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On exact locally conformally Kähler manifolds

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Abstract. In this article, we explore a distinguished class of complex manifolds known as d_θ-exact locally conformally Kähler (LCK) manifolds. These manifolds are characterized by the property that their fundamental 2-form ω can be expressed as $\omega = d_\theta \alpha$, where α is a 1-form on M and $d_\theta = d + \theta\wedge$. We establish a key result: if the 1-form α is holomorphic, then the Morse-Novikov cohomology $H^*_\theta(M)$ vanishes. Furthermore, we provide sufficient conditions under which a d_θ-exact LCK manifold admits a Vaisman structure. This work deepens the understanding of the interplay between geometric structures, cohomological properties, and special classes of LCK manifolds.

Keywords: Locally conformally Kähler manifolds, d_θ-exact LCK manifolds, Lee form, Morse-Novikov cohomology, Vaisman manifolds

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1 Introduction and main results

A Hermitian manifold (M, J, g) is called a **locally conformally Kähler (LCK) manifold** if it admits a closed 1-form θ, known as the **Lee form**, such that the fundamental 2-form $\omega(.,.) := g(J.,.)$ satisfies the integrability condition:

$$d\omega = \theta \wedge \omega.$$

LCK manifolds were introduced by I. Vaisman in the late 1970's as part of an effort to extend fundamental results from Kähler geometry to a broader class of manifolds. However, it soon became evident that many classical properties of Kähler geometry hold in the LCK setting only within certain restricted subclasses. A particularly notable example is the **Vaisman manifold** [7], an LCK manifold in which the Lee form remains parallel with respect to the Levi-Civita connection. These manifolds exhibit a version of the Hodge decomposition theorem for their cohomology groups, making them structurally closer to Kähler manifolds. Despite their importance, Vaisman structures are not stable under small deformations, limiting their applicability in the broader framework of LCK geometry.

To overcome this limitation, Ornea and Verbitsky introduced the class of **LCK manifolds with potential** [6], which encompasses Vaisman manifolds as a special case. This class remains stable under small deformations and allows embeddings into Hopf manifolds, significantly broadening the framework of LCK geometry.

For any LCK manifold, there exists an associated cohomology class known as the **Morse-Novikov class** $[\omega]_{MN}$, defined with respect to the well-known **Morse-Novikov differential** $d_\theta := d - \theta \wedge \cdot$, where d is the exterior derivative. This leads to the **Morse-Novikov cohomology**, associated with the cochain complex $(\Omega^*(M), d_\theta)$. Ornea and Verbitsky conjectured that if $[\omega]_{MN} = 0$, then M admits an LCK structure with potential. This conjecture was closely related to a twisted version of the $\partial_\theta\bar{\partial}_\theta$-lemma, analogous to its counterpart in Kähler geometry. However, Goto refuted this conjecture [10], demonstrating that LCK manifolds with potential form a strict subclass of LCK manifolds.

The conjecture and its refutation led to the identification of a new subclass: **d_θ-exact LCK manifolds**, where the fundamental 2-form satisfies $\omega = d_\theta\alpha$, for some 1-form α on M. These manifolds are of particular interest as they retain certain structural properties that merit further investigation. Notably, the subclasses of Vaisman manifolds, LCK manifolds with potential, and exact LCK manifolds are distinct and generally do not coincide.

In this paper, we provide sufficient conditions under which an exact LCK structure is Vaisman. In particular, we prove the following result which establishes a connection between curvature conditions and the Vaisman property in the context of exact LCK manifolds.

Theorem 1. *Let (M, J, ω) be a compact exact LCK manifold with $\omega = d_\theta\alpha$. If the Ricci curvature is non-negative and vanishes in the direction of α, then the LCK structure must be Vaisman.*

Finally, we prove that the Morse-Novikov cohomology of a compact exact LCK manifold with a holomorphic 1-form vanishes. This result is closely related to the geometric structure of the manifold.

Theorem 2. *Let (M, J, ω) be a compact exact LCK manifold with $\omega = d_\theta\alpha$, where $d_\theta = d + \theta\wedge$. If the 1-form α is holomorphic, then the Morse-Novikov cohomology vanishes:*

$$H^*_\theta(M) = \{0\}.$$

2 Preliminaries

In this section, we recall the foundational concepts and results necessary for understanding the main results of this paper. We begin with the definition of locally conformally Kähler (LCK) manifolds and their equivalent characterizations, followed by a discussion of Morse-Novikov cohomology and its properties. Finally, we introduce Vaisman manifolds and exact LCK manifolds, which are central to our work.

2.1 Locally Conformally Kähler Manifolds

A **Hermitian manifold** (M, J, g) consists of a smooth manifold M endowed with an almost complex structure J and a Hermitian metric g compatible with J. The associated fundamental 2-form ω is defined by:

$$\omega(X, Y) = g(JX, Y) \quad \text{for all vector fields } X, Y \text{ on } M.$$

Definition 1. A Hermitian manifold (M, J, g) is called a **locally conformally Kähler (LCK) manifold** if there exists a closed 1-form θ, called the **Lee form**, such that:

$$d\omega = \theta \wedge \omega.$$

This condition is equivalent to the existence of a Kähler structure on the universal cover $\widetilde{M}$ of M that is conformally equivalent to the pullback of ω. Specifically, there exists a smooth function $f : \widetilde{M} \to \mathbb{R}$ such that the metric $\tilde{g} = e^{-f}\tilde{\omega}$ is Kähler, where $\tilde{\omega}$ is the pullback of ω to $\widetilde{M}$.

Locally, an LCK manifold can be described as follows. Let $\{U_i\}_{i\in I}$ be an open cover of M, and let θ_i be the restriction of θ to U_i. If $d\theta_i = df_i \wedge \omega_i$ for some smooth functions $f_i : U_i \to \mathbb{R}$, then the local 1-forms df_i glue together to form a globally defined closed 1-form θ on M. Conversely, if θ is a closed 1-form on M, then by the Poincaré lemma, there exists an open cover $\{U_i\}_{i\in I}$ and smooth functions $f_i : U_i \to \mathbb{R}$ such that $\theta = df_i$ on U_i. The condition $d\omega = \theta \wedge \omega$ implies that $d\omega_i = df_i \wedge \omega_i$ on U_i. Multiplying by e^{-f_i}, we obtain:

$$d\left(e^{-f_i}\omega_i\right) = 0,$$

which shows that $e^{-f_i}g$ is a Kähler metric on U_i. This justifies the name "locally conformally Kähler," as the manifold resembles a Kähler manifold up to a conformal factor in local charts.

2.2 Morse-Novikov Cohomology

Let M be a compact differentiable manifold, and let θ be a closed 1-form on M. The **Morse-Novikov differential** d_θ is defined as:

$$d_\theta = d - \theta \wedge \cdot,$$

where d is the exterior derivative. Since θ is closed, then $d_\theta^2 = 0$, defining a cochain complex:

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d_\theta} \Omega^1(M) \xrightarrow{d_\theta} \cdots \xrightarrow{d_\theta} \Omega^n(M) \longrightarrow 0.$$

The cohomology of this complex, denoted by:

$$H^k_\theta(M) := \frac{\ker(d_\theta : \Omega^k(M) \to \Omega^{k+1}(M))}{\mathrm{im}(d_\theta : \Omega^{k-1}(M) \to \Omega^k(M))},$$

is called the **twisted cohomology** (or Morse-Novikov cohomology or Lichnerowicz cohomology) associated to the de Rham cohomology class $[\theta]_{dR}$. Importantly, $H^*_\theta(M)$ depends only on the de Rham cohomology class of θ.

If M is orientable, a version of **Poincaré duality** holds for Morse-Novikov cohomology:

$$H^k_\theta(M)^* \cong H^{n-k}_{-\theta}(M),$$

where $n = \dim_{\mathbb{R}} M$.

Remark 1. For an LCK manifold (M, J, g) with Lee form θ, the condition $d_\theta\omega = 0$ defines a cohomology class called the **Morse-Novikov class** $[\omega]_\theta \in H^2_\theta(M)$.

2.3 Vaisman Manifolds

A special subclass of LCK manifolds is that of **Vaisman manifolds**. An LCK manifold (M, J, g) is called Vaisman if its Lee form θ is parallel with respect to the Levi-Civita connection of g, i.e., $\nabla\theta = 0$. Vaisman manifolds exhibit several properties analogous to those of Kähler manifolds, including a version of the Hodge decomposition theorem for their cohomology groups.

Vaisman manifolds are also characterized by the existence of a distinguished vector field $A = \theta^\#$, called the **Lee vector field**, which satisfies:

$$\theta(X) = g(A, X) \quad \text{for all vector fields } X \text{ on } M.$$

This vector field plays a crucial role in the geometry of Vaisman manifolds.

For Vaisman manifolds the **Morse-Novikov cohomology vanishes** (cf. [2]), i.e.,

$$H^*_\theta(M) = \{0\}.$$

This vanishing result is a consequence of the parallelism of the Lee form θ.

Moroianu in [5] provides the following characterization of compact Vaisman manifolds of non-Kähler type:

Lemma 1. *Let (M, J, g, θ) be a compact, connected LCK manifold of complex dimension $n > 2$, of non-Kähler type. If M admits a non-trivial parallel vector field X, then g is Vaisman and $\theta^\# = X$.*

2.4 Exact LCK Manifolds

An LCK manifold (M, J, g) is called **exact** if its fundamental 2-form ω can be expressed as:

$$\omega = d_\theta \alpha$$

for some 1-form α on M. Exact LCK manifolds form an important subclass of LCK manifolds, as they exhibit special cohomological and geometric properties. In particular, the vanishing of the Morse-Novikov cohomology $H^*_\theta(M)$ is closely related to the exactness of ω.

The property of being d_θ-exact is conformally invariant, that is, if ω is d_θ-exact, then so is any conformal multiple of ω. This follows from the fact that $d_\theta\omega = 0$ defines the Morse-Novikov class $[\omega]_\theta$, and for exact LCK manifolds, this class vanishes.

The existence of exact LCK metrics on a compact complex manifold is constrained by geometric obstructions. In particular, when (M, J) contains a smooth compact complex submanifold Z of Kähler type with either $\dim_{\mathbb{C}} Z > 1$ or $Z \cong \mathbb{P}^1$, the manifold M cannot admit exact LCK metrics [8].

Hopf manifolds provide fundamental examples of strict LCK geometry, which are Exact. For $\lambda \in \mathbb{C}$ with $|\lambda| \neq 1$, the quotient $CH^n_\lambda = (\mathbb{C}^n \setminus \{0\})/\langle z \mapsto \lambda z\rangle$ is a compact LCK manifold (diffeomorphic to $S^1 \times S^{2n-1}$) that cannot be Kähler, as evidenced by its odd first Betti number $b_1 = 1$. The induced Boothby metric $h = |z|^{-2} \sum dz_j \otimes d\bar{z}_j$ makes CH^n_λ a Vaisman manifold with parallel Lee form $\theta = -d \log |z|^2$, yielding vanishing Morse-Novikov cohomology $H^*_\theta(CH^n_\lambda) = 0$ [2].

3 Proof of Main Results

In this section, we prove the main results of this paper. We begin by establishing the relationship between holomorphic 1-forms and conformal vector fields on exact LCK manifolds. We then prove a key result characterizing Vaisman manifolds under certain curvature conditions. Finally, we explore the vanishing of Morse-Novikov cohomology for exact LCK manifolds with holomorphic 1-forms.

We start by studying the properties of conformal vector fields on exact LCK manifolds. Specifically, we show that under certain conditions, the vector field dual to a holomorphic 1-form is conformal.

Proposition 1. *Given an LCK structure* (ω, θ) *on an LCK manifold, suppose that* $\omega = d_\theta \alpha$*. Then,* $J\alpha^{\#}$ *is a conformal vector field with potential function* $f = \theta(J\alpha^{\#}) - 1$*.*

Proof: To show that $J\alpha^{\#}$ is a conformal vector field, we need to check that the Lie derivative of the metric g with respect to $J\alpha^{\#}$ is proportional to g.

We know that

$$\mathcal{L}_{J\alpha^{\#}}\omega = d(\iota_{J\alpha^{\#}}\omega) + \iota_{J\alpha^{\#}}d\omega. \tag{1}$$

Since $\omega = d_\theta \alpha$, we have $d\omega = \theta \wedge \omega$, thus

$$\iota_{J\alpha^{\#}}d\omega = (\iota_{J\alpha^{\#}}\theta)\omega - \theta \wedge (\iota_{J\alpha^{\#}}\omega).$$

Substituting into 1 and taking into account that $\iota_{J\alpha^{\#}}\omega = -\alpha$, we get

$$\begin{aligned}\mathcal{L}_{J\alpha^{\#}}\omega =& d(\iota_{J\alpha^{\#}}\omega) + (\iota_{J\alpha^{\#}}\theta)\omega - \theta \wedge (\iota_{J\alpha^{\#}}\omega)\\ =&\theta(J\alpha^{\#})\omega - d\alpha + \theta \wedge \alpha = \theta(J\alpha^{\#})\omega - d_\theta\alpha\\ =&\theta(J\alpha^{\#})\omega - \omega = f\omega,\end{aligned}$$

where $f = \theta(J\alpha^{\#}) - 1$.

Thus, $J\alpha^{\#}$ is a conformal vector field, as required. ■

Next, we establish the equivalence of several conditions for an exact LCK manifold to be Vaisman. These conditions involve the relationship between the Lee form and the holomorphic 1-form.

Theorem 3. *Let* (M, J, θ, α) *be a compact exact LCK manifold. Then the following are equivalent:*

(1) M *is a Vaisman manifold,*

(2) $\alpha = -J\theta$*, where* $J \circ \theta := -\theta \circ J$*,*

(3) $d^c\alpha = 0$*, where* $d^c := J \circ d \circ J^{-1} = -J \circ d \circ J$*.*

Proof: 1) $\implies$ 2): If M is Vaisman, then θ is parallel and has constant norm. After rescaling, we can assume $\|\theta\| = 1$. Then it is easy to check that

$$\iota_{\theta^{\#}}\omega = J\theta \quad \text{and} \quad \iota_{\theta^{\#}}d\omega = \iota_{\theta^{\#}}(\theta \wedge \omega) = \omega - \theta \wedge J\theta.$$

So, by Cartan's formula for the Lie derivatives, we have

$$0 = \mathcal{L}_{\theta^\#}\omega = dJ\theta + \omega - \theta \wedge J\theta.$$

Thus

$$\omega = (-dJ\theta + \theta \wedge J\theta)\,.$$

Since $\omega = d_\theta \alpha$, we have:

$$d_\theta(\alpha + J\theta) = 0.$$

Since on a compact Vaisman manifold $H^*_\theta(M) = 0$, then $\alpha = -J\theta$.

2) $\implies$ 3): Trivial.

3) $\implies$ 1): Since $J\alpha^\#$ is a conformal vector field and $dJ\alpha = 0$, then $\nabla J\alpha$ is symmetric. Thus, $\nabla J\alpha = 0$, and by Lemma 1, M is Vaisman. ∎

Let (M, g) be a compact oriented Riemannian manifold and $\alpha, \beta \in \Omega^p(M)$. Following [1, §3.7] and denoting by $*$ the Hodge star operator, we consider the L^2-inner product on $\Omega^p(M)$:

$$\langle \cdot, \cdot \rangle : \Omega^p(M) \times \Omega^p(M) \to \mathbb{R}, \quad \langle \alpha, \beta \rangle := \int_M \alpha \wedge *\beta\,,$$

which satisfies the following fundamental identities:

(1) (Adjoint relation) $\langle d\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle$, for all $\alpha \in \Omega^{p-1}(M)$, $\beta \in \Omega^p(M)$, where δ denotes the co-differential;

(2) (Contraction duality) $\langle \iota_X \alpha, \beta \rangle = \langle \alpha, X^\flat \wedge \beta \rangle$, for all $X \in \mathfrak{X}(M)$, $\alpha \in \Omega^{p+1}(M)$ and $\beta \in \Omega^p(M)$, where $X^\flat$ denotes the 1-form on M dual to X by g ;

(3) (Lie derivative adjoint) $\langle \mathcal{L}_X \alpha, \beta \rangle = \langle \alpha, \mathcal{L}^*_X \beta \rangle$, for all $\alpha, \beta \in \Omega^p(M)$ where $\mathcal{L}^*_X = \delta L_{X^\flat} + L_{X^\flat}\delta$.

Here, $L_\beta(\alpha) := \beta \wedge \alpha$, for any differential form β on M. The commutation $[\mathcal{L}_X, d] = 0$ implies their adjoints satisfy:

$$\delta \mathcal{L}^*_X = \mathcal{L}^*_X \delta\,.$$

A direct computation yields the key identity:

$$(\mathcal{L}_X + \mathcal{L}^*_X)\alpha = (\delta\xi)\alpha + \sum_{r=1}^{p} g^{jk}(\mathcal{L}_X g)_{k i_r} \alpha_{i_1 \ldots \widehat{i_r} \ldots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}\,. \qquad (2)$$

In the special case where X is Killing, this simplifies to:

$$\mathcal{L}_X + \mathcal{L}_X^* = 0 \quad \text{on } \Omega^\bullet(M)\,.$$

It is known from [4] that any conformal vector field on a compact LCK manifold must necessarily be Killing with respect to the Gauduchon metric. We now examine this property in the specific case of exact LCK manifolds with holomorphic forms:

Lemma 2. *Let (M, J, ω) be a compact exact LCK manifold with $\omega = d_\theta\alpha$, where α is a holomorphic 1-form. Then the vector field $J\alpha^\#$ is Killing with respect to the Gauduchon metric g.*

Proof: Since α is holomorphic, then by Proposition 1, $A := J\alpha^\#$ is a conformal vector field with potential $\theta(A) - 1$, where the dualilty is with respect to the Gauduchon metric g.

From ([1]) A is conformal if and only if

$$\mathcal{L}_X\theta + \mathcal{L}_X^*\theta = (1 - \frac{2}{n})\delta\eta.\theta.$$

Since $\mathcal{L}_X^*$ commute with δ, we have

$$\delta\mathcal{L}_X\theta = (1 - \frac{2}{n})\delta(\delta\eta.\theta).$$

But $\mathcal{L}_X\theta = d\iota_A\theta$ and $\delta(\delta\eta.\theta) = \delta\eta.\delta\theta - g(\theta, d\delta\eta))$, then

$$\delta(d\iota_A\theta) = -(1 - \frac{2}{n})g(\theta, d(\delta\eta)).$$

Hence

$$\langle\delta\mathcal{L}_X\theta, \iota_A\theta\rangle = -(1 - \frac{2}{n})\langle g(\theta, d(\delta\eta)), \iota_A\theta\rangle,$$

or equivalently

$$\langle\mathcal{L}_X\theta, \mathcal{L}_X\theta\rangle = -(1 - \frac{2}{n})\int_M g(\theta, d(\delta\eta))\theta(A)dv,$$

i.e.

$$\|\mathcal{L}_X\theta\|^2 = -(1 - \frac{2}{n})\int_M g(\delta\theta, \delta\eta)\theta(A)dv = 0.$$

Thus $\mathcal{L}_X\theta = 0$.

On the other hand, we have $\theta(A) - 1 = -\frac{2}{n}\delta\eta$, this show that $d(\delta\eta) = 0$, that is $\delta\eta$ is constant. Since

$$\int_M \delta\eta dv = \langle 1, \delta\eta\rangle = \langle d1, \eta\rangle = 0,$$

we deduce that $\delta\eta = 0$, so A is a Killing vector field with respect to the Gauduchon metric g and $\theta(A) = 1$. ■

The following commutation relations were previously established for LCK manifolds with parallel Lee form θ [2]. We now generalize these results to the broader class of exact LCK manifolds:

Proposition 2. *Let (M, J, ω) be a compact exact LCK manifold with $\omega = d_\theta \alpha$. For the vector field $A = \alpha^\#$ dual to α, the following operator identities hold:*

(1) $\mathcal{L}_A = \iota_A d_\theta + d_\theta \iota_A + \mathrm{Id}$,

(2) $\delta \mathcal{L}_A = \mathcal{L}_A \delta$,

(3) $d_\theta \mathcal{L}_A = \mathcal{L}_A d_\theta$,

(4) $\delta_\theta \mathcal{L}_A = \mathcal{L}_A \delta_\theta$, *where δ_θ is the adjoint operator of d_θ.*

Proof:

(1) Starting with the interior product of the twisted differential:

$$\iota_A d_\theta = \iota_A (d - L_\theta) = \iota_A d - \iota_A L_\theta = \mathcal{L}_A - d\iota_A - \iota_A L_\theta.$$

But, for any $\alpha \in \Omega^k(M)$, we have

$$\iota_A(L_\theta \alpha) = \iota_A(\theta \wedge \alpha) = \theta(A)\alpha - \theta \wedge \iota_A \alpha = \alpha - L_\theta(\iota_A \alpha)$$

(since $\theta(A) = 1$). Substituting back yields:

$$\iota_A d_\theta = \mathcal{L}_A - d\iota_A - (\mathrm{Id} - L_\theta \iota_A) = \mathcal{L}_A - d_\theta \iota_A - \mathrm{Id}\,.$$

Rearranging gives the claimed decomposition:

$$\mathcal{L}_A = \iota_A d_\theta + d_\theta \iota_A + \mathrm{Id}$$

(2) Using the Killing property $\mathcal{L}_A^* = -\mathcal{L}_A$, we get by adjoint properties

$$\delta \mathcal{L}_A = -\delta \mathcal{L}_A^* = -\mathcal{L}_A^* \delta = \mathcal{L}_A \delta.$$

(3) We have

$$\mathcal{L}_A d_\theta = \mathcal{L}_A d - \mathcal{L}_A L_\theta = d\mathcal{L}_A - \mathcal{L}_A L_\theta.$$

But, for any $\alpha \in \Omega^k(M)$, we have

$$\mathcal{L}_A(L_\theta \alpha) = \mathcal{L}_A(\theta \wedge \alpha) = \mathcal{L}_A \theta \wedge \alpha + \theta \wedge \mathcal{L}_A \alpha = L_\theta(\mathcal{L}_A \alpha),$$

since $\mathcal{L}_A \theta = 0$. Thus:

$$\mathcal{L}_A d_\theta = d\mathcal{L}_A - L_\theta \mathcal{L}_A = d_\theta \mathcal{L}_A.$$

(4) Using the adjoint property again:

$$\delta_\theta \mathcal{L}_A = -\delta_\theta \mathcal{L}_A^* = -\mathcal{L}_A^* \delta_\theta = \mathcal{L}_A \delta_\theta.$$

■

Now, we are in a position to prove Theorems 1 and 2.

Proof of Theorem 1: Since α is holomorphic, then by Proposition 1 $J\alpha^\#$ is a non-vanishing conformal vector field on M. Applying the Goldberg formula [1, page 114] to $\eta := J\alpha$, we obtain

$$\langle \Delta\eta + \left(1 - \frac{2}{n}\right) d\delta\eta - Q\eta, \eta \rangle = 0,$$

where Q denotes the Ricci operator. Developing this, we get

$$\langle d\delta\eta + \delta d\eta, \eta \rangle + \left(1 - \frac{2}{n}\right) \langle d\delta\eta, \eta \rangle = \langle Q\eta, \eta \rangle.$$

Simplifying further, we obtain

$$\langle \delta\eta, \delta\eta \rangle + \langle d\eta, d\eta \rangle + \left(1 - \frac{2}{n}\right) \langle \delta\eta, \delta\eta \rangle = \langle Q\eta, \eta \rangle.$$

Thus

$$\langle d\eta, d\eta \rangle + 2\left(1 - \frac{1}{n}\right) \langle \delta\eta, \delta\eta \rangle = \langle Q\eta, \eta \rangle.$$

If Q is non-negative and $\langle Q\eta, \eta \rangle = 0$, we must have $\delta\eta = d\eta = 0$. Since $\eta^\#$ is conformal, this implies $\nabla\eta^\# = 0$, and by lemma 1, we conclude that $\theta = \eta = J\alpha$.

■

Proof of Theorem 2: Since α is holomorphic, $J\alpha$ is also holomorphic and, by Proposition 1, $A = (J\alpha)^\#$ is conformal vector field. Thus, by Lemma 2, it is Killing with respect to the Gauduchon metric, and $\theta(A) = 1$.

Since $H_\theta^*(M)$ is conformally invariant, we work with the Gauduchon metric. By the twisted Hodge decomposition theorem, we get

$$\Omega^k(M) = \mathcal{H}_\theta^k(M) \oplus d_\theta \left(\Omega^{k-1}(M)\right) \oplus \delta_\theta \left(\Omega^{k+1}(M)\right).$$

Let $\alpha \in \mathcal{H}_\theta^k(M)$. Then $d_\theta\alpha = 0$ and $\delta_\theta\alpha = 0$. Since A is Killing, $\mathcal{L}_A\alpha = 0$. It follows from the first identity of Proposition 2 that

$$\mathcal{L}_A\alpha = d_\theta(\iota_A\alpha) + \alpha.$$

Hence, $\alpha = 0$ and $H_\theta^k(M) \cong \mathcal{H}_\theta^k(M) = \{0\}$. ■

References

[1] S.I. Goldberg, *Curvature and Homology,* Academic Press, 2011.

[2] M. de León, B. López, J.C. Marrero and E. Padrón, *On the computation of the Lichnerowicz–Jacobi cohomology,* J. Geom. Phys., **44** (2003), 507–522.

[3] A. Banyaga, *Examples of non d_ω-exact locally conformal symplectic forms,* J. Geom., **87** (2007), 1–13.

[4] A. Moroianu and M. Pilca, *Conformal vector fields on LCK manifolds,* Math. Res. Letters, **30** (5) (2023), 1591–1608.

[5] A. Moroianu, *Compact LCK manifolds with parallel vector fields,* Complex Manifolds, **2** (1) (2015), 26–33.

[6] L. Ornea and M. Verbitsky, *Locally conformal Kähler manifolds with potential,* Math. Ann., **348** (1) (2011), 25–33.

[7] I. Vaisman, *Generalized Hopf manifolds,* Geom. Dedicata, **13** (1982), 231–255.

[8] N. Istrati, *A characterization of toric locally conformally Kähler manifolds,* J. Sympl. Geom., **17** (5) (2010), 1297–1316.

[9] N. Istrati, Conformal structures on compact complex manifolds, Doctoral thesis, Université Sorbonne Paris Cité, 2018.

[10] R. Goto, *On the stability of locally conformal Kähler structures,* J. Math. Soc. Japan, **66** (4) (2014), 1375–1401.