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Commuting statistical Jacobi operators

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Abstract. This paper extends the classical theory of Jacobi operators to statistical manifolds, integrating concepts from differential and information geometry. We analyze the commutation properties of statistical Jacobi operators and establish their implications for the geometry of statistical hypersurfaces. By generalizing results on commuting curvature operators, we derive new insights into the structure of statistical manifolds. Our findings contribute to a deeper understanding of the interplay between curvature, shape operators, and statistical connections.

Keywords: Jacobi operators, statistical manifolds

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Introduction

Jacobi operators play a fundamental role in differential geometry by characterizing the behavior of geodesics under curvature influences. Their extension to statistical manifolds provides a novel framework for analyzing geometric structures arising in probability theory, information geometry, and optimization. The pioneering work of Amari [1] established the foundations of statistical manifolds, introducing the concept of dual connections that encode statistical dependencies geometrically. They enable deeper insights into machine learning, probability, and information theory, extending classical differential geometry into the realm of statistical inference., provide a natural framework for analyzing statistical models using differential geometry. The interplay between the curvature structure of statistical manifolds and classical geometric operators is crucial in fields such as machine learning, probability theory, and mathematical physics. By extending Jacobi operator results to statistical manifolds, we gain deeper insights into their structure and potential applications. This work is motivated by previous research on commuting curvature operators, particularly the studies of Tsankov and Brozos-Vázquez and Gilkey, who examined classical Jacobi operators. We aim to generalize these results to statistical settings, showing that the commutativity of statistical Jacobi operators provides strong curvature constraints on the underlying manifold.

For a Riemannian curvature tensor R, Jacobi curvature operator J_X, which acts on a tangent vector V is defined as:

$$J_X(V) = R(V, X, X).$$

Brozos-Vázquez and Gilkey ([2]) established conditions under which commuting Jacobi operators imply constant sectional curvature. In the work of Tsankov [7] it is shown that if the classical Jacobi curvature operators J_X and J_Y commute on a hypersurfaceM^n for all orthonormal pairs (X, Y) and all points $p \in M^n$, then M^n must have constant sectional curvature.
A statistical manifold (M, g, ∇) consists of a Riemannian metric g and a torsion-free affine connection ∇, with a dual connection ∇^* satisfying

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z).$$

The curvature tensors R and R* associated with ∇ and ∇^* satisfy:

$$R(X, Y) + R^*(X, Y) = 2R_0(X, Y) + 2[K, K](X, Y),$$

$$R(X, Y) - R^*(X, Y) = 2dK^\flat(X, Y),$$

where R_0 denotes the classical Riemannian curvature tensor, $[K, K]$ encodes affine deformation effects, and $dK^\flat$ captures the differential of the shape operator.
Recent advances have connected dually flat statistical manifolds to optimization algorithms, Fisher-Rao geometry, and deep learning architectures, making curvature properties highly relevant in both theoretical and applied machine learning contexts. A dually flat manifold is a special type of statistical manifold characterized by the existence of a pair of dual affine connections, ∇ and ∇^*, that are both flat, i.e., their respective curvature tensors vanish ($R = 0$ and $R^* = 0$). These manifolds play a central role in information geometry, as they naturally arise in contexts such as exponential families and Bregman divergences. A key feature of dually flat manifolds is the existence of a globally defined potential function, ψ, such that the Riemannian metric g can be expressed as the Hessian of ψ, $g_{ij} = \frac{\partial^2 \psi}{\partial \theta^i \partial \theta^j}$, in the ∇-flat coordinate system. The dual coordinate system, associated with ∇^*, is related to the gradient of ψ, facilitating a dualistic structure that supports the study of optimization, statistical inference, and divergence measures in a geometrically consistent framework.
We will show that equality of dual statistical Jacobi operators implies that the ambient manifold is conjugate symmetric, which is on the other hand generalization of dually flat.

Opozda ([6]) introduced the notion of sectional curvature for statistical structures, paving the way for further exploration. For examining statistical hypersurfaces in holomorphic statistical manifolds, we will use Opozda's definition. In [4], the nonexistence of hypersurfaces satisfying $AX = \alpha X$ and $A^*X = \beta X$ is shown. Motivated by this result, in [5], the case with three constant eigenvalues is studied, and the information on holomorphic sectional curvature is obtained. We relate this result with commutativity of Jacobi operators condition, and show that two cases for holomorphic sectional curvature c are possible: $c = 0$ and $c < 0$. Our objective is to examine statistical Jacobi operators defined as:

$$J_X(V) = R(V, X, X), \quad J^*_X(V) = R^*(V, X, X).$$

We establish conditions under which these operators commute and explore their implications for the geometry of statistical hypersurfaces.
Our results provide a foundation for further exploration in multiple directions, particularly in the intersection of differential geometry, machine learning, and information theory. The connection between commuting statistical Jacobi operators and recent advances in optimization algorithms, neural network architectures, and statistical inference opens new avenues for research.
Structure of the paper is the following. Section 2 presents necessary background concepts and definitions related to statistical manifolds. Section 3 introduces statistical Jacobi operators and their commutativity properties. Section 4 discusses applications and open questions, including implications for optimization and machine learning.

1 Preliminaries

Given a Riemannian n-dimensional manifold (M, g) with Levi-Civita connection ∇^0, a symmetric TM-valued (0,2)-tensor $K : TM^2 \to TM$ defines a torsionless connection ∇:

$$\nabla_X Y = \nabla^0_X Y + K(X, Y),$$

where $K(X, Y) = K(X, Y)$.

The connection is statistical if the (0,3)-tensor defined by the contraction with the metric,

$$C(X, Y, Z) := g(K(X, Y), Z),$$

is symmetric in all three variables.

Then K is totally symmetric, and the dual connection ∇^*, called the conjugate connection, is given by:

$$\nabla^*_X Y = \nabla^0_X Y - K(X,Y).$$

The following relations hold:

$$\nabla_X g(Y,Z) = -2C(X,Y,Z) = -\nabla^*_X g(Y,Z),$$

$$K(X,Y) = \frac{1}{2}(\nabla_X Y - \nabla^*_X Y),$$

$$\nabla^0_X Y = \frac{1}{2}(\nabla_X Y + \nabla^*_X Y),$$

The curvature tensors for connections ∇ and ∇^* are respectively given by:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

and

$$R^*(X,Y)Z = \nabla^*{}_X \nabla^*{}_Y Z - \nabla^*{}_Y \nabla^*{}_X Z - \nabla^*{}_{[X,Y]} Z.$$

We define the following tensors:

$$R(X,Y,Z,W) = g(R(X,Y)W,Z)$$

and

$$R^*(X,Y,Z,W) = g(R^*(X,Y)W,Z)$$

Associated to K is the tensor $[K,K] : TM^2 \to L(TM;TM)$ defined by

$$[K,K](X,Y)Z := [K_X, K_Y](Z),$$

and $K^\flat : TM \to L(TM;TM)$ defined by musical isomorphism

$$K^\flat(X)(Y) = K(X,Y) = K_X Y.$$

Thus $dK^\flat$ (with respect to ∇^0) is the $L(TM;TM)$-valued 2-form on M given by

$$dK^\flat(X,Y) = \nabla^0_X K^\flat(Y) - \nabla^0_Y K^\flat(X).$$

The curvature tensors R and R^* of a pair of dual statistical connections (∇, ∇^*) on (M,g) satisfy the following relations:

$$R(X,Y) + R^*(X,Y) = 2R^0(X,Y) + 2[K,K](X,Y);$$

$$R(X,Y) - R^*(X,Y) = 2\nabla^0_X K(Y,\cdot) - 2\nabla^0_Y K(X,\cdot) = 2dK^\flat(X,Y);$$

$$R(X,Y,Z,W) = -R^*(X,Y,W,Z).$$

In particular, we may conclude:

$$R = R^0 + [K,K] + dK^\flat;$$

$$R^* = R^0 + [K,K] - dK^\flat.$$

The curvatures R, R^* and $[K,K]$ satisfy the following:

$$R(X,Y,Z,W) = -R(Y,X,Z,W);$$

$$R(X,Y,Z,W) + R(W,X,Z,Y) + R(Y,W,Z,X) = 0;$$

$$R(X,Y,Z,W) = -R^*(X,Y,W,Z);$$

$$[K,K](X,Y,Z,W) = -[K,K](Y,X,Z,W);$$

$$[K,K](X,Y,Z,W) + [K,K](W,X,Z,Y) + [K,K](Y,W,Z,X) = 0;$$

$$[K,K](X,Y,Z,W) = -[K,K](X,Y,W,Z).$$

Definition 1. (M,g,∇) is called a statistical manifold if:

(1) ∇ is torsion-free, and

(2) $(\nabla_X g)(Y,Z) = (\nabla_Y g)(X,Z)$ for $X,Y,Z \in \Gamma(TM)$.

∇^* is called the dual connection of ∇ with respect to g if

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z), \quad X,Y,Z \in \Gamma(TM).$$

We remark that if (M,g,∇) is a statistical manifold, so is (M,g,∇^*).

Definition 2. Let (M,g,∇) be a statistical manifold. For $X,Y,Z,V \in \Gamma(TM)$, we define

$$S(X,Y)Z := \frac{1}{2}\{R(X,Y)Z + R^*(X,Y)Z\}$$

and

$$S(X,Y,Z,V) := g(S(X,Y)Z,V).$$

Definition 3. A statistical manifold (M, g, ∇) is said to be of constant curvature $c \in \mathbb{R}$ if

$$S(X,Y)Z = c\,(g(Y,Z)X - g(X,Z)Y)\,, \quad X,Y,Z \in \Gamma(TM),$$

where $R^{\nabla}(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$.

A statistical manifold (M, ∇, g) is of constant curvature c if and only if (M, ∇^*, g) is of constant curvature c.

Definition 4.

Let (M, g, J) be a Kähler manifold and ∇ an affine connection on M. The manifold (M, g, ∇, J) is called a holomorphic statistical manifold if:

(1) (M, g, ∇) is a statistical manifold, and

(2) $\omega := g(\cdot, J\cdot)$ is a ∇-parallel 2-form on M.

We remark that for $X, Y \in \Gamma(TM)$, $\nabla_X(JY) = J\nabla^*_X Y$ holds in the holomorphic statistical manifolds case.

Definition 5. A holomorphic statistical manifold (M, g, ∇, J) is said to be of constant holomorphic sectional curvature c if

$$\begin{aligned} S(X,Y)Z = \frac{c}{4}\{&g(Y,Z)X - g(X,Z)Y \\ &+ g(JY,Z)JX - g(JX,Z)JY + 2g(X,JY)JZ\} \end{aligned}$$

for $X, Y, Z \in \Gamma(TM)$.

Statistical hypersurfaces. Let $(\overline{M}, \overline{g}, \overline{\nabla}, J)$ be a $2m$-dimensional holomorphic statistical manifold and let M be a hypersurface of $\overline{M}$. With ξ we denote a unit normal vector field of M. By U we denote a structure (tangent) vector field, defined by

$$J\xi = -U.$$

For $X \in \Gamma(TM)$, we have the following decomposition:

$$JX = PX + u(X)\xi.$$

Here, P is an endomorphism acting on TM and u is one-form on M that satisfy:

$$P^2X = -X + u(X)U;$$

$$g(X,U) = u(X), \quad g(U,U) = 1;$$

$$PU = 0, \quad u(PX) = 0.$$

The shape operators of M are defined by ([8]):

$$g(A_V X, Y) = \overline{g}(h^*(X, Y), V) \quad g(A^*_V X, Y) = \overline{g}(h(X, Y), V),$$

$X, Y \in \Gamma(TM)$, $V \in \Gamma^{\perp}(TM)$.
The Gauss and Weingarten equations are given by:

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

and

$$\overline{\nabla}_X \xi = -AX + s(X)\xi,$$

respectively.
Their dual equations are:

$$\overline{\nabla}^*_X Y = \nabla^*_X Y + h^*(X, Y),$$

and

$$\overline{\nabla}^*_X \xi = -A^* X + s^*(X)\xi.$$

One-forms s and s^* satisfy

$$s(X) = -s^*(X).$$

When an ambient manifold is of constant holomorphic sectional curvature c, the Gauss equation is:

$$\begin{aligned} S(X, Y)Z = {} & \frac{1}{2}\{g(A^*Y, Z)AX - g(A^*X, Z)AY + g(AY, Z)A^*X \\ & - g(AX, Z)A^*Y\} + \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y \\ & + g(PY, Z)PX - g(PX, Z)PY + 2g(X, PY)PZ\}. \end{aligned}$$

2 Statistical Jacobi operators

Let (M, g, ∇) be a statistical manifold.

Definition 6. On a statistical manifold M, K-Jacobi operator is defined by:

$$\mathcal{J}_K(X, Y) := [K, K](Y, X)X.$$

Definition 7. On M, a statistical Jacobi operator is defined as:

$$\mathcal{J}(X, Y) := R(Y, X)X = R^0(Y, X)X + \nabla^0_Y K(X, X) - \nabla^0_X K(Y, X) + \mathcal{J}_K(X, Y).$$

Following ideas of [3] (Lemma 2.2), we can prove the following two assertions.

Proposition 1. *A statistical manifold* (M, g, ∇) *is conjugate symmetric if and only if* $\mathcal{J} = \mathcal{J}^*$.

Proof. From the assumption that $\mathcal{J} = \mathcal{J}^*$, it follows

$$R(Z, X, Y, W) + R(Z, Y, X, W) = R^*(Z, X, Y, W) + R^*(Z, Y, X, W).$$

By (1), this is equivalent to

$$-R^*(Z, X, W, Y) - R^*(Z, Y, W, X) = -R(Z, X, W, Y) - R(Z, Y, W, X).$$

Combining this equation with

$$R(Z, X, W, Y) - R^*(Z, X, W, Y) = R^*(Z, W, X, Y) - R(Z, W, X, Y),$$

we conclude

$$R^*(Z, W, X, Y) - R^*(Z, Y, W, X) = R(Z, W, X, Y) - R(Z, Y, W, X).$$

Now, since $R(Z, W, X, Y) = -R^*(Z, W, Y, X) = R^*(W, Z, Y, X)$, we conclude

$$R^*(Z, W, X, Y) + R^*(Y, W, Z, X) = -R(Z, Y, W, X),$$

because of the first Bianchi identity.
Finally, since $R^*(Z, W, X, Y) = -R(Z, W, Y, X) = R(W, Z, Y, X)$, we have the assertion.
The opposite direction is obvious. QED

Proposition 2. *For a statistical Jacobi operator* $\mathcal{J}$ *and for a* K*-Jacobi operator* $\mathcal{J}_K$*, the following holds.*

(1) $\mathcal{J} = 0$ *if and only if* $R = 0$.

(2) $\mathcal{J}_K = 0$ *if and only if* $[K, K] = 0$.

Definition 8. We define a statistical Jacobi curvature operator with respect to S as:

$$\mathcal{J}_X^S(Y) := S(Y, X, X).$$

In the following let M be a hypersurface of dimension ≥ 3 in a holomorphic statistical manifold $\overline{M}$ of constant holomorphic sectional curvature c whose shape operators satisfy

$$AU = \alpha U, \quad AV = \beta V, \quad A^*X = \gamma X,$$

for all $X \in \Gamma(TM)$, $V \in U^\perp$, $\alpha \neq \beta$. In [5] we have shown the next assertion.

Lemma 1. *Let $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha \neq \beta$. Then one of the following holds:*

- $c = 0$, $A^* = 0$, $D_X\xi = 0$, $\beta = 0$,
- $c = 0$, $A^* = 0$, $D_V\xi = 0$ *for* $V \in U^\perp$, $\beta = 0$,
- $c = 2\gamma(\beta - \alpha) \neq 0$, $\beta = \gamma$, $D_X\xi = 0$,
- $c = 2\gamma(\beta - \alpha) \neq 0$, $\alpha = \gamma$,$D_X\xi = 0$,
- $c = 2\gamma(\beta - \alpha) \neq 0$, $\beta = \gamma$, $D_V\xi = 0$ *for* $V \in U^\perp$.

Proposition 3. *Let on M the condition $\mathcal{J}^S_U \mathcal{J}^S_X = \mathcal{J}^S_X \mathcal{J}^S_U$ holds for $X \in U^\perp$. Let the eigenvalues satisfy $\beta\gamma \neq 0$. Then one of the following holds:*

(1) $c = 0$;

  a. $\alpha = 0$;
  b. $\alpha = -\beta$;
  c. $\alpha = \gamma$.

(2) $c = -\beta\gamma$;

  a. $\alpha = -\frac{\beta}{2}$;
  b. $\alpha = \frac{\gamma \pm \sqrt{\gamma^2 + 2c}}{2}$ *and* $c < 0$, $|\gamma| > \sqrt{-2c}$.

Proof. For $Z \in \Gamma(TM)$, using the commutativity assumption, we obtain

$$\frac{c}{4}\gamma AZ - \frac{c}{4}\beta\gamma g(Z,X)X - \frac{\alpha^2}{4}(\gamma\alpha + \beta\gamma + \frac{c}{2})g(Z,U)U - \frac{\alpha\gamma}{4}(\alpha\gamma + \beta\gamma)g(Z,U)U$$
$$+ \frac{c}{8}\beta\gamma Z + \frac{c^2}{16}Z - \frac{c^2}{16}g(Z,X)X + \frac{3c^2}{16}g(Z,PX)PX - \frac{c}{8}\gamma\alpha g(Z,U)U$$
$$= -\frac{1}{2}\gamma^2\alpha^2 g(Z,U)U - \frac{1}{2}\alpha\gamma^2\beta g(Z,U)U + \frac{c}{8}\gamma AZ - \frac{c}{4}\alpha\gamma g(Z,U)U.$$

We replace $Z = PX$ in this equation, and obtain

$$\frac{c}{4}\beta\gamma PX + \frac{c^2}{4}PX = 0,$$

from which we conclude $c = 0$ or $c = -\gamma\beta$.
On the other hand, if we replace $Z = U$, we obtain

$$-\frac{\alpha^3\gamma}{4} - \frac{\beta\gamma\alpha^2}{8} + \frac{\alpha^2}{\gamma^2} - \frac{\beta^2\gamma^2}{16} = 0,$$

or

$$(\alpha + \frac{\beta}{2})\frac{\gamma}{4}(-\alpha^2 + \gamma\alpha - \frac{\gamma\beta}{2}) = 0.$$

From this we conclude $\alpha = -\frac{\beta}{2}$ or $\alpha = \frac{\gamma \pm \sqrt{\gamma^2 + 2c}}{2}$ and $c < 0$. In the special case when $\gamma^2 = -2c$, we conclude that α, β, γ are constants and $\alpha = \frac{\gamma}{2} = \beta$, which is not possible. Therefore, $\gamma^2 > -2c$, i.e. $|\gamma| > \sqrt{-2c}$. QED

Proposition 4. *Let $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha \neq \beta$. Let $\mathcal{J}_X^S \mathcal{J}_Y^S = \mathcal{J}_Y^S \mathcal{J}_X^S$ for $X, Y \in U^\perp$. Then $c = -\beta\gamma$.*

Proof. We assume that $\mathcal{J}_X^S \mathcal{J}_Y^S(Z) = \mathcal{J}_Y^S \mathcal{J}_X^S(Z)$ for X and Y in $U^\perp$ and $Z \in \Gamma(TM)$. Using the Definition 8 and eigenvalues assumption, we obtain

$$\begin{aligned}
&(\beta^2\gamma^2 + \frac{c}{2}\beta\gamma + \frac{c^2}{16})g(Z,X)g(X,Y)Y - (\frac{3c}{4}\beta\gamma + \frac{3c^2}{16})g(Z,PX)g(PX,Y)Y \\
&- (\frac{3c}{4}\beta\gamma + \frac{3c^2}{16})g(Z,X)g(X,PY)PY + \frac{9c^2}{16}g(Z,PX)g(PX,PY)PY \\
&= (\beta^2\gamma^2 + \frac{c}{2}\beta\gamma + \frac{c^2}{16})g(Z,Y)g(X,Y)X - (\frac{3c}{4}\beta\gamma + \frac{3c^2}{16})g(Z,PY)g(PY,X)X \\
&+ (\frac{3c}{4}\beta\gamma + \frac{3c^2}{16})g(Z,Y)g(PY,X)PX - \frac{9c^2}{16}g(Z,PY)g(P^2Y,X)PX.
\end{aligned}$$

Now, we replace Z by PX and multiply the obtained equation by X. The result is

$$(c + \beta\gamma)^2 g(PX,Y)g(X,Y) = 0.$$

From this we conclude $c = -\beta\gamma$. QED

Theorem 1. *Let $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha \neq \beta$. Let $\mathcal{J}_X^S \mathcal{J}_Y^S = \mathcal{J}_Y^S \mathcal{J}_X^S$ for $X, Y \in U^\perp$. Then one of the following holds:*

- $c = 0$, $A^* = 0$, $D_X\xi = 0$, $\beta = 0$,
- $c = 0$, $A^* = 0$, $D_V\xi = 0$ *for* $V \in U^\perp$, $\beta = 0$,
- $c < 0$, $\alpha = \pm\frac{3}{2}\sqrt{-c}$, $\beta = \gamma = \pm\sqrt{-c}$, $D_X\xi = 0$,
- $c < 0$, $\alpha = \gamma = \pm\sqrt{-\frac{3}{2}c}$, $\beta = \pm\sqrt{-\frac{2}{3}c}$, $D_X\xi = 0$,
- $c < 0$, $\alpha = \pm\frac{3}{2}\sqrt{-c}$, $\beta = \gamma = \pm\sqrt{-c}$, $D_V\xi = 0$ *for* $V \in U^\perp$.

Proof. The theorem follows from Lemma 1 and Propositon 4. QED

Theorem 2. *Let* $\alpha, \beta, \gamma \in \mathbb{R}$ *with* $\alpha \neq \beta$, $\beta\gamma \neq 0$. *Let* $\mathcal{J}_U^S \mathcal{J}_X^S = \mathcal{J}_X^S \mathcal{J}_U^S$ *for* $X \in U^{\perp}$. *Then* $c < 0$, $c = \frac{-\gamma^2 \pm \gamma\sqrt{\gamma^2+2c}}{3}$, $\alpha = \frac{\gamma \pm \sqrt{\gamma^2+2c}}{2}$, $\beta = \frac{2}{3}\alpha$, $|\gamma| > \sqrt{-2c}$.

Proof. The theorem follows from Lemma 1 and Propositon 3. QED

3 Conclusion and open problems

We have introduced the notion of commuting statistical Jacobi operators, extending classical results by incorporating statistical manifold structures. This refinement provides new insights into the interplay between curvature and statistical geometry, offering a broader framework for understanding geometric constraints in information geometry and related fields, and established conditions under which these operators provide constraints on the geometry of statistical hypersurfaces. Our results extend classical findings in Riemannian geometry to statistical settings, highlighting new structural properties of statistical curvature.

We give some open problems for future research:

(1) How do commuting statistical Jacobi operators impact optimization techniques in machine learning?

(2) What role do these operators play in defining new statistical divergences?

(3) Can computational algorithms be developed to efficiently verify commutativity conditions in practical applications?

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