} - y_{13}y_{24} + y_{12}y_{34} - y_0y_{1234}). \end{aligned}$$

Note that one possible way to obtain these equations is by taking 10 partial derivates of the E_6-invariant cubic. The Bruhat graph of $V(D_5, \omega_5)$ is displayed in Figure 1. Note that there are 12 maximal chains of length 11 in the poset $\Pi(Y)$, hence $\dim Y = 10$ and $\deg Y = 12$. Applying Theorem 4.6 we compute the Hilbert polynomial of Y:

$$\mathrm{HP}_Y = -\mathrm{HP}_7 + 8\mathrm{HP}_8 - 18\mathrm{HP}_9 + 12\mathrm{HP}_{10},$$

where $\mathrm{HP}_n = \binom{z+n}{n}$ denotes the Hilbert polynomial of $\mathbb{P}^n$. The Hilbert series of Y is

$$\mathrm{HS}_Y(t) = (1 + 5t + 5t^2 + t^3)/(1 - t)^{11}.$$

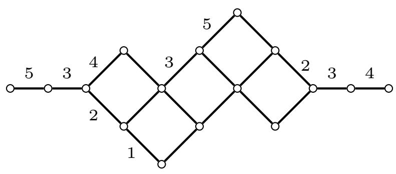

Figure 1. Bruhat graph of $V(D_5, \omega_5)$

5 Further known results about Grassmannians

The most detailed results on Hilbert polynomials and Hilbert series pertain the case of Grassmannians $Gr(k,n)$ and in particular of Grassmannians of planes $Gr(2,n)$ [Li42]. The Bruhat poset of the representation $V(A_n, \omega_2)$ is

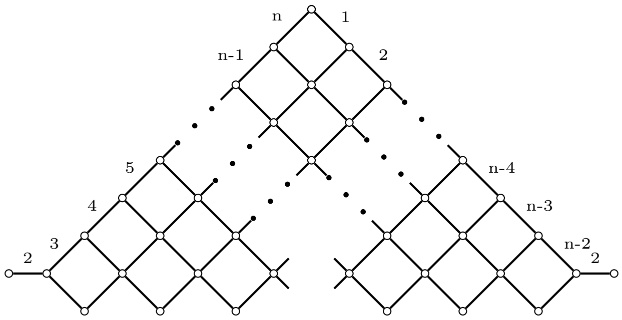

Figure 2. Bruhat graph of $V(A_n, \omega_2)$

In [Mu03] Mukai gave a closed formula for the Hilbert series of $Gr(2,n)$ by computing the Hilbert series of the semi-invariant ring $\mathbb{C}\left[\mathrm{Mat}(2,n)\right]^{SL(2)}$ using invariant theoretic methods (here $\mathrm{Mat}(2,n)$ is the vector space of $2 \times n$ matrices with the action of $SL(2)$ is induced from that on the columns of a matrix):

Theorem 5.1 (Ch. 8, [Mu03])**.**

$$\mathrm{HS}_{Gr(2,n)}(t) = \sum_{k=0}^{\infty} \left[\binom{n+k-1}{k}^2 - \binom{n+k}{k+1}\binom{n+k-2}{k-1} \right] t^k.$$

The coefficients of the h-polynomial of $Gr(2,n)$ turn out to be *Narayana numbers* [Nar59]:

$$N(n,j) := \frac{1}{j+1}\binom{n-1}{j}\binom{n}{j}.$$

These numbers carry enumerative information in several different contexts. One of their enumerative interpretations is in terms of Dyck paths. A *Dyck path* is a lattice path in $\mathbb{Z}^2$ from $(0,0)$ to (n,n) with allowed steps from $(i,j) \to (i+1,j)$ and $(i,j) \to (i,j+1)$, such that all points of the path satisfy $i \le j$. The Narayana numbers $N(n,j)$ count Dyck paths from which have exactly j ascents. Thus the Hilbert coefficients of $Gr(2,n)$ can be also obtained by counting Dyck paths. In light of Theorem 4.6 we can see that by rotating the Bruhat graph of $V(A_n, \omega_2)$ the maximal chains get identified with Dyck paths:

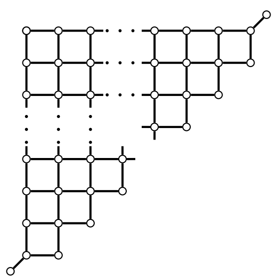

Figure 3. Rotated Bruhat graph of $V(A_n, \omega_2)$

Remark 5.2. A further connection to be made when considering $Gr(2,n)$ is with the *root poset* $\Psi(A_n)$ of type A_n. For arbitrary type Φ the root poset $\Psi(\Phi)$ consists of the set of positive roots R^+ endowed with the partial order defined by $\beta \leq \gamma$ when $\gamma - \beta$ can be written as a linear combination of simple roots with nonnegative coefficients. The number of antichains in the root poset $\Psi(\Phi)$ is given by Narayana numbers of type Φ (see [FR07], Thm. 5.2 and references therein and Thm. 5.9). Notice that in the particular case of $V(\omega_2, A_n)$ the underlying set of the Bruhat poset coincides with set of positive roots of A_n. Hence the Hilbert coefficients of $Gr(2,n)$ are also counted by the antichains in $\Psi(A_n)$.

The notion of two-dimensional Dyck path has been generalized as follows: Consider lattice paths in $\mathbb{Z}^k$ from the origin to $(n,\ldots,n)$ such that for each point $0 \leq x_1 \leq \ldots \leq x_n$ and the path has exactly j ascents. Counting these generalized Dyck paths gave rise to a generalization of Narayana numbers, called *k-Narayana numbers*, introduced by Sulanke in [Su04] and [Su05]:

$$N_k(r,j) := \sum_{l=0}^{j} (-1)^{j-l} \binom{kr+1}{j-l} \mathfrak{N}_k(r-1,l),$$

where

$$\mathfrak{N}_k(r,j) := \prod_{i=0}^{k-1} \binom{r+i+j-1}{i+j}.$$

Substituting $k=2$, $r=n-1$ in $\mathfrak{N}_k(r,j)$ one recovers the Hilbert polynomial of $Gr(2,n)$ obtained by Mukai. The *k-Narayana polynomial* and the *k-Narayana series* are defined as the generating series of $N_k(r,j)$ and $\mathfrak{N}_k(r,j)$:

$$N_{k,r} := \sum_{j=0}^{(k-1)(r-1)} N_k(r,j)t^j, \qquad \mathfrak{N}_{k,r} := \sum_{j\geq 0} \mathfrak{N}_k(r,j)t^j.$$

Analogously to the case $k=2$ the cone over $Gr(k,n)$ can be viewed as the quotient of $\mathrm{Mat}(k,n)$ by $G = SL(k)$, where the action of G is given by right multiplication (see e.g. Thm. 9.3.6, [LB15]). Thus G also acts on the coordinate ring $\mathbb{C}[\mathrm{Mat}(k,n)] = \mathbb{C}[x_{ij} \,|\, 1 \leq i \leq n, 1 \leq j \leq k]$. Therefore the homogeneous coordinate ring of $Gr(k,n)$ is precisely $\mathbb{C}[\mathrm{Mat}(k,n)]^{SL(k)}$ and their two Hilbert series coincide. Using this property Braun was able to generalize the result of Mukai for all Grassmannians:

Theorem 5.3 ([Br19])**.** The h-polynomial of $Gr(k,n)$ is the k-Narayana polynomial $N_{k,n-k+1}$ and its Hilbert series is the k-Narayana series $\mathfrak{N}_{k,n-k+1}$.

Remark 5.4. In [FR07] Fomin and Reading showed that the Narayana numbers $N(n,j)$ give the entries of the h-vector of the simplicial complex dual to an n-associahedron of type A. Motivated by this result they introduced generalized Narayana numbers for arbitrary root systems as the components of the h-vector of the dual complex of the corresponding generalized associahedron. It would be interesting to see if these generalized Narayana numbers play a role in the computation of invariants of a general G/P.

5.1 Schubert varieties in Grassmannians

Let V be a complex n-dimensional vector space with standard basis $\{e_1,\ldots,e_n\}$. We fix the complete flag of subspaces $V_0 = \emptyset \subseteq V_1 \subseteq \ldots \subseteq V_n = V$, where $V_i = < e_1,\ldots,e_i >$. Let $I_{k,n} = \{(\theta_1,\ldots,\theta_n) \,|\, 1 \leq \theta_1 < \ldots < \theta_n \leq n\}$. The Schubert varieties in $Gr(k,n)$ are indexed by $I_{k,n}$ and can be described as follows:

$$X(\theta_1,\ldots,\theta_k) := \{W \in Gr(k,n) |\, \dim(W \cap < e_1,\ldots,e_{\theta_j} >) \geq j, j = 1,\ldots,k\}.$$

Then we have the following result by Hodge and Pedoe (see also [Gho01]):

Theorem 5.5 (Thm. III, [HP52])**.** The Hilbert coefficients of the Schubert variety $X(\theta_1, \ldots, \theta_k)$ are given by

$$\{\theta_1, \ldots, \theta_k\}_j := \begin{vmatrix} [ccc]\binom{\theta_k+j-1}{j} & \cdots & \binom{\theta_1+j+k-2}{j+k-1} \\ \vdots & \ddots & \vdots \\ \binom{\theta_k+j-k}{j-k+1} & \cdots & \binom{\theta_1+j-1}{j} \end{vmatrix}.$$

Using the closed formula for the h-polynomial of the Schubert variety $X(\theta_1, \ldots, \theta_k)$ provided by Prop 2.9 of [Nan14], Braun showed that

Proposition 5.6 (Prop. 3, [Br19])**.** The i-th coefficient of the h-polynomial of $X(\theta_1, \ldots, \theta_k)$ is

$$h_i = \sum_{l=0}^{i} (-1)^l \{\theta_1, \ldots, \theta_k\}_{i-l} \binom{d}{l},$$

where $d = \dim X(\theta_1, \ldots, \theta_k)$.

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