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Analysis of the harmonic flow of geometric structures

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Abstract. We develop an analysis of the flow of harmonic H-structures with the strategy introduced by Chen-Struwe for the harmonic map heat equation when the target does not necessarily have negative sectional curvature, so the Eells-Sampson Theorem cannot apply.

In particular, this flow method enables us to find theoretical hypotheses under which the existence of a torsion-free H-structure is guaranteed and conditions for which the flow must blow up in finite time.

This extends results already known for some specific groups like $\mathrm{U}(n)$, G_2 or $\mathrm{Spin}(7)$.

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1 Holonomy and geometric structures

Berger's 1955 theorem gives the list of the seven possible holonomy groups of an irreducible, non-symmetric, simply-connected Riemannian n-dimensional manifold: $\mathrm{SO}(n)$ (the general case), $\mathrm{U}(\frac{n}{2})$ (Kähler), $\mathrm{SU}(\frac{n}{2})$ (Calabi-Yau), $\mathrm{Sp}(\frac{n}{4})$ (Hyper-Kähler), $\mathrm{Sp}(\frac{n}{4})\mathrm{Sp}(1)$ (Quaternionic-Kähler), G_2 ($n = 7$), $\mathrm{Spin}(7)$ ($n = 8$); plus the group $\mathrm{Spin}(9)$ for $n = 16$, before it was removed by Alekseevskij [1].

Moreover, each case is characterised by the existence of a parallel (multi-) tensor, with respect to the Levi-Civita connection of (M, g).

For $\mathrm{SO}(n)$, it is the unobstructed condition of a torsion-free connection, the Kähler case is equivalent to the existence of a parallel almost Hermitian structure, etc.

For G_2, the parallel three-form φ must, point-wise, be given in some frame of T_pM by

$$\varphi_p = e^{123} - e^1 \wedge (e^{45} - e^{67}) + e^2 \wedge (e^{46} - e^{57}) + e^3 \wedge (e^{47} - e^{56}),$$

while for Spin(7), $\Phi \in \Lambda^4(T^*M)$ and the local model is

$$\begin{aligned}\Phi_p = e^{0123} - e^{0145} - e^{0167} - e^{0246} + e^{0257} - e^{0347} - e^{0356} \\ + e^{4567} - e^{2367} - e^{2345} - e^{1357} + e^{1346} - e^{1256} - e^{1247}\end{aligned}$$

where

$$e^{0123} = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3.$$

The significance of holonomy is twofold. First, it informs on the curvature tensor as it must live in the holonomy Lie algebra and this implies for $\mathrm{SU}(\frac{n}{2})$, $\mathrm{Sp}(\frac{n}{4})$, G_2 and Spin(7) that the manifold is Ricci-flat.

On the other hand, manifolds with special holonomy are the cornerstone of higher-dimensional Gauge Theory. For example, take $n = 7$ and G_2, then a vector bundle $E \to M^7$ with connection ∇ has curvature $F^\nabla \in \Lambda^2$, so the condition $*F^\nabla = \lambda F^\nabla$ $(\lambda \in \mathbb{R})$ does not make sense anymore and we must introduce a three-form to counterbalance the degrees. If chosen closed, it will allow for Yang-Mills connections.

The disadvantage is that it is generally hard to construct manifolds with special holonomy, as the problem is highly non-linear, especially when the defining tensor determines the metric. For example, local solutions of G_2-manifolds first appear in 1984 [3], complete ones in 1986 [4] and compact constructions only as late as 1996-2000 with Joyce [15, 16].

An alternative, softer and more topological in nature, is to consider H-structures ($H \subset \mathrm{SO}(n)$), which are reductions of the structure group and can also be seen as sections of an ad-hoc twistor space constructed as the H-quotient of the unitary frame bundle $P_{\mathrm{SO}(n)}$.

Locally, H-structures are described by (multi-) tensors given at each point by a Euclidean prototype and stabilized by the group H.

It relies on a model (p,q)-tensor $\xi_0 \in T^{p,q}(\mathbb{R}^n)$ on the Euclidean space, stabilised by the group $H = \mathrm{Stab}(\xi_0) = \{\mathrm{g} \in \mathrm{GL}(n,\mathbb{R}) : \mathrm{g}.\xi_0 = \xi_0\}$ under the $\mathrm{GL}(n,\mathbb{R})$ right action

$$\mathrm{g}.\xi_0 = \xi^{i_1 \dots i_p}_{j_1 \dots j_q} \mathrm{g}^{-1} e_{i_1} \otimes \cdots \otimes \mathrm{g}^{-1} e_{i_p} \otimes \mathrm{g}^* e^{j_1} \otimes \cdots \otimes \mathrm{g}^* e^{j_q}. \qquad (1)$$

On the manifold (M,g), an H-structure ξ is a (multi-) tensor such that at any point $p \in M$ there is a frame of T_pM that identifies ξ_p and ξ_0 (or several ξ_0's in case of a multi-tensor).

In fact, this approach goes further than Berger's list and includes new cases such as almost contact structures ($H = \mathrm{U}(n) \times \mathrm{Id} \subset \mathrm{SO}(2n+1)$), parallelisms (i.e. global orthonormal frames for $H = \{1\}$) or unit vector fields ($H = \mathrm{SO}(n-1) \subset \mathrm{SO}(n)$) and others of varying degrees of interest.

Finally, one should add that H-structures with torsion, i.e. non-parallel, are key ingredients in several theories of Mathematical Physics, for example the models of Supersymmetric AdS4 compactifications of IIA supergravity of Lüst and Tsimpis [19]. This fact alone should be sufficient reason for their study.

2 Harmonic maps and harmonicity

For an abstract (smooth) map ϕ from a compact Riemannian manifold (M, g) to (N, h), the Dirichlet energy is

$$E(\phi) = \frac{1}{2} \int_M |d\phi|^2 \, v_g$$

where the energy density $\frac{1}{2}|d\phi|^2$ is the trace of $\frac{1}{2}\phi^* h$. Critical points of E are called harmonic maps [10] and the corresponding Euler-Lagrange equation is the vanishing of the tension field

$$\tau(\phi) = \operatorname{trace} \nabla d\phi.$$

The energy functional is conformally invariant when $\dim M = 2$ and harmonic maps generalise harmonic functions ($N = \mathbb{R}$), geodesics ($\dim M = 1$), totally geodesic maps (since $\nabla d\phi = 0$), minimal submanifolds (isometric immersions) and holomorphic maps between Kähler manifolds. Cf. [2, Chap. 3] for details and examples.

The groundbreaking work of Eells and Sampson [10] did not just introduced this notion but also proved the existence of a harmonic representative in each homotopy class when the target has non-positive sectional curvature. One should however keep in mind that if there only exists a single homotopy class, it will contain the trivially-harmonic constant maps.

Perhaps, as important was the heat-flow method they employed which has known a rich legacy, all the way to Hamilton's Ricci flow and Perelman's proof of the Poincaré conjecture.

This method is perhaps just as compelling by its failure when the curvature condition on the target is not met and the flow develops bubbles and explodes, as is the case of the sphere $\mathbb{S}^2$ to itself. This was superbly exploited by Sachs and Uhlenbeck with a reparametrization process enabling to restart the flow and accounting for the topological penalties in terms of energy quantization.

Paradoxically, for the two-sphere, these pitfalls of the heat flow can easily be circumvented by considering holomorphic maps and produce harmonic representatives in each and every homotopy class.

The best-known situation where harmonic maps do not exist is for degree-one maps from the two-torus to the two-sphere [11]. Indeed, such a map would have to be conformal, hence holomorphic, which is forbidden for degree one by Riemann-Roch.

In higher dimensions, one of the most often encountered example is the Hopf map from $\mathbb{S}^3$ to $\mathbb{S}^2$ and, as a Riemannian submersion with minimal fibres, it also happens to be a harmonic morphism.

In spite of decades of work on harmonic maps, relatively little has been done for sections, except perhaps for vector fields [7], and C. M. Wood was the first to consider almost Hermitian structures as sections of the twistor space and study them under the light of harmonicity [20].

The point here is not merely to compute the tension field of a particular map but more precisely to consider the Dirichlet energy of sections and search for its critical points among sections, in order to compare like with like.

On a philosophical level, there is no reason to limit this to the specific case of almost Hermitian structures and any type of reduction of structures on a Riemannian manifold can be the subjected to the same treatment.

The description of the infinitesimal variation of an H-structure ξ relies on the infinitesimal action of endomorphisms $A \in \Gamma(\mathrm{End}(TM))$ on $T^{p,q}(TM)$

$$\begin{aligned} A \diamond \xi &:= \left.\frac{d}{dt}\right|_{t=0} e^{tA}.\xi \\ &= \xi^{i_1\dots i_p}_{j_1\dots j_q}\Big(\sum_{r=1}^{p} -\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes A\frac{\partial}{\partial x^{i_r}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_p}}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_q} \\ &\qquad + \sum_{s=1}^{q}\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_p}}\otimes dx^{j_1}\otimes\cdots\otimes A^* dx^{j_s}\otimes\cdots\otimes dx^{j_q}\Big). \end{aligned}$$

If $A = (A^i_j) \in \mathfrak{gl}(n,\mathbb{R})$ then

$$(A\diamond\xi)^{i_1\dots i_p}_{j_1\dots j_q} = -\sum_{r=1}^{p} A^{i_r}_m \xi^{i_1\dots m\dots i_p}_{j_1\dots j_q} + \sum_{s=1}^{q} A^m_{j_s}\xi^{i_1\dots i_p}_{j_1\dots m\dots j_q}$$

and for a multi-tensor $\xi = (\xi_1,\dots,\xi_k)$, we define

$$A\diamond\xi := (A\diamond\xi_1,\dots,A\diamond\xi_k).$$

Cf. [12, Lemma 1.4] for more properties of the operator $\diamond$.

Assume $H \subset \mathrm{SO}(n)$ closed and connected so that the quotient $\mathrm{SO}(n)/H$ is a normal homogeneous Riemannian manifold with the metric induced by the canonical bi-invariant metric on $\mathrm{SO}(n)$ given by $\langle A,B\rangle = -\operatorname{trace}(AB)$. The H-module decomposition

$$\mathfrak{so}(n) = \mathfrak{h}\oplus\mathfrak{m}, \tag{2}$$

where $\mathfrak{m} := \mathfrak{h}^{\perp} \subset \mathfrak{so}(n)$ is the orthogonal complement of $\mathfrak{h} = \mathrm{Lie}(H)$ with respect to $\langle\cdot,\cdot\rangle$, is a *reductive* decomposition, i.e., it satisfies $\mathrm{Ad}_{\mathrm{SO}(n)}(H)\mathfrak{m} \subseteq \mathfrak{m}$.

Now suppose (M^n, g) admits a compatible H-structure $Q \subset \mathrm{Fr}(M,g)$. Since (2) is reductive, the H-structure induces an orthogonal H-module decomposition on the subbundle $\mathfrak{so}(TM) := \mathrm{Fr}(M,g) \times_{\mathrm{SO}(n)} \mathfrak{so}(n)$ of skew-symmetric endomorphisms in $\mathrm{End}(TM) = T^*M \otimes TM$:

$$\mathfrak{so}(TM) = \mathfrak{h}_Q \oplus \mathfrak{m}_Q, \quad \text{where} \tag{3}$$
$$\mathfrak{h}_Q := Q \times_H \mathfrak{h} \quad \text{and} \quad \mathfrak{m}_Q := Q \times_H \mathfrak{m}.$$

From (3), we get a corresponding H-module decomposition on $\Lambda^2(T^*M) \simeq \mathfrak{so}(TM)$:

$$\Lambda^2 = \Lambda^2_{\mathfrak{h}} \oplus \Lambda^2_{\mathfrak{m}}, \quad \text{with} \quad \Lambda^2_{\mathfrak{h}} \simeq \mathfrak{h}_Q \quad \text{and} \quad \Lambda^2_{\mathfrak{m}} \simeq \mathfrak{m}_Q.$$

Write $\Omega^2_{\mathfrak{h}} = \Gamma(\Lambda^2_{\mathfrak{h}})$ and $\Omega^2_{\mathfrak{m}} = \Gamma(\Lambda^2_{\mathfrak{m}})$, so

$$\Gamma(\mathrm{End}(TM)) \simeq \Omega^0 \oplus \Sigma^2_0 \oplus \Omega^2_{\mathfrak{h}} \oplus \Omega^2_{\mathfrak{m}},$$

where Ω^0 is the trivial submodule of $\Sigma^2(M)$ spanned by the Riemannian metric and Σ^2_0 denotes the space of traceless symmetric bilinear forms, hence

$$A = \frac{1}{\dim M}(\mathrm{trace}\, A)g + A_0 + A_{\mathfrak{h}} + A_{\mathfrak{m}}.$$

Lemma 2.1. Let $\sigma \in \Gamma(\mathrm{Fr}(M,g)/H)$ be a compatible H-structure:

(i) If $\xi \in \Gamma(\mathcal{T}^{p,q}(TM))$ is stabilised under the action of H, then $\Omega^2_{\mathfrak{h}} \subseteq \ker(\cdot\diamond\xi)$.

(ii) If $H = \mathrm{Stab}(\xi_\circ)$, so that σ corresponds to a geometric structure $\xi = (\xi_1, \dots, \xi_k)$ modelled on $\xi_\circ$, then

$$\Omega^2_{\mathfrak{h}} = \ker(\cdot \diamond \xi) = \ker(\cdot \diamond \xi_1) \cap \ldots \cap \ker(\cdot \diamond \xi_k).$$

(iii) If $H = \mathrm{Stab}_{\mathrm{SO}(n)}(\xi_\circ)$, so that σ corresponds to a geometric structure ξ modelled on $\xi_\circ$, which is compatible with g and vol_g, then

$$\Omega^2_{\mathfrak{h}} = \ker(\cdot \diamond \xi) \cap \Omega^2.$$

Example 2.2. (1) When $H = \mathrm{U}(m) = \mathrm{Stab}_{\mathrm{SO}(2m)}(J_\circ)$, where $J_\circ$ is the standard complex structure on $\mathbb{R}^{2n}$, the complement $\mathfrak{m} = \mathfrak{u}(m)^{\perp} = \{A \in \mathfrak{so}(n) : AJ_\circ = -J_\circ A\}$ is irreducible, and for any compatible $\mathrm{U}(m)$-structure $\xi = J$ on (M^{2m}, g), using [12, Lemma (1.4)] we can compute, for all $A, B \in \Omega^2_{\mathfrak{m}}(M)$,

$$\begin{aligned}\langle A \diamond J, B \diamond J\rangle &= \langle [A,J],[B,J]\rangle = \langle 2AJ, (-2)JB\rangle = 4\,\mathrm{trace}(AJJB)\\ &= 4\langle A, B\rangle.\end{aligned}$$

(2) When $H = \mathrm{G}_2 \subset \mathrm{SO}(7)$, the complement $\mathfrak{m} = \Lambda^2_7 \subset \mathfrak{so}(7)$ is irreducible, and if φ is a G_2-structure on M^7 then [18, §2.2]

$$\langle A \diamond \varphi, B \diamond \varphi \rangle = 6 \langle A, B \rangle, \quad \forall A, B \in \Omega^2_{\mathfrak{m}}(M).$$

(3) When $H = \mathrm{Spin}(7) \subset \mathrm{SO}(8)$, the complement $\mathfrak{m} = \Lambda^2_7 \subset \mathfrak{so}(8)$ is irreducible, and if Φ is a Spin(7)-structure on M^8 then [17, Proposition 2.5]:

$$\langle A \diamond \Phi, B \diamond \Phi \rangle = 16 \langle A, B \rangle, \quad \forall A, B \in \Omega^2_{\mathfrak{m}}(M).$$

(4) When $H = \{1\} \subset \mathrm{SO}(n)$ is the trivial subgroup, we have $\mathfrak{h} = \{0\}$ and the reducible $\{1\}$-module $\mathfrak{m} = \mathfrak{so}(n)$ splits completely into the trivial one-dimensional representations generated by each element of the standard basis of $\mathfrak{so}(n)$. In this case,

$$\lambda_1 = \ldots = \lambda_{\dim \mathfrak{so}(n)} = 1.$$

Indeed, a compatible $\{1\}$-structure is simply a global oriented orthonormal frame $\xi = (\xi_1, \ldots, \xi_n)$ of TM, and $A \diamond \xi = (-A\xi_1, \ldots, -A\xi_n)$ for every $A \in \Omega^2_{\mathfrak{m}}(M) = \Omega^2(M)$, so

$$\langle A \diamond \xi, B \diamond \xi \rangle = \sum_{j=1}^{n} \langle A\xi_j, B\xi_j \rangle = \langle A, B \rangle, \quad \forall A, B \in \Omega^2_{\mathfrak{m}}(M) = \Omega^2(M).$$

(5) For $H = \mathrm{SU}(m) \subset \mathrm{SO}(2m)$, $m \geqslant 2$, $\mathfrak{m}$ is reducible and splits into two non-trivial irreducible submodules $\mathfrak{m}_1$ and $\mathfrak{m}_2$, with $\lambda_1(m) \neq \lambda_2(m)$. We describe the group $\mathrm{SU}(m)$ with the standard coordinates $(x^i, y^i,)_{i=1,\ldots,m}$ on $\mathbb{R}^{2m}$, the canonical complex structure $J_\circ$, the Euclidean metric $g_\circ$ and the fundamental 2-form $\omega_\circ$:

$$J_\circ \tfrac{\partial}{\partial x^p} = \tfrac{\partial}{\partial y^p}, \quad J_\circ \tfrac{\partial}{\partial y^p} = -\tfrac{\partial}{\partial x^p},$$
$$g_\circ = \sum_{p=1}^{m} \left(dx^p \otimes dx^p + dy^p \otimes dy^p \right), \quad \text{and} \quad \omega_\circ = \sum_{p=1}^{m} dx^p \wedge dy^p.$$

Then, SU(m) is the subgroup of $\mathrm{GL}(2m, \mathbb{R})$ preserving $g_\circ$, $J_\circ$ (and $\omega_\circ$) and the *complex volume form*

$$\Upsilon_\circ = dz^1 \wedge \ldots \wedge dz^m \in \Lambda^m_{\mathbb{C}}(\mathbb{C}^m)^*.$$

So the model structure is $\xi_\circ = (J_\circ, \Upsilon_\circ)$ and $\mathrm{SU}(m) = \mathrm{Stab}_{\mathrm{SO}(2m)}(\xi_\circ)$.

Now $\mathfrak{m} = \mathfrak{su}(m)^\perp \subset \mathfrak{so}(2m)$ is a reducible H-module, since

$$\mathfrak{so}(2m) = \mathfrak{u}(m) \oplus \mathfrak{u}(m)^\perp = \mathfrak{su}(m) \oplus \langle J_\circ \rangle \oplus \mathfrak{u}(m)^\perp,$$

and we have the orthogonal decomposition of $\mathfrak{m}$ into the irreducible submodules

$\mathfrak{m}_1 = \langle J_\circ \rangle \subset \mathfrak{so}(2m)$ and $\mathfrak{m}_2 = \mathfrak{u}(m)^\perp = \{A \in \mathfrak{so}(2m) : AJ_\circ = -J_\circ A\}$.

Direct computations show that

$\lambda_1 = m2^{m-1}$ and $\lambda_2 = 4 + 2^{m-1}$, cf. [12] for details of the computations.

The fundamental object governing the behaviour of the covariant derivative of an H-structure is the torsion T

Lemma 2.3. Let ξ be a compatible H-structure, where $H = \mathrm{Stab}_{\mathrm{SO}(n)}(\xi)$. Then

$$\nabla_X \xi = T_X \diamond \xi, \quad \forall X \in \mathscr{X}(M), \tag{4}$$

where $T \in \Omega^1(M, \mathfrak{m})$ denotes the torsion of the H-structure ξ. In particular, there are constants $c, \tilde{c} > 0$, depending only on (M,g) and H, such that

$$\tilde{c}|T|^2 \leqslant |\nabla\xi|^2 \leqslant c|T|^2.$$

If furthermore there is $c > 0$ such that $\langle A \diamond \xi, B \diamond \xi \rangle = c\langle A, B\rangle$, for all $A, B \in \Omega^2_{\mathfrak{m}}(M)$, i.e. if $c = \lambda_1 = \ldots = \lambda_k$ (e.g. if $\mathfrak{m}$ is an irreducible H-module), then

$$|\nabla\xi|^2 = c|T|^2. \tag{5}$$

Example 2.4. (1) When $H = \mathrm{U}(m)$, Lemma 2.3 gives

$$\nabla_X J = (T_X \diamond J) = -[T_X, J] = 2JT_X,$$

since $T_X \in \Omega^2_{\mathfrak{u}(m)^\perp} \simeq \{A \in \mathfrak{so}(M) : AJ = -JA\}$. Thus, we have

$$T_X = -\frac{1}{2} J \nabla_X J. \tag{6}$$

In particular, $|\nabla J|^2 = 4|T|^2$.

(2) When $H = \mathrm{G}_2$ and given a G_2-structure φ on M^7, the space of 3-forms decomposes into irreducible G_2-modules $\Omega^3 = \Omega^3_1 \oplus \Omega^3_7 \oplus \Omega^3_{27}$. We know that $\ker(\cdot \diamond \varphi) = \Omega^2_{\mathfrak{g}_2}$ and, by dimension counting, $\cdot \diamond \varphi$ maps $\Omega^2_{\mathfrak{m}}$ isomorphically into Ω^3_7. Since $T_X \in \Omega^2_{\mathfrak{m}}$, we see that $\nabla_X \varphi \in \Omega^3_7$ and $|\nabla\varphi|^2 = 6|T|^2$. Because of this description of $\Omega^2_{\mathfrak{m}}(M)$, it is common in G_2-geometry to identify the intrinsic torsion $T \in \Omega^1(M, \Lambda^2_{\mathfrak{m}})$ with the endomorphism $\mathcal{T}_{lm}$ defined by $T_{l;ij} =: -\frac{1}{3}\mathcal{T}_{lm}\varphi_{mij}$. So $|T|^2 = \frac{2}{3}|\mathcal{T}|^2$ and

$$\mathcal{T}_{pq} = \frac{1}{24} \nabla_p \varphi_{ijk} \psi_{qijk}.$$

where $\psi = *\varphi$ is the 4-form dual to φ.

(3) When $H = \mathrm{Spin}(7) \subset \mathrm{SO}(8)$, a Spin(7)-structure Φ on M^8 induces a decomposition on the space of 4-forms into irreducible Spin(7)-submodules $\Omega^4 = \Omega^4_1 \oplus \Omega^4_7 \oplus \Omega^4_{27} \oplus \Omega^4_{35}$. Then, arguing as in the previous example, $\cdot \diamond \Phi$ maps $\Omega^2_{\mathfrak{m}}$ isomorphically into Ω^4_7, and (4) yields $\nabla_X \Phi = T_X \diamond \Phi$, implying that $\nabla_X \Phi \in \Omega^4_7$, moreover

$$|\nabla \Phi|^2 = 16 |T|^2,$$

and

$$T_{m;ab} = \frac{1}{96} (\nabla_m \Phi_{ajkl}) \Phi_{bjkl}.$$

The L^2-norm of the torsion is precisely the vertical part of the energy density of the map σ into the ad-hoc twistor space and, with respect to a fixed metric on M, taking variations through H-structures is identical to choosing an element in $\Omega^2_{\mathfrak{m}}$. This (vertical) Dirichlet energy functional on the space of H-structures assigns to each H-structure ξ the L^2-norm of its torsion T, with respect to its induced metric g:

$$\mathrm{E}(\xi) = \frac{1}{2} \int_M |T|^2 \,\mathrm{vol}_g \,. \tag{7}$$

We compute the general first variation of $\mathrm{E}(\xi)$:

Corollary 2.5. If $\{\xi(t)\}$ is a smooth family of isometric H-structures, inducing the fixed Riemannian metric g, with $\xi(0) = \xi$ and $\frac{d}{dt}|_{t=0} \xi(t) = C \diamond \xi$, for $C \in \Omega^2_{\mathfrak{m}}$, then

$$\frac{d}{dt}\Big|_{t=0} \mathrm{E}(\xi) = - \int_M \langle \operatorname{div} T, C \rangle \,\mathrm{vol}_g, \tag{8}$$

where T is the torsion of the H-structure ξ.

This computation leads naturally to the next definition

Definition 2.6. Let (M^n, g) be an oriented Riemannian n-manifold admitting a compatible H-structure ξ. We say that ξ is *harmonic* when it has divergence-free torsion:

$$\operatorname{div}_g T = 0.$$

Example 2.7. For $H = \mathrm{U}(m)$, a direct computation based on Equation (6) yields

$$\operatorname{div}_g T = -\frac{1}{4} [\nabla^* \nabla J, J].$$

To emulate the proof of Eells-Sampson for sections we introduce a tailor-made parabolic equation which will evolve according to the gradient of steepest descent, i.e. the vector field $\operatorname{div} T$.

Definition 2.8. Let (M^n, g) be an oriented Riemannian n-manifold admitting a compatible H-structure. A family of compatible H-structures $\{\xi(t)\}_{t\in I}$ on (M, g), parametrised by a non-degenerate interval $I \subset \mathbb{R}$, is a solution to the *harmonic flow of H-structures* (or *harmonic H-flow* for short) if the following evolution equation holds for every $t \in I$:

$$\frac{\partial}{\partial t}\xi(t) = \operatorname{div} T(t) \diamond \xi(t), \tag{HF}$$

where $T(t)$ denotes the torsion of $\xi(t)$. Given a compatible H-structure ξ_0 on (M^n, g), a solution to the harmonic flow of H-structures with *initial condition* (or *starting at*) ξ_0 is a solution of (HF) defined for every $t \in [0, \tau_0)$, for some $0 < \tau_0 \leqslant \infty$, and such that $\xi(0) = \xi_0$.

3 The Heat flow

The ultimate objective of the heat flow of geometric structures is to provide conditions ensuring the existence of an absolute minimiser of the Dirichlet energy and answer the question of torsion-free geometric structures in a given homotopy class.

With a target of positive curvature, one cannot hope to apply Eells-Sampson's original proof and must turn to Chen-Struwe's arguments. In the end, this approach leads to a threshold for the Dirichlet energy of the initial data under which convergence will happen to a torsion-free solution. By contraposition, this lower bound can also be used to obtain singularity results.

This method is based on a series of more or less technical results but first, one should recall that short-time existence is an automatic consequence of the harmonic map equation. The main equation we will work with is the Bochner formula which essentially turns the heat flow equation, living in the pull-back of the vertical space of the twistor space, i.e. the bundle associated with the isotropic representation, into a scalar expression at the cost of becoming an inequality.

The Shi-type estimates control, along the flow, the norms of the derivatives of the torsion (basically the coderivatives of the geometric structure) in terms of the torsion itself and the derivatives of the curvature.

Both these results are valid for a general group $H \subset SO(n)$ (and even more cf. [9]).

The cornerstone of the whole strategy is the Almost Monotonicity Formula (AMF) for a time-dependent functional which integrates, over a geodesic ball, the product of the energy density and the backward heat kernel with singularity at the centre of the ball (along with localisation by a cut-off function).

This AMF will prompt the ϵ-regularity of the heat flow which draws a pointwise bound of the energy density from a bound on the integral of the function involved in the AMF.

This opens the gates, not only for energy gap results but also for long-time existence and finite-time singularity statements.

Indeed, by a refined treatment of the Bochner formula, due to Chen-Ding [5], controlling the existence interval of the harmonic flow, one can prove that if the energy density remains bounded over the maximal existence interval then the flow exists for all time and converges to a harmonic solution.

On the other hand, if it explodes then the maximal existence-time must be less than (the square-root of) the initial energy.

To conclude, bounds on the Dirichlet energy and L^∞-norm of the initial data exclude explosion of the energy density over the maximal existence interval, hence force long-time existence and convergence toward a torsion-free solution.

Additionally, if no torsion-free H-structure exists in a given homotopy class while the infimum of the Dirichlet energy is indeed zero, small enough initial Dirichlet energy will make the flow develop a finite-time singularity.

Versions of these results for specific groups H can be found in [14, 8] but note that groups like $\mathrm{SU}(n)$ are not covered by these results.

Assume (M^n, g) to be an oriented Riemannian n-manifold of bounded geometry. In §1 we saw that a compatible H-structure ξ on (M^n, g) corresponds to a section σ of $\pi : \mathrm{Fr}(M, g)/H \to M$. Now, there is a natural isomorphism between $\pi : \mathrm{Fr}(M, g)/H \to M$ and the associated bundle $\mathrm{Fr}(M, g) \times_{\mathrm{SO}(n)} \mathrm{SO}(n)/H$, which, fibrewise, is an isometry with respect to the bi-invariant metric on $\mathrm{SO}(n)$. The induced one-to-one correspondence between sections $\sigma \in \Gamma(\mathrm{Fr}(M, g)/H)$ and $\mathrm{SO}(n)$-equivariant maps $s : \mathrm{Fr}(M, g) \to \mathrm{SO}(n)/H$ identifies solutions to the harmonic section flow with $\mathrm{SO}(n)$-equivariant solutions to the classical harmonic map heat flow for maps $\mathrm{Fr}(M, g) \to \mathrm{SO}(n)/H$, where the target space $\mathrm{SO}(n)/H$ is considered with its normal homogeneous Riemannian manifold structure.

As the short-time existence of the heat flow of harmonic maps is a standard consequence of the ellipticity of the tension field and the harmonic H-flow is merely the vertical part of the heat flow for harmonic maps from M to $Fr(M)/H$, its short-time existence is assured.

Proposition 3.1 (Short time existence)**.** Given any smooth compatible H-structure ξ_0 on (M^n, g), there is a maximal time $0 < \tau(\xi_0) \leqslant \infty$ such that the harmonic H-flow (HF) with initial condition ξ_0 admits a unique smooth solution $\xi(t)$ for $t \in [0, \tau)$.

To be able to work with the heat flow equation (HF) and control its behaviour, one needs to transform it into a scalar equation. This is the role of the Bochner formula.

Lemma 3.2 (Bochner-type estimate)**.** There is a uniform constant $c > 0$, depending only on (M, g) and H, such that if $\{\xi(t)\}_{t\in I}$ is a solution to the harmonic H-flow (HF) on $B_r(y) \subset (M, g)$, then

$$(\partial_t - \Delta)e(\xi) \leqslant c(e(\xi)^2 + 1),$$

where $e(\xi)$ denotes $|T|^2$.

The influence of the geometry of the manifold on the higher derivatives of a solution of (HF) is explained by the Shi-type estimates, originally established for the Ricci flow [Shi].

Proposition 3.3 (Shi-type estimates)**.** Let $\kappa \geqslant 1$ and $\{\xi(t)\}_{t\in[0,\kappa^{-4}]}$ be a solution of the harmonic H-flow (HF). Assume that there are constants B_j such that

$$|\nabla^j Rm| \leqslant B_j \kappa^{j+2}, \quad \forall j \geqslant 0.$$

If $|T| \leqslant \kappa$, then, for each $m \in \mathbb{N}$, there is a constant $c_m = c_m(M, g, H)$ such that

$$|\nabla^m T| \leqslant c_m \kappa t^{-m/2}, \quad \forall t \in \left[0, \frac{1}{\kappa^4}\right].$$

While the above results rely on general principles and are valid for any group $H \subset \mathrm{SO}(n)$, henceforth we will assume the representation theoretical condition (2.1):
Let $H = \mathrm{Stab}_{\mathrm{SO}(n)}(\xi_\circ)$, where $\xi_\circ$ is an element of a r-dimensional $\mathrm{SO}(n)$-submodule $V \leqslant \oplus \mathcal{T}^{p,q}(\mathbb{R}^n)$, where $V = V_1 \oplus \ldots \oplus V_k$ with $V_i \leqslant \mathcal{T}^{p_i,q_i}(\mathbb{R}^n)$. We suppose that H is such that $\lambda_1 = \ldots = \lambda_k$, i.e. there is $c > 0$ such that

$$\langle A \diamond \xi, B \diamond \xi \rangle = c\langle A, B\rangle, \quad \forall A, B \in \Omega^2_{\mathfrak{m}}(M). \tag{9}$$

This condition is satisfied by $\mathrm{U}(m)$, G_2, $\mathrm{Spin}(7)$, the quaternion-Kähler case $\mathrm{Sp}(k)\mathrm{Sp}(1) \subset \mathrm{SO}(4k)$ [13] and the trivial subgroup case $H = \{1\}$. However, this is not the case for $\mathrm{SU}(m)$.

An essential ingredient of the Chen-Struwe [6] scheme is the Almost Monotonicity Formula. It controls the time-growth of a functional which only differs from the Dirichlet energy by the insertion of the Euclidean backward heat kernel localised around a given singularity.

Let $r_M > 0$ be a lower bound to the injectivity radius of (M^n, g) such that there is a uniform constant $c > 0$ such that $\forall y \in M$, the components g_{ij} in normal coordinates $x = (x^1, \ldots, x^n)$ on the geodesic ball $B_{r_M}(y)$ satisfy:

$$\frac{1}{4}\delta_{ij} \leqslant g_{ij} \leqslant 4\delta_{ij}, \quad \text{(as bilinear forms)}$$
$$|g_{ij} - \delta_{ij}| \leqslant c|x|^2 \quad \text{and} \quad |\partial_k g_{ij}| \leqslant c|x|,$$

where $|x|$ is the Euclidean distance in $B_{r_M}(0) \subset T_xM \cong \mathbb{R}^n$. The constants $r_M > 0$ and $c > 0$ can be chosen to depend only on the injectivity radius and the curvature of g.

Let $\{\xi(t)\}$ be a solution to the harmonic H-flow (HF), with initial condition $\xi(0) = \xi_0$ and maximal interval of existence and uniqueness $[0, \tau)$, fix any $\tau_0 \in (0, \tau)$ and a cut-off function

$$\phi \in C_c^\infty(B_{r_M}(0)) \quad \text{with} \quad \phi|_{B_{(r_M)/2}(0)} \equiv 1.$$

For all $t \in (0, \tau_0)$ and $0 < r \leqslant \min\{\sqrt{\tau_0}/2, r_M\}$, we define the following functions associated to the energy functional $\mathcal{E}$:

$$\Theta(t) = (\tau_0 - t) \int_{\mathbb{R}^n} |T|^2(\cdot, t) G_{(0,\tau_0)}(\cdot, t)\phi^2 \sqrt{\det(g)}dx,$$
$$\Psi(r) = \int_{\tau_0-4r^2}^{\tau_0-r^2} \int_{\mathbb{R}^n} |T|^2 G_{(0,\tau_0)}\phi^2 \sqrt{\det(g)}dxdt,$$

where, for any $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$ we denote by

$$G_{(x_0,t_0)}(x, t) = (4\pi(t_0 - t))^{-n/2} \exp\left(-\frac{|x - x_0|^2}{4(t_0 - t)}\right)$$

the Euclidean backward heat kernel with singularity at (x_0, t_0). The quantities $\Theta(t)$ and $\Psi(r)$ are invariant under parabolic rescaling, cf. [12].

Theorem 3.4. For any $\tau_0 - \min\{\tau_0, 1\} < t_1 \leqslant t_2 < \tau_0$ and $N > 1$, the following almost-monotonicity formula holds:

$$\Theta(t_2) \leqslant e^{c(f(t_2)-f(t_1))}\left(\Theta(t_1) + c\left(N^{n/2}(E_0 + \sqrt{E_0}) + \frac{1}{\ln^2 N}\right)(t_2 - t_1)\right),$$

where $c = c(M, g) > 0$ is a constant, E_0 denotes the energy of $\xi(0)$ and $f(t) = \hat{f}(\tau_0 - t)$ with

$$\hat{f}(x) = -x(\ln^4(x) - 4\ln^3(x) + 13\ln^2(x) - 26\ln(x) + 26).$$

Its proof relies on particular identities of the time-derivative and space-gradient of this backward heat kernel, the Bianchi identity and a careful choice of the function f.

Having a control over an integral quantity along (HF), we need to deduce point-wise information from it, in the form of the ϵ-regularity.

The following result along the harmonic H-flow generalises the case $H = \mathrm{U}(m)$ in [14, Theorem 3.3].

Theorem 3.5 (ε-regularity)**.** For any $E_0 \in (0,\infty)$, there exists a constant $\varepsilon_0 > 0$, depending only on (M^n, g), H and E_0, such that if $\{\xi(t)\}$ is a solution to the harmonic H-flow (HF) on $B_{r_M}(y) \times [0,\tau) \subset M \times [0,\tau)$, with $\tau \leqslant r_M^2$ and initial energy bounded by E_0, and fix any $\tau_0 \in (0,\tau)$.

If, for some $0 < R < \min\{\varepsilon_0, \sqrt{\tau_0}/2\}$,

$$\Psi(R) < \varepsilon_0,$$

then

$$\sup_{P_{\delta R}(0,\tau_0)} e(\xi) \leqslant 4(\delta R)^{-2},$$

where the constant $\delta > 0$ depends only on (M^n, g) and H, and possibly on E_0 and $\min\{1, R\}$.

The ϵ-regularity and the Shi-type estimates imply an energy gap theorem for harmonic H-structures by a relatively short contradiction argument.

Proposition 3.6 (Energy gap)**.** There is a constant $\varepsilon_0 > 0$, depending only on (M^n, g) and the group H, such that, if ξ is a compatible harmonic H-structure whose Dirichlet energy satisfies $\mathrm{E}(\xi) = \frac{1}{2}\|\nabla\xi\|^2_{L^2(M)} < \varepsilon_0$, then ξ is actually torsion-free, i.e. $\nabla\xi = 0$.

Proof. By contradiction, if there is (ξ_k) a sequence of harmonic H-structures with $\mathcal{D}(\xi_k) \to 0$ but $\nabla\xi_k \neq 0$ for all k, by the ε-regularity and Shi-type estimates applied to each ξ_k (as a static harmonic H-flow), for $k \gg 1$, $|\nabla^m \xi_k|$ is uniformly bounded, so (ξ_k) subconverges in the smooth topology to a torsion-free H-structure ξ. But since ξ_k is harmonic, $|\Delta\xi_k| \leqslant c|\nabla\xi_k|^2$, and since $\nabla\xi = 0$, we get

$$|\Delta(\xi_k - \xi)| \leqslant c|\nabla(\xi_k - \xi)|^2.$$

Integrating by parts gives

$$\int_M |\nabla(\xi_k - \xi)|^2 \leqslant c\|\xi_k - \xi\|_{L^\infty(M)} \int_M |\nabla(\xi_k - \xi)|^2.$$

Since $\xi_k \to \xi$ in the smooth topology, the above yields $\nabla(\xi_k - \xi) = 0$, i.e. $\nabla\xi_k = \nabla\xi = 0$, for all $k \gg 1$. *QED*

To obtain long-time existence and convergence of the (HF), we will need hypotheses to control both the L^∞-norm and Dirichlet energy of the initial data. That these two conditions must be packaged together is illustrated the example of the seven-torus $\mathbb{T}^7$ and the group G_2.

Example 3.7 (Finite-time singularity)**.** Let $M = \mathbb{T}^7 = \mathbb{S}^1 \times \ldots \times \mathbb{S}^1$ be the 7-torus, endowed with the standard G_2-structure $\varphi_\circ$ inducing the flat metric $g_\circ$. Then the frame bundle $\mathrm{Fr}(\mathbb{T}^7, g_\circ)$ is trivialised so

$$\mathrm{Fr}(\mathbb{T}^7, g_\circ)/\mathrm{G}_2 \cong \mathbb{T}^7 \times \mathbb{RP}^7.$$

Any G_2-structure $\varphi \in [[\varphi_\circ]]$ can be thought of as a map from $\mathbb{T}^7$ to $\mathbb{RP}^7$ and the torsion-free compatible G_2-structures correspond to constant maps. Moreover, the isometric homotopy class of φ is the (unrestricted) homotopy class of the corresponding map.
We claim the existence of a smooth G_2-structure $\varphi \in [[\varphi_\circ]]$ which coincides with the constant map $\varphi_\circ \equiv y_0 \in \mathbb{RP}^7$ outside $B(p, r_0)$, but whose isometric homotopy class $[\varphi]$ is nontrivial, $[\varphi] \neq 0 = [\varphi_\circ]$.

Considering a quotient map

$$q : (\overline{B}(p, r_0), \partial\overline{B}(p, r_0)) \to (\mathbb{S}^7, x_0)$$

Recall that the set $[\mathbb{S}^7, \mathbb{RP}^7]$ of unrestricted homotopy classes of maps $\mathbb{S}^7 \to \mathbb{RP}^7$, can be identified with the quotient of $\pi_7(\mathbb{RP}^7, y_0)$ by the usual action of $\pi_1(\mathbb{RP}^7, y_0)$, i.e.

$$[\mathbb{S}^7, \mathbb{RP}^7] = \pi_7(\mathbb{RP}^7, y_0)/\pi_1(\mathbb{RP}^7, y_0).$$

But this action of $\pi_1(\mathbb{RP}^n)$ on $\pi_n(\mathbb{RP}^n) \cong \mathbb{Z}$ is trivial when n is odd, so the set $[\mathbb{S}^7, \mathbb{RP}^7]$ is countably infinite. Choose a nontrivial element $0 \neq [f] \in [\mathbb{S}^7, \mathbb{RP}^7]$, $f : (\mathbb{S}^7, x_0) \to (\mathbb{RP}^7, y_0)$, which in turn induces the G_2-structure $\overline{\varphi} : \mathbb{T}^7 \to \mathbb{RP}^7$ isometric to $\varphi_\circ$ given by

$$\overline{\varphi}(x) = \begin{cases} \varphi_\circ(x) = y_0, & \text{if } x \in \mathbb{T}^7 \setminus B(p, r_0), \\ f(q(x)), & \text{if } x \in B(p, r_0). \end{cases}$$

Moreover, $[\overline{\varphi}] \neq 0$ since

$$\deg(\overline{\varphi} : \mathbb{T}^7 \to \mathbb{RP}^7)\mathrm{vol}(\mathbb{RP}^7) = \int_{\mathbb{T}^7} \overline{\varphi}^* \,\mathrm{vol} = \int_{B(p,r_0)} (f \circ q)^* \,\mathrm{vol} = \int_{\mathbb{S}^7} f^* \,\mathrm{vol} \neq 0.$$

By Whitney's approximation theorem, we then find a smooth map $\varphi : \mathbb{T}^7 \to \mathbb{RP}^7$ which coincides with the constant map $\varphi_\circ \equiv y_0 \in \mathbb{RP}^7$ outside $B(p, r_0)$, and is sufficiently C^0-close to $\overline{\varphi}$ so that we still have $[\varphi] \neq 0$. This proves the claim.

Next, for each $r \in (0, r_0)$, let φ_r be the G_2-structure on $\mathbb{T}^7$ defined by

$$\varphi_r(x) = \begin{cases} \varphi_\circ(x) = y_0, & \text{if } x \in \mathbb{T}^7 \setminus \overline{B}(p, r), \\ \varphi\left(\frac{x r_0}{r}\right), & \text{if } x \in B(p, r_0) \simeq B(0, r_0) \subset \mathbb{R}^7, \end{cases}$$

where the smooth map $x \mapsto \varphi\left(\frac{xr_0}{r}\right)$ is defined using the normal coordinates giving the isometric identification $B(p,r_0) \simeq B(0,r_0)$. Note that φ_r is a smooth G_2-structure on $\mathbb{T}^7$, isometric to $\varphi_\circ$ and such that $\varphi_r \in [\varphi] \neq 0 = [\varphi_\circ]$ and its Dirichlet energy is

$$\mathrm{E}(\varphi_r) = r_0^{-5} r^5 \mathrm{E}(\varphi).$$

Therefore

$$\inf_{\tilde{\varphi} \in [\varphi]} \mathrm{E}(\tilde{\varphi}) = 0.$$

On the other hand, since $[\varphi] \neq 0$, this class cannot contain a torsion-free G_2-structure, which would correspond to a constant map from $\mathbb{T}^7$ to $\mathbb{RP}^7$.

Therefore, for small enough $r \ll 1$, the harmonic G_2-flow starting at φ_r has a finite-time singularity, as guaranteed by [12, Theorem G], otherwise the flow $\{\varphi(t)\}$ with $\varphi(0) = \varphi_r$ would exist for all time $t > 0$, and, since $r \ll 1$ and thus $\mathcal{D}(\varphi_r) \ll 1$, it would follow that $\varphi(t)$ converges smoothly as $t \to \infty$ to a G_2-structure $\varphi_\infty \in [\varphi_r] = [\varphi]$ with divergence-free torsion, hence torsion-free by the energy gap Proposition. It is also noteworthy that however much φ_r may have arbitrarily small energy $\mathcal{D}(\varphi_r) \to 0$ as $r \to 0$, the L^∞-norm of its torsion is actually blowing-up:

$$\|\nabla \varphi_r\|_{L^\infty(M)} = r_0 r^{-1} \|\nabla \varphi\|_{L^\infty(B(p,r_0))} \to \infty \quad \text{as } r \to 0.$$

This also exemplifies why a general result of long-time existence for the harmonic flow under small initial energy should take into account the L^∞-norm of the initial torsion.

Theorem 3.8 (Long-time existence under small initial energy)**.** For any given constant $\kappa > 0$, there exists a universal constant $\varepsilon(\kappa) > 0$, such that, if ξ_0 satisfies

(i) $\|\nabla \xi_0\|_{L^\infty(M)} \leqslant \kappa$ and

(ii) $\mathrm{E}(\xi_0) = \frac{1}{2}\|T|^2_{L^2(M)} < \varepsilon(\kappa)$,

then the harmonic H-flow with initial condition ξ_0 exists for all time $t \geqslant 0$ and subconverges smoothly to a torsion-free H-structure as $t \to \infty$. Moreover, the universal constant can be chosen of the form

$$\varepsilon(\kappa) = \min\left\{\varepsilon_*, c\left(\arctan \frac{1}{2\kappa^2}\right)^{n-2}\right\},$$

where $\varepsilon_*, c > 0$ are constants depending only on (M^n, g) and H.

The proof depends on an explicit formula for the maximal time-interval of existence and uniqueness, first proved by Chen and Ding [5, Lemma 2.1] and [14, Lemma 3.3].

Lemma 3.9. Let $\delta = 1/c > 0$, then, for any $t_0 \in [0, \tau)$,

$$t_0 + \delta \arctan \frac{1}{2\overline{e}_0} < \tau, \quad \text{with} \quad \overline{e}_0 = \overline{e}(t_0),$$

and

$$\overline{e}(t) \leqslant \frac{\overline{e}_0 + \tan c(t - t_0)}{1 - \overline{e}_0 \tan c(t - t_0)} \quad \forall t \in \left[t_0, t_0 + \delta \arctan \frac{1}{\overline{e}_0}\right).$$

In particular,

$$\overline{e}(t) \leqslant 2\overline{e}_0 + \frac{1}{\overline{e}_0}, \quad \forall t \in \left[t_0, t_0 + \delta \arctan \frac{1}{2\overline{e}_0}\right].$$

Then, this estimate, the AMF, ϵ-regularity and energy gap theorems combine into results on the existence and convergence under uniform bounded torsion, and finite-time singularity under unbounded torsion.

Lemma 3.10 (Existence and convergence under uniformly bounded torsion)**.** Let $\{\xi(t)\}_{[0,\tau)}$ be the maximal unique solution to the harmonic H-flow (HF) with initial condition $\xi(0) = \xi_0$, and suppose that

$$\sup\{\overline{e}(t) : t \in [0, \tau)\} < \infty.$$

Then $\tau = \infty$, and the flow $\{\xi(t)\}$ subconverges smoothly when $t \to \infty$. Moreover, any such subsequential limit ξ_∞ satisfies $\mathrm{E}(\xi_\infty) \leqslant \mathrm{E}(\xi_0)$ and has divergence-free torsion:

$$\operatorname{div} T(\xi_\infty) = 0.$$

If furthermore $\mathrm{E}(\xi_0) < \varepsilon_0$, then any subsequential limit ξ_∞ is torsion-free.

Lemma 3.11 (Finite-time singularity under unbounded torsion)**.** There are constants $\varepsilon_1 > 0$ and $c_1 > 0$, such that if $\{\xi(t)\}_{[0,\tau)}$ is the maximal unique solution to the harmonic H-flow and

$$\sup\{\overline{e}(t) : t \in [0, \tau)\} = \infty,$$

if $\varepsilon = \mathrm{E}(\xi_0) < \varepsilon_1$, then

$$\tau^{\frac{n-2}{2}} \leqslant c_1 \sqrt{\varepsilon}.$$

We shall require the group H to satisfy the supplementary condition

$$\pi_{\mathfrak{h}} \left(\frac{\partial}{\partial t} T_m \right) = c_H \pi_{\mathfrak{h}} \left([T_m, \operatorname{div} T] \right), \tag{10}$$

for some constant $c_H \in \mathbb{R}$ depending only on H, where T is the torsion of any solution $\{\xi(t)\}$ of the harmonic H-flow. The groups $H = \mathrm{U}(m)$, G_2, Spin(7), Sp(k)Sp(1) or $\{1\}$ satisfies (10) for some $c_H \in \mathbb{R}$. Under this assumption, we generalise results already known for $H = \mathrm{G}_2$ and Spin(7) and obtain long-time existence under small initial torsion.

Theorem 3.12 (Long time existence under small initial torsion)**.** For every $\delta > 0$, there exists $\varepsilon = \varepsilon(\delta, M^n, g, H) > 0$ such that if $\|\nabla\xi_0\|_{L^\infty(M)} < \varepsilon$, then the harmonic H-flow starting from ξ_0 exists for all time $t \geqslant 0$ and converges smoothly to a harmonic H-structure ξ_∞, which furthermore satisfies $\|\nabla\xi_\infty\|_{L^\infty(M)} < \delta$.

Put together with the energy gap theorem, this yields two stability results

Theorem 3.13 (Stability of torsion-free structures under the harmonic flow)**.**

(i) There is a constant $\kappa_0 = \kappa_0(M, g, H) > 0$ such that, if ξ_0 is a compatible H-structure satisfying $\|\nabla\xi_0\|_{L^\infty(M)} < \kappa_0$, then the harmonic H-flow (HF) starting at ξ_0 exists for all $t \geqslant 0$ and converges smoothly to a torsion-free H-structure ξ_∞, as $t \to \infty$.

(ii) If (M^n, g) admits a compatible torsion-free H-structure $\overline{\xi}$, then for every $\delta > 0$, there exists $\overline{\varepsilon}(\delta, M, g, H) > 0$ such that, for any compatible H-structure ξ_0 with $\|\xi_0 - \overline{\xi}\|_{C^2(M)} < \overline{\varepsilon}$, the harmonic H-flow (HF) with initial condition ξ_0 exists for all $t \geqslant 0$, satisfies the estimate $\|\xi_t - \overline{\xi}\|_{C^1(M)} < \delta$ for all $t \geqslant 0$, and converges smoothly to a torsion-free H-structure ξ_∞ as $t \to \infty$.

This statement could be compared with the traditional direct method of the calculus of variations where the main obstacle of the compacity of the minimising sequence is replaced by its harmonicity.

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