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Potential theory and applications in conformal geometry

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Abstract. In this paper, we want to give an exposition of our recent work on linear and nonlinear potential theory and their applications in conformal geometry. We use potential theory to study linear and quasilinear equations arising from conformal geometry. We establish the asymptotic behavior near singularities and derive applications in conformal geometry. In particular, we establish some Huber's type theorems and Hausdorff dimension estimates of the ends in conformal geometry in general dimensions.

Keywords: Riesz potentials, Wolff potentials, capacities, thin sets, scalar curvature equations, Q-curvature equations, p-Laplace equations, Huber's type theorems, Hausdorff dimensions

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1 Introduction

This article is written based on the talk delivered at the Second International Conference on Differential Geometry ICDG-FEZ'2024. The talk was to report some recent results on linear and nonlinear potential theory and their applications in conformal geometry.

The subject of conformal geometry here in many ways is developed from theories of analysis and geometry in 2 dimensions. We know that the integral of the Gauss curvature directly gives the Euler number from the Gauss-Bonnet. Perhaps it is slightly less well known that the differential equation about Gauss curvature facilitated the differential geometric approach to the uniformization theorem for Riemann surfaces. The theory of supharmonic functions and the Gauss curvature equation in 2 dimensions became essential to the theory of open surfaces [17]. The study of Gauss curvature equation also led to the differential geometric approach to the Teichmüller theory of Riemann surfaces

[39]. Indeed the Gauss curvature equation has been shown to be the powerful analytic tool to study the theory of conformal structure on surfaces. Another seminal achievement is exemplified by the works on the isospectral problems in 2 dimensions in [32, 33, 34].

To emulate these remarkable achievements in 2 dimensions to higher dimensions, one way is to look for analogs of the Gauss curvature equation and analytic tools to study these geometric differential equations. In this article we will review some well known results and report some recent developments.

We will start in Section 2 with the introduction of generalizations of the Gauss curvature equation in general dimensions systematically. Particularly we will introduce a family of p-Laplace equations about Ricci curvature tensor as quasilinear generalizations of the Gauss curvature equation. The introduction of superharmonic as well as p-superharmonic functions paves the way to the uses of linear and nonlinear potential theories.

In Section 3 we will review motivations in the study of geometric differential equations in conformal geometry. In higher dimensions, it started with the Yamabe problem [37]. The Yamabe problem is to search for constant scalar curvature metrics in the hope to find the extension of uniformization theorem in higher dimensions. From this perspective, we want to provide an overview of current research on geometric differential equations in conformal geometry.

In Section 4 and 5, we will focus on our recent work on applications of potential theory in the study of singular solutions of differential equations that are analogs of the Gauss curvature equation in higher dimensions. We then will report analogs of Huber theorem [17, 27, 28] in higher dimensions, and other geometric and topological consequences in conformal geometry [29, 25, 26].

2 Partial differential equations in conformal geometry

Let us first quickly introduce curvature on Riemannian manifolds. In a local coordinate (U, ϕ) on a Riemannian manifold (M^n, g),

$$x = (x_1, x_2, \cdots x_n) : \Omega \subset \mathbb{R}^n \to U \subset M^n,$$

the Riemannian metric and Christoffel symbols are

$$g_{ij} = g(\partial_{x_i}, \partial_{x_j}) \text{ and } \Gamma_{ij}^k = \frac{1}{2} g^{kl}(\partial_{x_i} g_{jl} + \partial_{x_j} g_{il} - \partial_{x_l} g_{ij}).$$

Therefore the Levi-Civita covariant differentiation is given as

$$\nabla_{\partial_{x_i}} \partial_{x_j} = \Gamma_{ij}^k \partial_{x_k}$$

and Riemann curvature tensor is

$$R_{ijk}{}^{l} = -\partial_{x_i}\Gamma^l_{jk} + \partial_{x_j}\Gamma^l_{ik} - \Gamma^m_{jk}\Gamma^l_{mi} + \Gamma^m_{ik}\Gamma^l_{mj}.$$

Then Ricci curvature tensor and (Ricci) scalar curvature are

$$R_{ik} = g^{jl}R_{ijkl} = R_{ijk}{}^{j} \text{ and } R = g^{ik}R_{ik} = R_i{}^{i}.$$

In conformal geometry one often encounters the Schouten curvature tensor

$$A_{ik} = \frac{1}{n-2}(R_{ik} - \frac{1}{2(n-1)}R\, g_{ik}) \tag{1}$$

and uses the notation $J = g^{ik}A_{ik} = \frac{R}{2(n-1)}$.

2.1 Semilinear equations and superharmonic functions

In dimension 2, let $R_{1212} = K$. Then the celebrated Gauss curvature equation is

$$-\Delta u + K = K_{e^{2u}g}e^{2u}. \tag{2}$$

In dimensions greater than 2, the scalar curvature equation is

$$P_2 u = \frac{n-2}{4(n-1)}R_{u^{\frac{4}{n-2}}g}u^{\frac{n+2}{n-2}} \tag{3}$$

where $P_2 = -\Delta + \frac{n-2}{4(n-1)}R$ is the conformal Laplacian. (3) is often referred as the Yamabe equation and has been extensively studied in connection to the well known Yamabe probelm in conformal geometry.

One may recall that the Paneitz operator is

$$P_4 = \Delta^2 + \operatorname{div}(4A\cdot\nabla - (n-2)J\nabla) + \frac{n-4}{2}Q_4$$

and the associated Q-curvature is

$$Q_4 = -\Delta J + \frac{n}{2}J^2 - 2|A|^2.$$

The Q-equation is

$$P_4 u + Q_4 = (Q_4)_{e^{2u}g}e^{4u} \tag{4}$$

in dimension 4 and

$$P_4 u = \frac{n-4}{2}(Q_4)_{u^{\frac{4}{n-4}}g}u^{\frac{n+4}{n-4}} \tag{5}$$

in dimensions higher than 4. In fact, there are higher order analogs, the so-called GJMS operators P_{2k} and corresponding curvature Q_{2k} (cf. [7, 8, 11]). They all satisfy the following characteristic properties.

- The conformal covariance:

$$(P_{2k})_{u^{\frac{4}{n-2k}}g}(\phi) = u^{-\frac{n+2k}{n-2k}}(P_{2k})_g(u\cdot\phi)$$

- On Euclidean space, $P_{2k} = (-\Delta)^k$, which is the leading order term in general.

- The associated scalar curvature of higher order is $Q_{2k} = P_{2k}(1)$ when $2k < n$.

And

$$P_n u + Q_n = (Q_n)_{e^{2u}g}e^{nu} \quad \text{when } n \text{ is even and}$$

$$P_{2k}u = (Q_{2k})_{u^{\frac{4}{n-2k}}g}u^{\frac{n+2k}{n-2k}} \quad \text{when } 2k < n.$$

Moreover there are fractional analogs coming from the scattering theory of Poincré-Einstein manifolds (cf. [12, 14]). Let $(M^n, [g])$ be the conformal infinity of a Poincaré-Einstein manifold (X^{n+1}, g^+), where $[g]$ stands for the class of conformal metrics on M^n. For a representative g, the regularized scattering operator $(P_\alpha)_g$ of order $\alpha \in (0, n]$ and the associated nonlocal "curvature"

$$(Q_\alpha)_g = (P_\alpha)_g 1$$

of order α behave similarly

$$(P_n)_g u + (Q_n)_g = (Q_n)_{e^{2u}g}e^{nu} \text{ when } \alpha = n \text{ is odd, and}$$

$$(P_\alpha)_g u = (Q_\alpha)_{u^{\frac{4}{n-\alpha}}g}u^{\frac{n+\alpha}{n-\alpha}} \text{ for } \alpha \in (0, n).$$

P_α is the family of fractional powers of Laplacians that includes GJMS operators P_{2k} (cf. [12, 9, 35, 36, 14, 10]).

We will consider solutions to the equation

$$(-\Delta)^{\frac{\alpha}{2}}u = \mu \text{ on Euclidean space } \mathbb{R}^n \tag{6}$$

or

$$P_\alpha u = \mu \text{ on } M^n \text{ in general}$$

for a nonnegative Radon measure μ as generalized superharmonic functions and appeal to the linear potential theory in our study of the asymptotic behavior near singularities.

2.2 p-Laplace equations and p-superharmonic functions

Here, we want to call the attention to the intermediate Schouten curvature tensor (cf. [25, 26])

$$A^{(p)} = (p-2)A + J\,g \tag{7}$$

for $p \in (1, \infty)$. For $\bar{g} = u^{\frac{4(p-1)}{n-p}} g$ and $p \neq n$,

$$\begin{aligned} A^{(p)}_{\bar{g}} = A^{(p)} &- \frac{2(p-1)}{n-p}\left[\frac{\Delta u}{u} g + (p-2)\frac{D^2 u}{u}\right] \\ &+ \frac{2(p-1)}{n-p}[(1-(n+p-4)\frac{p-1}{n-p})\frac{|\nabla u|^2}{u^2} g \\ &+ (p-2)(1+\frac{2(p-1)}{n-p})\frac{\nabla u \otimes \nabla u}{u^2}]. \end{aligned} \tag{8}$$

Multiplying $u|\nabla u|^{p-2}\frac{u_i}{|\nabla u|}\frac{u_j}{|\nabla u|}$ to and summing up on both sides, we arrive at the p-Laplace equations in conformal geometry

$$-\Delta_p u + \frac{n-p}{2(p-1)} S^{(p)}(\nabla u) u = \frac{n-p}{2(p-1)}(S^{(p)}(\nabla u))_{\bar{g}} u^q \tag{9}$$

where

$$S^{(p)}(\nabla u) = |\nabla u|^{p-2} A^{(p)}(\nabla u), \quad q = \frac{2p(p-1)}{n-p} + 1,$$

and $A^{(p)}(\nabla u)$ is the $A^{(p)}$ curvature in the direction ∇u. When $p = n$, we realize $A^{(n)} = Ric$, and the n-Laplace equation is

$$-\Delta_n \phi + |\nabla \phi|^{n-2} Ric(\nabla\phi) = (|\nabla\phi|^{n-2} Ric(\nabla\phi))_{\bar{g}} e^{n\phi} \tag{10}$$

where $\bar{g} = e^{2\phi} g$. Recall

$$\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u).$$

For $p = n = 2$, (10) goes back to the Gauss curvature equation

$$-\Delta\phi + K = K_{\bar{g}} e^{2\phi},$$

where $\bar{g} = e^{2\phi} g$. For $p = 2$ and $n \geq 3$, the intermediate Schouten curvature goes back to the scalar curvature and the p-Laplace equation (9) goes back to the scalar curvature equation

$$-\Delta u + \frac{n-2}{4(n-1)} R u = \frac{n-2}{4(n-1)} R_{\bar{g}} u^{\frac{n+2}{n-2}},$$

where $\bar{g} = u^{\frac{4}{n-2}} g$. For $p > n$, the p-Laplace equation (9) for the intermediate Schouten curvature is still valid for $\bar{g} = u^{-\frac{4(p-1)}{p-n}} g$ and $q = -\frac{2p(p-1)}{p-n} + 1 < 0$.

And, when taking $p \to \infty$, we arrive at the infinite Laplace equation on Schouten curvature A

$$- \Delta_\infty u - \frac{1}{2}|\nabla u|^2 A(\nabla u) u = -\frac{1}{2}(|\nabla u|^2 A(\nabla u))_{\bar{g}} u^{-7} \tag{11}$$

for $\bar{g} = u^{-4}\, g$. Recall $\Delta_\infty u = u_{ij} u_i u_j$.

We will consider solutions to the equation

$$- \Delta_p u = \mu \tag{12}$$

for a nonnegative Radon measure μ as p-superharmonic functions and appeal to nonlinear potential theory in our study of the asymptotic behavior near singularities.

3 Motivating problems in conformal geometry

3.1 Motivations from the surface theory

Much of the developments in conformal geometry have been motivated by the tremendous success in the study of surfaces. We want to start with the uniformization theorem (Klein 1883 Poincaré 1882 Koebe 1907), which states, the universal covering of a closed Riemann surface is conformally equivalent to one of the three: $\mathbb{D}^2$, $\mathbb{R}^2$, $\mathbb{S}^2$.

By the differential-geometric approach, this is to say that, on a closed Riemannian surface (M^2, g), there is a conformal metric $e^{2u}g$ whose Gauss curvature is constant. Equivalently, one solves

$$-\Delta u + K = \kappa e^{2u}$$

for $\kappa = -1, 0, 1$.

On the theory of open surfaces, there is the important theorem, proved by Huber in [17], which states, if a complete noncompact surface (M, g) satisfies

$$\int_M K^- d\mu_g < +\infty,$$

then it is conformally equivalent to a compact Riemann surface with finitely many points removed. Here $K^- = \max\{-K, 0\}$ is the negative part of Gaussian curvature K and $d\mu_g$ is the volume form of the metric g. Analytically this includes the statement:

Theorem. Let the conformal metric $\bar{g} = e^{2u}g$ on $\Omega \setminus S$ be geodesically complete near S. Suppose that $\int_{\Omega\setminus S}(K^- dvol)_{\bar{g}} < \infty$. Then S consists of at most finitely many points.

3.2 Interplays of analysis and conformal geometry in higher dimensions

There have been many interesting work to establish theorems of Huber-type in higher dimensions. To motivate, we want to make two remarks. First, in two dimensions, open surfaces are well understood topologically. However, in higher dimensions, the topology as well as the local and global conformal structure become much more complicated. Secondly, in higher dimensions, Ricci curvature tensor is genuinely no longer a scalar. Therefore, for applications in geometry, we will focus on locally conformally flat manifolds, particularly those whose development maps are injective.

One may focus on locally conformally flat manifolds as they are more closely related to theories of surfaces. Suppose that (M^n, g) is a locally conformally flat manifold and that the conformal immersion from a covering $(\tilde{M}^n, \tilde{g})$ to $(\mathbb{S}^n, g_{\mathbb{S}})$ is injective.

$$\begin{array}{ccc} \tilde{M}^n & \overset{\phi}{\text{———————}} & \mathbb{S}^n \\ \pi \Big| & & \\ M^n & & \end{array}$$

Then, on $\phi(\tilde{M}^n) \subset \mathbb{S}^n$, there is a complete conformal metric $(\phi^{-1})^*\tilde{g} = e^{2u} g_{\mathbb{S}}$. One is interested in the size of $\mathbb{S}^n \setminus \phi(\tilde{M}^n)$, or specifically, the Hausdorff dimension of $\mathbb{S}^n \setminus \phi(\tilde{M}^n)$. We want to confirm that, in these situations, there are strong correlations between the Hausdorff dimension of $\mathbb{S}^n \setminus \phi(\tilde{M}^n)$, the topology of M^n, and the curvature of the metric g. Specifically, we will introduce notions of the positivity of curvature tensors and their impact to the homology groups and homotopy groups.

One way to measure the positivity of curvature and derive topological consequences is to consider (scalar) curvatures of higher order and nonlinear nature and the associated curvature equations like the semilinear and quasilinear equations mentioned in the previous section in the spirit of the uniformization theorem on surfaces. The Yamabe problem and its generalizations have been introduced and generated huge interest, which has been one of the most significant and influential subjects in geometric analysis and geometric partial differential equations. We believe the newly introduced p-Laplace equations in conformal geometry will add fuel and contribute further developments.

For instance, as a way to gauge the positivity of Ricci curvature tensor, we propose here to use the cones

$$\mathcal{A}^{(p)} = \{\lambda \in \mathbb{R}^n : \min_k \{(p-2)\lambda_k + \sum_{i=1}^n \lambda_i\} \geq 0\} \tag{13}$$

for $p \in (1, \infty)$ to describe the positivity of the intermediate Schouten curvature tensor $A^{(p)}$ when $\{\lambda_i\}$ stands for the eigenvalues of the Schouten curvature tensor A.

To illustrate how effective this approach can lead to vanishing theorems on topology, we recall the cones

$$\mathcal{R}^{(r)} = \{\lambda \in \mathbb{R}^n : \min\{(n-r)\sum_{k=1}^r \lambda_{i_k} + r \sum_{k=r+1}^n \lambda_{i_k}\} \geq 0\} \tag{14}$$

introduced for dealing with the terms in the Bochner formula on r-forms on locally conformally flat n-manifolds, where min is taken for all possible rearrangements $\{\lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_n}\}$ of $(\lambda_1, \lambda_2, \cdots, \lambda_n)$. We observe that

Lemma 1. *([25])*

$$\mathcal{A}^{(p)} \subset \mathcal{A}^{(q)} \quad \text{for } 2 \leq q < p < \infty$$
$$\mathcal{R}^{(s)} \subset \mathcal{R}^{(r)} \quad \text{for } 0 < s \leq r \leq \frac{n}{2}$$

and

$$\mathcal{A}^{(p)} \subset \mathcal{R}^{(r)} \quad \text{for } \frac{n-p}{2} + 1 \leq r \leq \frac{n}{2}.$$

Obviously $\mathcal{A}^{(2)} = \mathcal{R}^{(\frac{n}{2})}$ *is the baseline.*

Consequently, we obtain

Theorem 1. *([25]) Let (M^n, g) be a compact locally conformally flat manifold with $A^{(p)} \geq 0$ for $p \in [2, n)$. Suppose that the scalar curvature is positive somewhere on M^n. Then, for $\frac{n-p}{2} + 1 \leq k \leq \frac{n+p}{2} - 1$, the Betti numbers $\beta_k = 0$, unless $(\tilde{M}^n, g) \overset{isometric}{\sim} \mathbb{H}^k \times \mathbb{S}^{n-k}$.*

The proof uses The Bochner formula on r-forms (cf. [13, 31])

$$\Delta \omega = \nabla^* \nabla \omega + \mathcal{R}(\omega)$$
$$\mathcal{R}(\omega) = ((n-r)\sum_{i=1}^r \lambda_i + r \sum_{i=r+1}^n \lambda_i)\omega$$

for $\omega = \omega_1 \wedge \omega_2 \cdots \wedge \omega_r$ and $\{\omega_k\}$ is the orthonormal basis under which the Schouten curvature tensor A is diagonalized on locally conformally flat n-manifolds.

In the light of (9), we therefore want to use asymptotic behavior p-superharmonic functions to estimate the size of singularities and derive consequences on the homotopy groups when assuming $A^{(p)} \geq 0$ for $p \in [2, \infty)$.

4 Linear potentials and applications in conformal geometry

In this section, we give an exposition of our recent work on linear potential theory in conformal geometry. We apply linear potential theory to study partial differential equations arising from conformal geometry and, in particular, the problems related to the dimension of the boundary of a domain that admits certain complete conformal metric.

4.1 Riesz potential, capacity, and thin set

Let Ω be a bounded open subset in the Euclidean space $\mathbb{R}^n$. Then, for $x \in \Omega$, let

$$R_\mu^{\alpha,\Omega}(x) = \begin{cases} \int_\Omega \frac{1}{|x-y|^{n-\alpha}} d\mu(y) & \text{when } \alpha \in (1,n) \\ \int_\Omega \log \frac{D}{|x-y|} d\mu(y) & \text{when } \alpha = n \end{cases} \tag{15}$$

for a Radon measure μ on Ω, where D is the diameter of Ω. Let E be a subset in Ω and Ω be a bounded open subset in $\mathbb{R}^n$. For $\alpha \in (1,n]$, we define the Riesz capacity by

$$C_{\mathcal{L}}^\alpha(E,\Omega) = \inf\{\mu(\Omega) : \begin{array}{c} \mu \geq 0 \text{ Radon measure on } \Omega \\ R_\mu^{\alpha,\Omega}(x) \geq 1 \text{ for all } x \in E \end{array}\}. \tag{16}$$

The following basic properties are easy to prove (cf. [30])

Lemma 2. *Let $C_{\mathcal{L}}^\alpha$ be the Riesz capacity defined as in* (16) *for $\alpha \in (1,n]$. Then*

(1) $C_{\mathcal{L}}^\alpha$ is nondecreasing in E, that is

$$C_{\mathcal{L}}^\alpha(E_1,\Omega) \leq C_{\mathcal{L}}^\alpha(E_2,\Omega)$$

when $E_1 \subset E_2 \subset \Omega \subset \mathbb{R}^n$.

(2) $C_{\mathcal{L}}^\alpha$ is countably subadditive, that is

$$C_{\mathcal{L}}^\alpha(\cup_{i=1}^\infty E_i,\Omega) \leq \sum_{i=1}^\infty C_{\mathcal{L}}^\alpha(E_i,\Omega)$$

for subsets $E_i \subset \Omega$.

(3) For a positive number λ, *let*

$$A_\lambda = \{\lambda x : x \in A\}$$

for any subset A *of* $\mathbb{R}^n$. *Then, for* $\alpha \in (1, n]$,

$$C^\alpha_{\mathcal{L}}(E_\lambda, \Omega_\lambda) = \lambda^{n-\alpha} C^\alpha_{\mathcal{L}}(E, \Omega).$$

(4) Suppose

$$\Phi : \Omega \to \Omega$$

is a contractive map, that is

$$|\Phi(x) - \Phi(y)| \le |x - y|$$

for all $x, y \in \Omega$. *Then, for* $\alpha \in (1, n]$,

$$C^\alpha_{\mathcal{L}}(\Phi(E), \Omega) \le C^\alpha_{\mathcal{L}}(E, \Omega)$$

for any subset $E \subset \Omega$.

(5) For $\alpha \in (1, n]$,

$$C^\alpha_{\mathcal{L}}(\partial B_1(0), B_2(0)) = c(n, \alpha)$$

for some positive constant $c(n, \alpha)$.

Thin sets with respect to the Riesz capacity $C^\alpha_{\mathcal{L}}$ are defined through dyadic annuli:

$$\omega^\delta_i(p) = \{x \in \mathbb{R}^n : |x - p| \in [2^{-i}\delta, 2^{-i+1}\delta]\}$$
$$\Omega^\delta_i(p) = \{x \in \mathbb{R}^n : |x - p| \in (2^{-i-1}\delta, 2^{-i+2}\delta)\}.$$

Definition 1. Let E be a subset in the Euclidean space $\mathbb{R}^n$ and $p \in \mathbb{R}^n$. The subset E is said to be α-thin with respect to the Riesz capacity $C^\alpha_{\mathcal{L}}$ at the point p for $\alpha \in (1, n)$ if

$$\sum_{i \ge 1} \frac{C^\alpha_{\mathcal{L}}(E \bigcap \omega^\delta_i(p), \Omega^\delta_i(p))}{C^\alpha_{\mathcal{L}}(\partial B_{2^{-i}\delta}(p), B_{2^{-i+1}\delta}(p))} < \infty$$

for some small $\delta > 0$. The subset E is said to be n-thin at p if

$$\sum_{i \ge 1} i C^n_{\mathcal{L}}(E \bigcap \omega^\delta_i(p), \Omega^\delta_i(p)) < \infty$$

for some small $\delta > 0$.

A simple and important fact (see also [30]) is

Lemma 3. *Let* E *be a subset in the Euclidean space* $\mathbb{R}^n$ *and* $p \in \mathbb{R}^n$. *Suppose that* E *is* α*-thin at the point* p *for* $\alpha \in (1, n]$. *Then there is always a ray from* p *that avoids* E *at least within some small ball at* p.

4.2 Singularities of superharmonic functions under the fine topology

What we want to derive is the asymptotic behavior of superharmonic functions near singularities outside thin sets.

Theorem 2. *Suppose μ be a finite nonnegative Radon measure on a bounded domain $\Omega \subset R^n$. Then, for $p \in \Omega$, where $R_\mu^{\alpha,\Omega}(p) = \infty$, there is a subset A that is α-thin at p such that*

$$\lim_{x \to p \text{ and } x \in R^n \setminus A} \frac{R_\mu^{\alpha,\Omega}(x)}{|x-p|^{\alpha-n}} = \mu(\{p\}).$$

for $\alpha \in (1, n)$ and

$$\lim_{x \to p \text{ and } x \in R^n \setminus A} \frac{R_\mu^{n,\Omega}(x)}{\log \frac{1}{|x-p|}} = \mu(\{p\})$$

for $\alpha = n$.

To obtain the Hausdorff dimension estimates we use the help from the following

Lemma 4. *([22]) Let μ be a nonnegative Radon measure on a complete Riemannian manifold (M^n, g) and let*

$$G_d^\infty = \{x \in M^n : \limsup_{r \to 0} r^{-d} \mu(B_r(x)) = +\infty\}$$

for any $d \in [0, n]$. Then

$$\mathcal{H}_d(G_d^\infty) = 0$$

where $\mathcal{H}_d$ is the Hausdorff measure of dimension d.

Consequently we have

Theorem 3. *([29] Suppose that μ is a finite nonnegative Radon measure on a bounded domain $\Omega \subset \mathbb{R}^n$. Let S be a compact subset in Ω such that its Hausdorff dimension is greater than d, where $d < n - \alpha$ and $\alpha \in (1, n)$. Then there is a point $p \in S$ and a subset E that is α-thin at p such that*

$$R_\mu^{\alpha,\Omega}(x) \leq \frac{C}{|x-p|^{n-\alpha-d}}$$

for some constant C and all $x \in B_\delta(p) \setminus E$ for some small $\delta > 0$.

4.3 Consequences in conformal geometry

We first derive the Hausdorff dimension estimates from the scalar curvature equation (3).

Theorem 4. *([29]) Let S be a compact subset and D be a bounded open neighborhood of S. Suppose that $\bar{g} = u^{\frac{4}{n-2}} g$ is a conformal metric on $D \setminus S$ and is geodesically complete near S. Then the Hausdorff dimension*

$$dim_{\mathcal{H}}(S) \leq \frac{n-2}{2},$$

provided that $R^-[\bar{g}] \in L^{\frac{2n}{n+2}}(D \setminus S, \bar{g}) \bigcap L^p(D \setminus S, \bar{g})$ for some $p > n/2$.

Obviously the integrability condition holds if $R \geq 0$. Therefore this improves some early result of Schoen-Yau 1988 ([38]). Using (5) we have

Theorem 5. *[29] Let S be a compact subset and D be a bounded open neighborhood of S. Suppose that $\bar{g} = u^{\frac{4}{n-4}} g$ is a conformal metric on $D \setminus S$ with nonnegative scalar curvature and is geodesically complete near S. And suppose also that*

$$(Q_4^-)_{\bar{g}} \in L^{\frac{2n}{n+4}}(D \setminus S, \bar{g}).$$

Then

$$dim_{\mathcal{H}}(S) \leq \frac{n-4}{2}.$$

We also were able to show a Huber's type theorem in dimension 4.

Theorem 6. *[28]) Let S be a compact subset and D be a bounded open neighborhood of S. Suppose that $\bar{g} = e^{2u} g$ is a conformal metric on $D \setminus S$ with nonnegative scalar curvature and is geodesically complete near S. And suppose that*

$$\int_D (Q_4^- dvol)_{\bar{g}} < \infty.$$

Then S consists of at most finitely many points.

This can be compared with some early result of Chang-Q-Yang 2000 ([6]).

5 Nonlinear potentials and applications

In this section we give an exposition of our recent work on nonlinear potential theory in conformal geometry. We apply nonlinear potential theory to study p-Laplace equations arising from conformal geometry and, in particular, the problems related to the asymptotic behavior near and the size of singularities in conformal geometry.

5.1 Wolff potentials, p-superharmonic functions, and their singular behavior

Let us first give the definitions of p-harmonic and p-superharmonic functions. In this section we always assume $1 < p < n$ unless specified otherwise. Let us recall again what is the p-Laplace operator

$$\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u).$$

Definition 2. ([24, Definition 2.5]) We say that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a weak solution of the p-harmonic equation in Ω, if

$$\int \langle |\nabla u|^{p-2}\nabla u, \nabla \eta\rangle dx = 0$$

for each $\eta \in C_0^\infty(\Omega)$. If, in addition, u is continuous, we then say u is a p-harmonic function.

Definition 3. ([24, Definiton 5.1]) A function $v : \Omega \to (-\infty, \infty]$ is called p-superharmonic in Ω, if

- v is lower semi-continuous in Ω;
- $v \not\equiv \infty$ in Ω;
- for each domain $D \subset\subset \Omega$ the comparison principle holds, that is, if $h \in C(\bar{D})$ is p-harmonic in D and $h|_{\partial D} \leq v|_{\partial D}$, then $h \leq v$ in D.

As stated in [18, Theorem 2.1], for u to be a p-superharmonic function in Ω, there is a nonnegative Radon measure μ in Ω such that

$$-\Delta_p u = \mu. \tag{17}$$

For more about p-superharmonic functions and nonlinear potential theory, we refer to [19, 16, 15, 1, 24] and references therein. The most important tool is the following Wolff potential

$$W^\mu_{1,p}(x, r) = \int_0^r \left(\frac{\mu(B(x, t))}{t^{n-p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} \tag{18}$$

for any nonnegative Radon measure μ and $p \in (1, n]$. The fundamental estimate for the use of the Wolff potential in the study of p-superharmonic functions is as follows:

Theorem 7. *([19, Theorem 1.6]) Suppose that u is a nonnegative p-superharmonic function satisfying* (17) *for a nonnegative finite Radon measure μ in $B(x, 3r)$. Then*

$$c_1 W^{\mu}_{1,p}(x, r) \le u(x) \le c_2(\inf_{B(x,r)} u + W^{\mu}_{1,p}(x, 2r)) \tag{19}$$

for some constants $c_1(n, p)$ and $c_2(n, p)$ for $p \in (1, n]$.

To introduce nonlinear potential theory and present the results on asymptotic behavior of the Wolff potentials at singularities we first recall some definitions and basics.

Definition 4. ([19, Section 3.1]) For a compact subset K of a domain Ω in $\mathbb{R}^n$, we define

$$cap_p(K, \Omega) = \inf\{\int_{\Omega} |\nabla u|^p dx : \ u \in C_0^{\infty}(\Omega) \text{ and } u \ge 1 \text{ on } K\}. \tag{20}$$

Then p-capacity for arbitrary subset E of Ω is

$$cap_p(E, \Omega) = \inf_{\text{open } G \supset E \ \& \ G \subset \Omega} \ \sup_{\text{compact } K \subset G} cap_p(K, \Omega). \tag{21}$$

Analogously, the capacity cap_p shares the same basic properties in Lemma 2 as the Riesz capacity does (cf. [16, 1, 25]). The notions of thinness in the potential theory are vitally important. The readers are referred to [2, 3, 15, 19, 27, 29, 28] for detailed discussions and references therein. To study the singular behavior of p-superharmonic functions, like [27, Definition 3.1] (see also [30, Section 2.5]), we propose a thinness that is less restrictive than that by the Wiener type integral when $p \in (2, n)$. Recall the dyadic annuli are

$$\begin{aligned} \omega_i(x_0) &= \{x \in \mathbb{R}^n : 2^{-i} \le |x - x_0| \le 2^{-i+1}\}; \\ \Omega_i(x_0) &= \{x \in \mathbb{R}^n : 2^{-i-1} \le |x - x_0| \le 2^{-i+2}\}. \end{aligned}$$

Definition 5. A set $E \subset \mathbb{R}^n$ is said to be p-thin for singular behavior for $p \in (1, n)$ at $x_0 \in \mathbb{R}^n$ if

$$\sum_{i=1}^{\infty} \frac{cap_p(E \cap \omega_i(x_0), \Omega_i(x_0))}{cap_p(\partial B(x_0, 2^{-i}), B(x_0, 2^{-i+1}))} < +\infty. \tag{22}$$

Meanwhile a set E is said to be n-thin at $x_0 \in \mathbb{R}^n$ if

$$\sum_{i=1}^{\infty} i^{n-1} \, cap_n(E \cap \omega_i(x_0), \Omega_i(x_0)) < +\infty. \tag{23}$$

Notice that these thin sets are with respect to the p-capacity cap_p in contrast to the ones with respect to the Riesz capacity in Definition 1. Like Lemma 3, the most important fact about p-thinness for singular behavior to us due to Lemma 2 is the following

Lemma 5. *Let E be a subset in the Euclidean space $\mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$ be a point. Suppose that E is p-thin for singular behavior at the point x_0 for $p \in (1, n]$. Then there is a ray from x_0 that avoids E at least within some small ball at x_0.*

The key estimates on the Wolff potential are the following

Theorem 8. *([25, 26]) Suppose μ is a nonnegative finite Radon measure in Ω. Assume that, for a point $x_0 \in \Omega$ and some number $m \in (0, n-p)$,*

$$\mu(B(x_0, t)) \leq Ct^m \tag{24}$$

for all $t \in (0, 3r_0)$ with $B(x_0, 3r_0) \subset \Omega$. Then, for $\varepsilon > 0$, there are a subset $E \subset \Omega$, which is p-thin for singular behavior at x_0, and a constant $C > 0$ such that

$$W^{\mu}_{1,p}(x,\, r_0) \leq C|x - x_0|^{-\frac{n-p-m+\varepsilon}{p-1}} \quad \text{for all } x \in \Omega \setminus E \tag{25}$$

for $p \in [2, n)$.

And

Theorem 9. *([25, 26]) Let μ be a nonnegative finite Radon measure in Ω and $B(x_0, 3r_0) \subset \Omega$. Then there is a subset E that is p-thin for the singular behavior at x_0 such that*

$$\lim_{x \to x_0 \text{ and } x \notin E} |x - x_0|^{\frac{n-p}{p-1}} W^{\mu}_{1,p}(x,\, r_0) = \frac{p-1}{n-p} \mu(\{x_0\})^{\frac{1}{p-1}}$$

for $p \in (1, n)$. Similarly, there is a subset E that is n-thin for the singular behavior at x_0 such that

$$\lim_{x \to x_0 \text{ and } x \notin E} \frac{W^{\mu}_{1,n}(x,\, r_0)}{\log \frac{1}{|x - x_0|}} = \mu(\{x_0\})^{\frac{1}{n-1}}.$$

It turns out the notion of thinness indeed facilitates the fine topology (cf. [19, 30]) in the following context, which gives the equivalent analytic definition of thinness.

Theorem 10. *([25, 26]) Suppose that E is a subset that is p-thin for the singular behavior at the origin according to Definition 5 for $p \in (1, n]$. And suppose that the origin is in $\bar{E} \setminus E$. Then, when $p \in (1, n)$, there is a Radon measure μ in a neighborhood of the origin such that, for some fixed $r_0 > 0$,*

$$\lim_{x \to 0 \text{ and } x \in E} |x|^{\frac{n-p}{p-1}} W^{\mu}_{1,p}(x, r_0) = \infty.$$

Similarly, when $p = n$, there is a Radon measure μ in a neighborhood of the origin such that, for some fixed $r_0 > 0$,

$$\lim_{x\to 0 \text{ and } x\in E} \frac{W^{\mu}_{1,n}(x, r_0)}{\log \frac{1}{|x|}} = \infty.$$

In fact, we use Theorem 7 to derive the estimates for p-superharmonic functions. From Theorem 9, we have

Theorem 11. *([26]) Suppose that u is a nonnagetive p-superharmonic function in $\Omega \subset \mathbb{R}^n$ satisfying*

$$-\Delta_p u = \mu \text{ in } \Omega$$

for a nonnegative finite Radon measure μ on Ω and $p \in (1, n]$. Then, for $x_0 \in \Omega$, there is a subset E that is p-thin for singular behavior at x_0 such that

$$\lim_{x\to x_0 \text{ and } x\notin E} \frac{u(x)}{G_p(x, x_0)} = m = \begin{cases} \frac{p-1}{n-p}\left(\frac{\mu(\{0\})}{|\mathbb{S}^{n-1}|}\right)^{\frac{1}{p-1}} & \text{when } p \in (1, n) \\ \left(\frac{\mu(\{0\})}{|\mathbb{S}^{n-1}|}\right)^{\frac{1}{n-1}} & \text{when } p = n \end{cases}, \tag{26}$$

where

$$G_p(x, x_0) = \begin{cases} |x - x_0|^{-\frac{n-p}{p-1}} & \text{when } p \in (1, n) \\ -\log |x - x_0| & \text{when } p = n \end{cases}. \tag{27}$$

Moreover $u(x) \geq mG_p(x, x_0) - c_0$ for some c_0 and all x in a neighborhood of x_0.

From Theorem 10, we have

Theorem 12. *([26]) Suppose that E is a subset that is p-thin for the singular behavior at the origin according to Definition 5 for $p \in (1, n]$. And suppose that the origin is in $\bar{E} \setminus E$. Then, when $p \in (1, n)$, there is a p-superharmonic function u in a neighborhood of the origin such that*

$$\lim_{x\to 0 \text{ and } x\in E} |x|^{\frac{n-p}{p-1}} u(x) = \infty.$$

Similarly, when $p = n$, there is a n-superharmonic function u in a neighborhood of the origin such that

$$\lim_{x\to 0 \text{ and } x\in E} \frac{u(x)}{\log \frac{1}{|x|}} = \infty.$$

5.2 Consequences in conformal geometry

In this subsection we will use the p-Laplace equation (9) and Theorem 8 to derive the consequence of the curvature condition $A^{(p)} \geq 0$. In the light of Lemma 4 from Subsection 4.2, we apply Theorem 8 to prove

Theorem 13. *([25, 26]) Suppose that S is a compact subset of a bounded domain $\Omega \subset \mathbb{R}^n$. And suppose that there is a metric $\bar{g}$ on $\Omega \setminus S$ such that*

- *it is conformal to the Euclidean metric $g_{\mathbb{E}}$;*
- *it is geodesically complete near S.*

Assume that $A^{(p)}[\bar{g}] \geq 0$ for some $p \in [2, n)$. Then

$$dim_{\mathcal{H}}(S) \leq \frac{n-p}{2}$$

for $p \in [2, n)$.

As a consequence,

Theorem 14. *([25]) Suppose that S is a closed subset of the sphere $\mathbb{S}^n$. And suppose that there is a metric $\bar{g}$ on $\mathbb{S}^n \setminus S$ that is conformal to the standard round metric $g_{\mathbb{S}}$. Assume that it is geodesically complete near S and that $A^{(p)}[\bar{g}] \geq 0$ for some $p \in [2, n)$. Then*

$$dim_{\mathcal{H}}(S) \leq \frac{n-p}{2}.$$

Historically, in [5, 27, 28], n-Laplace equations and applications in hypersurfaces and conformal geometry were first investigated in connection to find Huber's type theorems in general dimensions. The study of asymptotic behavior of n-superharmonic functions at singularities was also carried out by using on nonlinear potential theory in [25, 26] (cf. Theorem 11 in previous section).

Theorem 15. *([28]) For $n \geq 3$, let D be a bounded domain in the Euclidean space $(\mathbb{R}^n, |dx|^2)$ and let $S \subset D$ be a subset which is closed in $\mathbb{R}^n$. Suppose that, on $D \backslash S$, there is a conformal metric $g = e^{2v}|dx|^2$ satisfying*

$$\lim_{x \to S} v(x) = +\infty \text{ and } Ric_g^- |\nabla v|^{n-2} e^{2v} \in L^1(D \backslash S, |dx|^2).$$

Then S is a finite point set.

One consequence of Theorem 15 is the following corollary.

Corollary 1. *([28]) For $n \geq 3$, let Ω be a domain in the standard unit round sphere $(\mathbb{S}^n, g_{\mathbb{S}})$. Suppose that, on Ω, there is a complete conformal metric $g = e^{2u} g_{\mathbb{S}}$ satisfying either Ric_g is nonnegative outside a compact subset or*

(1) $Ric_g^- \in L^1(\Omega, g) \cap L^\infty(\Omega, g)$

(2) $R_g \in L^\infty(\Omega, g)$ *and* $|\nabla^g R_g| \in L^\infty(\Omega, g)$.

Then $\partial\Omega = \mathbb{S}^n \setminus \Omega$ *is a finite point set.*

5.3 Applications to fully nonlinear elliptic equations

In this subsection we collect some corollaries of Theorem 11 on the solutions to fully nonlinear elliptic equations. It is interesting to compare the intermediate positivity cones $\mathcal{A}^{(p)}$ with those in the study of fully nonlinear equations. Recall

$$\Gamma^k = \{\lambda = \{(\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n : \sigma_1(\lambda) \geq 0, \sigma_2(\lambda) \geq 0, \cdots, \sigma_k(\lambda) \geq 0\}$$

for $k = 1, 2, \cdots, n$, where σ_l is the elementary symmetric functions for $l = 1, 2, \cdots, n$. It is easily seen that $\mathcal{A}^{(2)} = \Gamma^1$ and $\mathcal{A}^{(p)}$ approaches Γ^n as $p \to \infty$. Hence, for any positive cone Γ between Γ^1 and Γ^n, we may consider

$$p_\Gamma = \max\{p : \Gamma \subset \mathcal{A}^{(p)}\}. \tag{28}$$

p_Γ is useful when one uses p-superharmonic functions to study solutions to a class of fully nonlinear elliptic equations. We first realize

Lemma 6. *([25, 26]) Suppose that* u *is nonnegative and that* $u \in C^2(\Omega \setminus S)$ *for a compact subset* S *of a bounded domain* Ω *in* $\mathbb{R}^n$*. And suppose* $\lim_{x\to S} u(x) = +\infty$*. Assume* -$\lambda(D^2u(x)) \in \Gamma$ *for* $p_\Gamma \in (1, n]$*. Then* u *is a* p_Γ*-superharmonic function in* Ω*.*

We remark that, from the proof of [27, Lemma 3.2 and 3.3] (see also [4, Proposition 1.1] when S is an isolated point), it is easily seen that $-\Delta_{p_\Gamma} u$ is a Radon measure under even somewhat weaker assumptions. Consequently, we have

Corollary 2. *([25, 26]) Suppose that* u *is nonnegative and that* $u \in C^2(\Omega \setminus S)$ *for a compact subset* S *of a bounded domain* Ω *in* $\mathbb{R}^n$*. And suppose* $\lim_{x\to S} u(x) = +\infty$*. Assume* -$\lambda(D^2u(x)) \in \Gamma$ *for* $p_\Gamma \in (1, n]$*. Then* S *is of Hausdorff dimension not greater than* $n - p_\Gamma$ *and, for* $x_0 \in S$*, there are a subset* E *that is* p_Γ*-thin for the singular behavior at* x_0 *and a nonnegative number* m *such that*

$$\lim_{x\to x_0 \text{ and } x\notin E} \frac{u(x)}{G_{p_\Gamma}(x, x_0)} = m.$$

Moreover $u(x) \geq mG_{p_\Gamma}(x, x_0) - c_0$ *in some neighborhood of* x_0*.*

It seems surprising that we have a rather effective way to calculate p_Γ for a cone associated with a homogeneous, symmetric, convex function of n-variables.

Lemma 7. *([25, 26]) Suppose that* Γ *is a cone given by a homogeneous, symmetric, convex function* $F(\lambda)$ *on* $\mathbb{R}^n$*. Let* $(-\frac{n-1}{p-1}, 1, 1, \cdots, 1) \in \partial\Gamma = \{\lambda \in \mathbb{R}^n : F(\lambda) = 0\}$*. Then* $\Gamma \subset \mathcal{A}^{(p)}$ *and* $p_\Gamma = p$*.*

Consequently, we can calculate p_{Γ^k} easily.

Corollary 3. *([25, 26]) For the positive cone Γ^k, we have*

$$p_{\Gamma^k} = \frac{n(k-1)}{n-k} + 2 \in [2, n] \tag{29}$$

for $1 \leq k \leq \frac{n}{2}$.

Proof. This simply is because

$$\sigma_k(-\frac{n-k}{k}, 1, 1, \cdots, 1) = -\frac{n-k}{k}\binom{k-1}{n-1} + \binom{k}{n-1} = 0.$$

QED

Remarkably, we are able to derive an asymptotic estimates that extends [23, Theorem 3.6] significantly.

Corollary 4. *([25, 26]) Suppose that u is nonnegative and that $u \in C^2(\Omega \setminus S)$ for a compact subset S inside a bounded domain Ω in $\mathbb{R}^n$. And suppose $\lim_{x \to S} u(x) = +\infty$. Assume $-\lambda(D^2 u(x)) \in \Gamma^k$ for $1 \leq k \leq \frac{n}{2}$. Then S is of Hausdorff dimension not greater than $n - p_\Gamma$ and, for $x_0 \in S$, there are a subset E that is p_{Γ^k}-thin for the singular behavior at x_0 and a nonnegative number m such that*

$$\lim_{x \to x_0 \text{ and } x \notin E} \frac{u(x)}{\mathcal{G}^k(x, x_0)} = m.$$

Moreover $u(x) \geq m\mathcal{G}^k(x, x_0) - c_0$ in some neighborhood of x_0, where

$$\mathcal{G}^k(x, x_0) = \begin{cases} |x - x_0|^{2-\frac{n}{k}} \text{ when } 1 \leq k < \dfrac{n}{2} \\ -\log|x - x_0| \text{ when } k = \dfrac{n}{2} \end{cases}.$$

In general it takes a lot more to rule out the thin set E in the above two corollaries, even for isolated singularities (cf. [20, 21, 40, 23]).

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