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Semi-Riemannian metrics on compact simple Lie Groups

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Abstract. This is a survey on left invariant semi-Riemannian metrics on compact Lie groups.

1 Introduction

Let K be a Lie group endowed with a semi-Riemannian metric g. There is in general two fundamental questions that one can ask in comparing the general situation to the Riemannian one:

(1) Is the geodesic flow of K complete, that is every geodesic in K is defined for all time (as this is the case when g is Riemannian)?

(2) Is the isometry group $\mathsf{Isom}(K, g)$-acting properly on K? This means that $\mathsf{Isom}(K, g)$ preserves some auxiliary Riemannian metric, say $\bar{g}$.

Let us consider the two additional natural following questions:

(3) By definition (of being a left invariant metric), K is a subgroup of $\mathsf{Isom}(K, g)$, but, then, what is the full isometry group of (K, g)? In particular, is the isotropy of $1 \in K$ made by automorphisms of K?

(4) When is the conformal group $\mathsf{Conf}(K, g)$ essential, that is its action on K does not preserves a metric in the conformal class of g? Observe in fact in this case (of left invariant metrics) that non-essential means exactly $\mathsf{Conf}(K, g) = \mathsf{Isom}(K, g)$.

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1.1

In this note, we will survey this topic, by focusing on the case where K is a compact Lie group. As said above, $\mathsf{Isom}(K, g)$ is an extension in the diffeomorphism group $\mathsf{Diff}(K)$ of K (where K is seen as a subgroup of $\mathsf{Diff}(K)$, acting by left multiplication on itself). Let us observe however that existence of such extensions of K, say by a non-compact group G, is not a surpriszing matter. Indeed if a semi-simple G is a Lie group, and K is its maximal compact, then K acts simply transitively on G/B where B is a Borel subgroup of G. Thus the left K-left-action of K on itself identifies with the the K-action on G/B (as $K \subset G$). Thus the G-action on G/B is an extension of the K-action. This action preserves some geometric structure, surely of parabolic type. All the question now is to see if this G-action on K can preserve a (conformal) semi-Riemannian structure (this is specially related to Item (4) above)?

1.2 Results

The recent and classical literature are summarized in the following results:

1.2.1 The geodesic flow

Marsden [12] proved that the geodesic flow of a compact semi-Riemannian homogeneous space (M, g) is complete. It was observed in [8], that there is a Riemannian metric $\bar{g}$ on M, which once seen as a scalar function on the tangent bundle TM, is a first integral of the geodesic flow of g. So, not only the geodesic flow is complete, but its orbits are uniformly bounded.

1.2.2 Left Riemannian metrics

Regarding Item (3), T. Ochiai and T. Takahashi proved in [13], that if K is a compact simple Lie group, and the metric g is Riemannian, then the identity component of the isotropy group acts by automorphisms. So, up to a finite cover, $\mathsf{Isom}(K, g)$ is contained in $K \times K$, acting by the left and the right on K.

This beautiful proof, of topological-algebraic nature, will be recalled in some details in §2. This result is no longer true in the general semi-simple case, see the example in §5.1 due to Ozeki [14] who proved a generalization of [13] which can also help to handle the semi-simple case.

1.2.3 Non-Riemannian case

Regarding Item (2), again if K is simple, but g has any signature, it was recently proved by Z. Chen, K. Liang and F. Zhu, that $\mathsf{Isom}(K, g)$ is compact. In particular, this group preserves a left invariant Riemannian metric, and hence satisfies the previous description.

The beautiful proof uses deep results from Gromov's rigid transformation groups theory. We will show in §4 that this result also follows from the simpler and direct techniques by Baues-Globke-Zeghib [1] (and [2]). We will also partially sketch this approach.

Semi-Riemannian compact semi-simple non-simple groups can however have non-compact isometry groups, see §5 for examples. Actually, results of [1] give many details about the semi-simple case, and tend to show that this construction is essentially the unique way to get examples (of non-compact isometry groups for left invariant metrics on compact semisimple groups).

1.2.4 Maximally symmetric metric

The two previous results can be formulated as follows:

Theorem 1. *Let K be a compact simple Lie group. Consider g_K, its left invariant (in fact also right invariant) metric determined by the Killing form (defined on the Lie algebra $\mathfrak{k}$). Then, g_K is maximally symmetric among left invariant metrics, that is, for any left invariant metric g on K, $\mathsf{Isom}^0(K, g) \subset \mathsf{Isom}^0(K, g_K)$.*

1.2.5 The conformal group

The new contribution of the present article concerns the conformal question (Item (4)). Based on the current project on the Lichnerowicz conformal semi-Riemannian conjecture, in a homogeneous setting [3, 4, 5], we get that if $\mathsf{Conf}(K, g)$ is essential, then (K, g) is conformally flat. This happens rarely:

Theorem 2. *Let K be a compact semisimple Lie group. Assume that $\mathsf{Conf}(K, g)$ is essential. Then, up to a finite cover, K is $\mathsf{SU}(2)$ or $\mathsf{SU}(2) \times \mathsf{SU}(2)$ The conformal group is (up to finite cover) respectively: $\mathsf{SO}(1, 3)$ and $\mathsf{SO}(4, 4)$.*

It would be really interesting to see if this result can be proved “algebraically”, that is without using the results on the homogeneous Lichnerowicz conjecture.

1.2.6 Terminology: Supergroup extensions

In light of §1.1 and §1.2.2, it becomes natural to call a supergroup (or maybe a supergroup extension) of K, a group G that contains K such that K acts freely and transitively on some homogeneous space G/H.

The result mentioned in §1.2.2 means exactly that a compact simple Lie group has a maximal compact supergroup which is $K \times K$ (up to finite covers).

As example, $\mathsf{SL}_2(\mathbb{R})$ is a supergroup extension of $\mathsf{SO}(2)$, since $\mathsf{SO}(2)$ acts simply transitively on the circle which is a homogenous space of $\mathsf{SL}(2,\mathbb{R})$. However, $\mathsf{SL}_3(\mathbb{R})$ is not a supergroup of $\mathsf{SO}(2)$ since it can not act (non-trivially) on the circle (it is know that the unique simple Lie groups acting non-trivially the circle are covers of $\mathsf{PSL}(2,\mathbb{R})$).

One general construction of supergroups goes as follows (for any K). Let $\rho : K \to \mathsf{GL}(F)$ be a representation, where F is a finite dimensional subspace of functions on K. In other words F is a K-invariant finite dimensional subspace in the space of all smooth functions on K (endowed with its usual action). Let $\mathbb{T}$ be a circle in K, say given by a one parameter subgroup $t \to \exp tu$. Consider the F action on K, defined by $f.k = k \exp f(k)u \in K$. For f constant, we get the $\mathbb{T}$-action by the right on K. Combining with the left K-action, we get a transitive action of the semi-direct product $K \ltimes_\rho F$.

A similar construction is available with $\mathbb{T}$ replaced by a higher dimension torus $\mathbb{T}^d$.

This is an example of a supergroup which always exists. It is interesting to see when this could preserve a semi-Riemannian metric? In fact, results of [1] say essentially that all semi-Riemannian supergroups of a compact semi-simple (non necessarily simple) group, are of this type.

2 Riemannian case

Theorem 3. *[13] Let K be a connected compact simple Lie group. Let G be a supergroup of K, that is G acts faithfully on K, and contains a copy of K whose action identifies to the left action of K on itself. If G is connected and compact, then a finite cover of G is a subgroup of $K \times K$.*

The $K \times K$-action on K, by the usual rule $(k_1, k_2)x = k_1 x k_2^{-1}$, has as kernel $Z \times Z$, where Z is the center of K. Therefore, the theorem says that a supergroup is contained in $K \times K/Z \times Z$.

Sketch of proof.

• Let H be the stabilizer of 1. Then G is naturally homeomorphic to the product $K \times H$. Indeed, $g.1$ equals $k \in K$, and hence $k^{-1}g = h$, for some $h \in H$. This coherently defines a bijective map $g \to (k, h)$, which is naturally continuous.

• As a compact group, the universal cover of G decomposes as a direct product of compact simple groups and an abelian group covering a toral factor. Recall here that a compact group has a finite fundamental group exactly if it is semi-simple. Since K is simple, it has no non-trivial homomorphism to an abelian group, and hence, its universal cover is contained in the product of simple factors. If we change G accordingly, that is we remove the toral factor from it, we do not change our problem, that is this new G is still a supergroup of K. So, we will henceforth assume that G is semi-simple.

• We have equality of homotopy groups: $\pi_i(G) = \pi_i(K) \times \pi_i(H)$.

We deduce first that $\pi_1(H)$ has a finite fundamental group and is thus semi-simple.

At this stage, we can, and will, assume that all groups G, H and K are simply connected (and hence in particular admit direct decompositions into simple factors).

• Now, we recall that, for a simply connected simple Lie group, $\pi_2 = 1$, and $\pi_3 = \mathbb{Z}$. This was proved by R. Bott as a corollary of the main results of [6]), and as application of Morse theory to the topology of Lie groups. We don't know if a direct proof is available.

Thus, for a simply connected compact semi-simple group, its π_3-group is $\mathbb{Z}^d$, where d is the number of its simple factors.

Therefore, if G has d factors, then H has $(d-1)$ factors.

• Write $G = G_1 \times \ldots \times G_d$. If K projects trivially on some factor G_i, then we can remove this factor without changing our problem (that is we will still have a supergroup of K). So assume K projects injectively in each G_i, in particular for any i, $\dim G_i \geq \dim K$, and G_i isomorphic to K, in case of equality of dimensions.

• We have $\dim H = \Sigma \dim G_i - \dim K$. So, if $\dim G_i > \dim K$, then $\dim G_i + \dim H > \dim G$, which implies that $G_i \cap H \neq 1$ (this happens at the Lie algebra level, and then applies to groups too).

• Any non-trivial $G_i \cap H$ will be a non-trivial normal subgroup of H, and is hence product of factors of H.

• Change notations and write $G = A_1 \times \ldots \times A_a \times B_1 \times \ldots \times B_b$, where $\dim A_i > \dim K$, and B_i is isomorphic to K, for any i.

• Each $A_i \cap H$ is a product of factors of H.

• Consider the set Σ of factors H_j of H that are contained in $A_1 \times \ldots \times A_a$. There are at least a elements of Σ. Their contribution in $\dim H$ is at most $\Sigma \dim A_i$.

Remember that H has $d-1 = a+b-1$ factors, so it remains at most $b-1$ factors H_j not in Σ. Any such H_j projects non-trivially on some B_i, and hence is isomorphic to K. The total contribution of such factors in $\dim H$ is thus at most $(b-1)\dim K$. But $\dim H = \dim G - \dim K = \Sigma \dim A_i + (b-1)\dim K$. It follows that $A_1 \times \ldots \times A_a$ is contained in H.

• Remember however that H is the isotropy of the G-action on K, and by the faithfulness (tacit) hypothesis, H contains no normal subgroup of G. We infer from all this, that all the G-factors are isomorphic to K and none of them intersects non-trivially H. Say $G = A^d$, with A isomorphic to K

• H has $(d-1)$ factors, all embed in A, but since $\dim H = \dim G - \dim A = (d-1)\dim A$, each of these factors is isomorphic to A, that is H is isomorphic to K^{d-1}.

• To fix ideas, assume $d = 3$. So $G = K^3$, and $H \cong K^2$ embeds in K^3. This consists in two copies of K in K^3 which commutes. So their projections on each factor of K^3 commute. But such a projection is either $\{1\}$ or K. It cannot be K since K is not commutative. This implies these two copies cannot have same non-trivial projection on a factor of K^3. Hence at least one of these copies is a factor of K^3. But then, the isotropy H contains a normal subgroup of G which contradicts faithfulness. This argument applies in a similar way to any situation $d \geq 3$.

• It remains to consider the case $d = 2$, so $G = K \times K$, and K and H embeds "obliquely" in $K \times K$, and so each of them is the graph of a homomorphism $K \to K$. The same applies to their Lie sub-algebras. They are graphs in $\mathfrak{k} \oplus \mathfrak{k}$ of derivations $d_1, d_2 : \mathfrak{k} \to \mathfrak{k}$. The intersection of these graphs consists of vectors of the forms $u \oplus d_1(u) \in \mathfrak{k} \oplus \mathfrak{k}$, such that $d_1(u) = d_2(u)$. But $d_1 - d_2$ is a derivation of $\mathfrak{k}$, and since $\mathfrak{k}$ is semi-simple, $d_1 - d_2 = \mathsf{ad}_w$, for some $w \in \mathfrak{k}$. In particular $d_1(w) = d_2(w)$, and the two graphs have a non-trivial intersection. This contradicts that fact that G equals KH which implies the sub-algebras of K and H are transversal.

• All this implies that K is in fact a factor of G. So, all the other factors of G commute with K, and thus their action consist in right multiplication, and so, the G-action on K transits via $K \times K$ (up to a cover). ♢

3 Conformal Group, Proof of Theorem 2

Recall that $\mathsf{Eins}^{p,q}$ is the substratum of the flat conformal semi-Riemannian geometry of signature (p, q). One model of it can be defined as follows. Consider the pseudo-Euclidean space $\mathbb{R}^{p+1,q+1}$, and $\mathcal{C}^{p+1,q+1}$ its light cone (the space of isotropic vectors). Then, $\mathsf{Eins}^{p,q}$ is the quotient of the light cone by the radial $\mathbb{R}^+$-action.

One sees in particular that $\mathsf{Eins}^{p,q}$ has the topology of $\mathbb{S}^p \times \mathbb{S}^q$.

The orthogonal group $\mathsf{O}(p+1,q+1)$ of $\mathbb{R}^{p+1,q+1}$ acts conformally on $\mathsf{Eins}^{p,q}$, and in fact equals it full conformal group (for $p+q>2$).

A semi-Riemannian conformally flat manifold of signature (p,q) is modelled on $\mathsf{Eins}^{p,q}$, and conversely. In other words being conformally flat is equivalent of having a (G,X)-structure, for $X=\mathsf{Eins}^{p,q}$, and $G=\mathsf{O}(p+1,q+1)$. .

By the results of [3, 4, 5], if $\mathsf{Conf}(K,g)$ is essential, then (K,g) is conformally flat. So, we have a developing map $\tilde{K}\to\widetilde{\mathsf{Eins}^{p,q}}$, where (p,q), $q\geq p$, is the signature of g.

Since semisimple, K has a finite fundamental group, so up to a cover, we can assume K is simply connected.

The developing map is a local diffeomorphism, and K is compact and simply connected, if follows that it is a covering, and that K is the universal cover of $\mathsf{Eins}^{p,q}$. This implies that $p\neq 1$, since $\mathsf{Eins}^{1,q}$ has a non-compact universal cover, and that d is a diffeomorphism, since $\mathsf{Eins}^{p,q}$ is simply connected for $p\neq 1$.

So, from the topological viewpoint, K is a semi-simple Lie group diffeomorphic to $\mathbb{S}^p\times\mathbb{S}^q$ $p\leq q$).

As recalled in Section 2, a semi-simple Lie group K satisfies $\pi_2(K)=1$, and $\pi_3(K)\neq 0$ [6]. In fact, $\pi_3(K)=\mathbb{Z}^k$, where k is the number of simple factors of K.

It follows that either (p,q) is either $(0,3)$ or $p=3$ and $q\geq 3$. Let us consider the case $p=3, q\geq 3$, the other case being easier to handle. Therefore, either $q>3$, and then K is simple, or $q=3$, and K has two simple factors.

(K,g) is conformally isomorphic to $\mathsf{Eins}^{p,q}$, so its conformal group embeds in $\mathsf{O}(p+1,q+1)$. In particular, the K-left action on itself gives a transitive action on $\mathsf{Eins}^{p,q}$, say via an embedding $h:K\to\mathsf{O}(p+1,q+1)$. Up to conjugacy, h has values in the maximal compact subgroup $\mathsf{O}(p)\times\mathsf{O}(q)$. Write $h=(h_1,h_2)$.

Assume $q>3$, so $\pi_3(K)=\mathbb{Z}$ (since diffeomorphic to $\mathbb{S}^p\times\mathbb{S}^q$). Hence K is simple. Necessarily, one of the homomorphisms h_1 or h_2 is trivial. But then $h(K)$ does not act transitively on $\mathbb{S}^3\times\mathbb{S}^q$.

From this, we infer that $q=3$, and K has two simple factors. In this case $h=(h_1,h_2)$ maps K to $\mathsf{SO}(4)\times\mathsf{SO}(4)$. If K acts transitively on $\mathbb{S}^3\times\mathbb{S}^3$, then none of the h_i is trivial. Recall here that up to finite cover $\mathsf{SO}(4)=\mathsf{SO}(3)\times\mathsf{SO}(3)$. As $h(K)$ has exactly two simple factors, each of factors must be, up to finite covers, $\mathsf{SO}(3)$. One can also see that, up to finite covers, $h(K)$ is a product of two copies of $\mathsf{SO}(3)$, each contained in one factor $\mathsf{SO}(4)$ (of the maximal compact). $\square$

4 Non Riemannian case

Compact simply connected homogeneous semi-Riemannian manifolds were studied in [1] (and also the unpublished [2] which becomes then part of [1]). The principal result is stated as follows:

Theorem 4. *Let M be a connected and simply connected pseudo-Riemannian homogeneous space of finite volume, $G = \mathsf{Isom}(M)^\circ$, and let H be the stabilizer subgroup in G of a point in M. Let $G = CR$ be a Levi decomposition, where R is the solvable radical of G. Then:*

(1) M is compact.

(2) C is compact and acts transitively on M.

(3) R is abelian. Let A be the maximal compact subgroup of R. Then $A = \mathrm{Z}(G)^\circ$, the identity component of the center of G. More explicitly, $R = A \times V$ where $V \cong \mathbb{R}^n$ and $V^C = 0$ (that is the C-representation has no factor where it acts trivially).

(4) H is connected. If $\dim R > 0$, then $H = (H \cap C)E$, where E and $H \cap C$ are normal subgroups in H, $(H \cap C) \cap E$ is finite, and E is the graph of a non-trivial homomorphism $\varphi : R \to K$, where the restriction $\varphi|_A$ is injective.

4.1 Sketch

Let us give some hints on the proof of this result, especially the fact that G has no non-compact semisimple factor (Item (2) in the Theorem). So G acts on M transitively. For X in the Lie algebra $\mathfrak{g}$, let $\bar{X}$ be the associated vector field on M. For $x \in M$, we define a degenerate metric $m(x)$ on $\mathfrak{g}$, using the pull back by evaluation map $X \in \mathfrak{g} \to \bar{X}(x) \in T_xM$, that is $m(x)(X,Y) = g_x(\bar{X}(x), \bar{Y}(x))$ (g is the given semi-Riemannian metric on M). Let also $\mathfrak{h}(x)$ be the Lie algebra of the stabilizer of x. Observe that the Kernel of $m(x)$ is exactly $\mathfrak{h}(x)$. Like this, one define maps $m : M \to \mathsf{Sym}(\mathfrak{g})$ and $l : M \to \mathcal{L}(\mathfrak{g})$, where $\mathsf{Sym}(\mathfrak{g})$ is the space of quadratic forms on $\mathfrak{g}$ and $\mathcal{L}(\mathfrak{g})$ is the Grassmann of linear d-subspaces of $\mathfrak{g}$, where $d = \dim \mathfrak{h}(x)$ (for any x).

The point is that m and l are equivariant with respect to the action of G on M, and its natural action of $\mathsf{Sym}(\mathfrak{g})$ and $\mathcal{L}(\mathfrak{g})$.

Let us assume here that M is compact (instead of the slightly more general hypothesis M of finite volume as in the theorem).

The image $Z = m(M) \subset \mathsf{Sym}(\mathfrak{g})$ is in particular invariant under the linear G-action. So we are in a situation of a compact set Z in a linear space, say $\mathbb{R}^N$, invariant by a subgroup $G \subset \mathsf{GL}(N, \mathbb{R})$. Let $p(t) = e^{tA}$ be a one parameter group in G. Then, for any $z \in Z$, $p(t)z$ is bounded when $t \to \infty$. Assume A nilpotent, then $p(t) = 1 + tA + (t^2/2)A^2 + \ldots + (t/N!)A^N$. Clearly, $p(t)z$ bounded, implies $A(z) = 0$, that is $p(t)z = z, \forall t$. If S a subgroup of G is generated by such one parameter groups, with A nilpotent, then S acts trivially on Z. This applies in particular to the semi-simple factor of G of non-compact type, as well as to the nilradical of G.

The case of the map l is more complicated since it has values in a Grassmann space V, which is compact. In this case, one uses another dynamical idea. Again assuming A nilpotent, then a point v is recurrent if there is a sequence $t_n \to \infty$, such that $p(t_n)v \to v$. One concludes in this case that $p(t)v = v$. Since the G-action on M preserves the semi-Riemannian measure, there is a G-invariant measure with full support in the image of l. By Poincaré recurrence Theorem, almost all points are recurrent. Therefore, we have the same conclusion that S acts trivially on the image of l, once it is generated by one parameter groups with nilpotent infinitesimal generator.

Let S be a semi-simple factor of non-compact type, and $\mathfrak{s}$ its Lie algebra. The last conclusion translates in terms of brackets to $[\mathfrak{s}, \mathfrak{h}(x)] \subset \mathfrak{h}(x)$ (for any x). By considering the projection $\pi : \mathfrak{g} \to \mathfrak{s}$, one sees that if the projection of $\mathfrak{h}(x)$ is non-trivial, then this projection is an ideal of $\mathfrak{s}$. In particular, since there are only finitely many ideals of $\mathfrak{s}$, we get a factor $\mathfrak{l}$ contained in the projection of all $\mathfrak{h}(x)$, $\forall x$. By semi-simplicity, this gives $\mathfrak{l}' \cong \mathfrak{l}$, a subalgebra of $\mathfrak{g}$ contained in all the $\mathfrak{h}(x)$, contradicting the faithfulness of the G-action. Therefore $\mathfrak{h}(x)$ is contained in $\mathfrak{c} + \mathfrak{r}$, where $\mathfrak{c}$ is a compact semi-simple factor and $\mathfrak{r}$ is the radical.

At the Lie group level, let $G = S.C.R$, then, for the isotropy $H \subset C.R$. So, we have a well defined map $M = G/H = (S.C.R)/H \to S$. By compactness of M, S must be trivial. □

4.2 Simple case

Recall the result of [15]:

Theorem 5. *[15] A left invariant semi-Riemannian metric on a compact simple group, has a compact isometry group.*

We will deduce this result from Theorem 4, without using neither [10, 7] nor [13].

Proposition 1. *If* $\dim R > 0$*, then there is no simple subgroup* $K \subset G$ *which acts transitively on* M*.*

Proof. Let C_0 be the kernel of the representation of C in V. This is a normal subgroup of C and we have a splitting $C = C_0 C_1$. From Proposition 9. 6 (see also the proof of Lemma 10.3) of [1], we have that $H \subset C_0 R$. In particular if $C_0 = 1$, then $H \subset R$, but this is impossible since C acts transitively on M, unless $H = R$, which is also impossible since the G action is (tacitly!) assumed to be faithful.

Assume now that G contains a simple Lie group K acting transitively on M. Then up to conjugacy, K is a subgroup of C, and by simplicity, it is either in C_0 or in C_1.

Assume $K \subset C_1$. Consider $G_1 = K \ltimes R$. The isotropy $H_1 = G_1 \cap H$ is contained in R, since $H \subset C_0 R$. But K acts transitively on M, so the isotropy H_1 must be equal to R, which contradicts faithfulness.

Therefore $K \subset C_0$. Consider the direct product $G_2 = K \times R$. So, on M, the R-action commutes with the transitive action of K.
If the K-action on M is free, in which case M is identified to K acting by left translation on itself, then R must act on the right via a homomorphism in K. So the $K \times R$-action extends to a $K \times K$-action, and thus the isometry group is compact. Finally, the case where the K-action on M is not free, works similarly, with a slightly more complicated notations. QED

5 Non-simple examples

5.1 Riemannian Non-simple example

[14] Let L be any group. Embed it in L^3 as $A = \{(x, x, 1)/x \in L\}$, and embed L^2 as $B = \{(x, y, x)/l_1, l_2 \in L\}$. Any element of L^3 can be uniquely decomposed as a product of an element of A and an element of B. So L^3 is acts on L^3/A, which is identified to B, and on L^3/B which is identified to A. Observe however that the L^3-action on A is not faithful. The L^3-action on $B \cong L^2$ is however faithful and this supergroup of $B = L^2$ is not contained (up to covers) in $B \times B$.

5.2 Non-Riemannian non-simple example

[1] Let $G_1 = (\widetilde{\mathsf{SO}}(3) \ltimes \mathbb{R}^3) \times \mathbb{T}^3$, where $\widetilde{\mathsf{SO}}(3)$ acts on $\mathbb{R}^3$ by the adjoint representation ($\mathbb{R}^3 \cong \mathfrak{so}(3)$) and let $\Phi : \mathbb{R}^3 \to \mathbb{T}^3$ be a homomorphism with discrete kernel. Note V_1 (resp. V_0) the Lie algebra of $\mathbb{R}^3$ (resp. $\mathbb{T}^3$).

Put $H = \{(1, v, \Phi(v)) \mid v \in \mathbb{R}^3\}$. Its Lie sub-algebra is $\mathfrak{h} = \{(0, v, \varphi(v)), v \in V_1\}$, where $\varphi : V_1 \to V_0$ is the derivative of Φ. Define a pseudo-product $\langle, \rangle$ on $\mathfrak{g}_1$ by:

- $\mathfrak{so}(3)$ and $V_0 \oplus V_1$ are (totally) isotropic.

- if $u \in \mathfrak{so}(3), v_0 \in V_0, v_1 \in v_1$, then $\langle u, v_0 + v_1 \rangle = \kappa(u, v_1 + \varphi^{-1}(v_0))$, where κ is the Killing form of $\mathfrak{so}(3)$, and V_0 and V_1 are identified to $\mathfrak{so}(3)$.

One can check that the kernel of this product is exactly $\mathfrak{h}$, and that the so defined product on $\mathfrak{g}_1/\mathfrak{h}$ has signature $(3, 3)$. Also, this product is $\mathsf{Ad}(H)$-invariant. All these properties are a particular case of the following general construction. Let L any group, with $\mathfrak{l}$ its Lie algebra and $\mathfrak{l}^*$ its dual. Consider the semi-direct product $P = L \ltimes \mathfrak{l}^*$. Its Lie algebra as a vector space is $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{l}^*$.

The paring of $\mathfrak{l}$ and its dual $\mathfrak{l}^*$, that is $k(x,\alpha)=\alpha(x), x\in\mathfrak{l}, \alpha\in\mathfrak{l}^*$, determines a pseudo-scalar product on $\mathfrak{p}$ of signature (d,d), $d=\dim L$, which is in fact $\mathsf{Ad}(P)$-invariant.

From this scalar product on $\mathfrak{g}_1/\mathfrak{h}$, we get a G_1-invariant semi-Riemannian metric of signature $(3,3)$ on $M_1=G_1/H$. This M_1 is identified to $\widetilde{\mathsf{SO}}(3)\times\mathbb{T}^3$.

and thus obtain a G_1-invariant pseudo-Riemannian metric of signature $(3,3)$ on the quotient $M_1=G_1/H=\widetilde{\mathsf{SO}}(3)\times\mathbb{T}^3$. Here, M_1 is a non-simply connected manifold with a non-compact connected stabilizer.

In order to obtain a simply connected example, embed $\mathbb{T}^3$ in a simply connected compact semisimple group C_0, for example $C_0=\widetilde{\mathsf{SO}}(6)$, so that G_1 is embedded in $G=(\widetilde{\mathsf{SO}}(3)\ltimes\mathbb{R}^3)\times C_0$.

The previously defined scalar product on $\mathfrak{g}_1$ can be extended to $\mathfrak{g}$ as follows. Choose $\mathfrak{t}'\subset\mathfrak{c}_0$, as a $\mathbb{T}^3$-invariant supplementary subspace of the Lie algebra of $\mathbb{T}^3$ in that of C_0, and endow it with a positive scalar product. Then, equip $\mathfrak{g}=\mathfrak{g}_1\oplus\mathfrak{t}'$, with the direct sum of scalar products. This is $\mathsf{Ad}(H)$-invariant. We therefore get $M=G/H\cong\widetilde{\mathsf{SO}}(3)\times C_0$. Therefore $\widetilde{\mathsf{SO}}(3)\times\widetilde{\mathsf{SO}}(6)$ admits a left invariant semi-Riemannian metric having a non-compact isometry group.

6 More results and questions

Let us end with the following questions, some of which are good exercises.

6.0.1 Finite isometry groups

For a compact homogeneous semi-Riemannian manifold, the isometry group has finitely many connected components. Observe that all results here concern the identity component. For instance, for a simple group K, with a left invariant metric g, a priori, it might happen that the isotropy at the identity contains a finite group acting by isometry that are not automorphisms?

6.0.2 Non-simple case

There is in fact in [1] more details about isometry groups of compact simply connected semi-Riemannian spaces, which might allow one to an optimal classification of compact simply connected homogeneous semi-Riemannian manifolds, in particular in the case where M is identified to a compact semi-simple Lie group. Also Ozeki's [14] and Koszul's [11] results might be helpful in this regard.

6.0.3 Non-group case

Our proof of Proposition 1 applies also to semi-Riemannian homogeneous spaces of simple groups, that is, manifolds $M = K/P$, where K is a compact simple group. Their isometry group is compact.

6.0.4 Non semi-simple case

The conformal Theorem 2 generalizes, up to a slight modification, to the case where K is compact but not necessarily semi-simple. So K is, up to a finite cover, the product of a semi-simple by a torus. As example, we have $\mathsf{SO}(2) \times \mathsf{SU}(2)$ whose conformal group is $\mathsf{SO}(2,4)$.

6.0.5 The non-compact Riemannian case

For a semi-simple group S endowed with a left invariant Riemannian metric g, S is co-compact in $\mathsf{Isom}(S,g)$. But if S contains no compact factor, then it is cocompact only in groups of the form $S \times L$, with L compact. In particular S cannot be co-compact in another different semi-simple group without compact factors (see [9] for proofs)

6.0.6 The non-compact semi-Riemannian case

If S is simple non-compact, a semi-Riemannian left invariant metric can have a large isometry group, say where S is not compact, the is the isotropy (at the identity) is not compact. As example, the Killing form, determines a bi-invariant metric, the identity component of its isometry group is $S \times S$, modulo the center. Here, one can ask if it is a maximally symmetric metric as in Theorem 1.

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