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# BGG Sequences — A Riemannian perspective

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**Abstract.** BGG resolutions and generalized BGG resolutions from representation theory of semisimple Lie algebras have been generalized to sequences of invariant differential operators on manifolds endowed with a geometric structure belonging to the family of parabolic geometries. Two of these structures, occur as conformal structures and projective structures occur as weakenings of a Riemannian metric respectively of a specified torsion-free connection on the tangent bundle. In particular, one obtains BGG sequences on open subsets of $\mathbb{R}^n$ as very special cases of the construction. It turned out that several examples of the latter sequences are of interest in applied mathematics, since they can be used to construct numerical methods to study operators relevant for elasticity theory, numerical relativity and related fields.

This article is intended to provide an intermediate level between BGG sequences for parabolic geometries and the case of domains in $\mathbb{R}^n$. We provide a construction of conformal BGG sequences on Riemannian manifolds and of projective BGG sequences on manifolds endowed with a volume preserving linear connection on their tangent bundle. These constructions do not need any input from parabolic geometries. Except from standard differential geometry methods the only deeper input comes from representation theory. So one can either view the results as a simplified version of the constructions for parabolic geometries in an explicit form. Alternatively, one can view them as providing an extension of the simplified constructions for domains in $\mathbb{R}^n$ to general Riemannian manifolds or to manifolds endowed with an appropriate connection on the tangent bundle.

**Keywords:** Bernstein-Gelfand-Gelfand sequences, BGG sequences, BGG machinery, geometric prolongation, invariant differential operators

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## 1 Introduction

The origin of BGG sequences lies in pure algebra. For an irreducible representation $\mathbb{V}$ of a complex semisimple Lie algebra $\mathfrak{g}$, I.N. Bernstein, I.M. Gelfand and S.I. Gelfand constructed in [7] a resolution of $\mathbb{V}$ by homomorphisms of Verma modules. In [29], this was generalized by J. Lepowsky to a resolution by homomorphisms of generalized Verma modules associated to a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$. These results have a connection to geometry via a duality relating homomorphisms between certain induced modules to invariant differential operators acting on sections of homogeneous vector bundles over a homogeneous space. Via this duality, the homomorphisms showing up in the resolutions constructed in [7] correspond to invariant differential operators between sections of homogeneous line bundles over the full flag manifold of a complex Lie group $G$ with Lie algebra $\mathfrak{g}$. Likewise, in the setting of [29] there is a relation to invariant differential operators acting on sections of homogeneous vector bundles over the generalized flag manifold $G/P$, that are induced by irreducible representations of an appropriate parabolic subgroup $P \subset G$.

Via complexification, these results are also related to invariant differential operators on homogeneous vector bundles over real generalized flag manifolds $G/P$. For appropriate choices of $G$ and $P$, these generalized flag manifolds are the homogeneous model for geometric structures, which are of broad interest in differential geometry. In particular, for the connected component $G := SO_0(n+1,1)$ of the identity in the orthogonal group of a Lorentzian inner product in dimension $n+2$, there is just one parabolic subgroup $P$ up to conjugation, namely the stabilizer of an isotropic line in $\mathbb{R}^{n+1,1}$. It then turns out that $G/P \cong S^n$ with the action identifying $G$ with the group of all orientation preserving conformal isometries of $S^n$. Thus there is a relation of Lepowsky's generalized BGG resolutions to conformally invariant differential operators, which was exploited for example in the work [22] of M.G. Eastwood and J.W. Rice. In the 1970's and 80's it became also clear, that other generalized flag manifolds are related to interesting geometric structures. In particular, the unitary group $SU(n+1,1)$ in a similar way leads to the homogeneous model of strictly pseudo-convex CR structures of hypersurface type, which is the starting point for several parallel developments in CR geometry and conformal geometry, that turned out to be very fruitful.

Motivated by this, a general study of geometries with homogeneous model a real or complex generalized flag manifold was initiated in the 1990's under the name "parabolic geometries", see [17] for an introduction. One of the early successes of this theory was a general version of the BGG construction in the setting of differential operators in [19]. This construction does not only work for the homogeneous model (and provide resolutions of locally constant sheaves there) but for arbitrary curved geometries, where it leads to sequences of differential operators that are intrinsic to the geometric structure in question. Proving existence of such operators in the curved setting is a major problem that is solved by this construction in many cases. The basis for this construction is an equivalent description of the geometric structures in the family as Cartan geometries.

Tools derived from this description are an essential ingredient, which makes the construction not easily accessible. While these tools are not needed in what follows, we have to discuss them briefly to explain the spirit and applications of BGG sequences and motivate the further developments.

The Cartan description gives rise to a special class of natural vector bundles called *tractor bundles*. These are rather exotic objects since the action of morphisms on them depends on higher order jets in a point, but each such bundle comes with a linear connection canonically associated to the geometry. These so-called *tractor connections* can be coupled to the exterior derivative to obtain the twisted de Rham sequence of differential forms with values in the given tractor bundle. By construction, this is a sequence of differential operators of order one defined on sections of natural bundles which are intrinsically associated to the geometry. On the other hand, Lie algebra theory gives rise to algebraic structures on the bundles of tractor-bundle-valued forms, basically a filtration by smooth subbundles and a tensorial operation mapping $k$-forms to $(k-1)$-forms. These can then be used to define natural subquotient bundles that are associated to Lie algebra homology spaces and are more standard geometric objects. Moreover, the twisted de Rham sequence can be "compressed" to a sequence of higher order operators acting on sections of these subquotient bundles. By construction, these so-called *BGG-operators* also are naturally associated to the geometry.

On geometries locally isomorphic to the homogeneous model $G/P$, all tractor connections are flat. Hence each of the twisted de Rham sequences is a complex and a fine resolution of the sheaf of local parallel sections of the tractor bundle in question. Analyzing the construction carefully, one concludes that also the induced BGG sequence is a complex and computes the same cohomology and thus also is a fine resolution. For general geometries, the curvature of any tractor connection equivalently encodes the Cartan curvature, so one does not obtain complexes. But there still is a close relation between the two sequences which allows one to switch between a picture of simple operators on complicated bundles and one of complicated operators on simple bundles. This is a crucial input for many applications of BGG sequences on non-flat geometries. From the perspective of the current paper, one should in particular mention applications to the study of conformally compact metrics and in particular Poincaré-Einstein metrics, see e.g. [14, 23, 25] and the analogs of these concepts in projective differential geometry, for example [12, 13].

Another line of applications of BGG sequences on curved geometries concerns the first operators in BGG sequences, which turn out to always define an overdetermined system of PDE. These contain many important examples like Killing and conformal Killing operators on all types of tensor bundles. Here one still gets a close relation to parallel sections of the tractor bundle in question. Modifying the tractor connections in a way that preserves the relation between the sequences, one can interpret the construction as *geometric prolongation* which allows to equivalently rewrite the overdetermined system defined by the first BGG operator in first order closed form and hence describe its solutions as parallel sections for a connection. There are several nice applications of this idea, see e.g. [9, 21], and it was developed systematically in [8], which also contains several steps in the direction of the current article.

The second important input for this article came rather unexpected. One example of a BGG sequence coming from projective differential geometry is a version of the Riemannian deformation sequence, which (for the flat metric on a domain in $\mathbb{R}^3$) is known as the *Calabi complex* or the *fundamental complex of linear elasticity*, see [20]. Studying analytical properties of this complex with the ultimate goal to develop efficient numerical methods for its study is of considerable interest in the applied mathematics community. Finite element methods for differential forms were developed between the 1970s and 1990s and in the first decade of the 2000's, a more complete, efficient picture for the numerical analysis of (scalar) differential forms was developed under the name *finite element exterior calculus*, see e.g. [1]. This raised some interest in the geometric BGG construction (that is based on differential forms), which in turn led to interaction between the communities and progress, see [2, 3], in that period. Unfortunately, there was not much interaction between the two communities in the subsequent years and while there was quite a bit of further activity on BGG-like constructions in the applied community, this was usually based on ad-hoc constructions and did no use representation theory.

The interaction between the two communities was taken up again around 2020 in discussions of D. Arnold, K. Hu, and myself. After a longer period of developing a common language and exchanging ideas and points of view, this led to the joint article [15] of K. Hu and myself. In this article, we develop a simplified version of the BGG construction in the setting of the flat metric and flat connection on Lipschitz domains in $\mathbb{R}^n$, which also applies in low (Sobolev) regularity. This is done in an abstract setting of Hilbert complexes with the BGG sequences coming from representation theory as a major example. In a followup article [16], we have shown that this approach can be used to carry over constructions of Poincaré operators with good analytical properties for scalar differential forms to the setting of BGG complexes (again in low regularity).

This sets the stage for the current paper. I'll present a version of the BGG construction on Riemannian manifolds which avoids all the technical input on parabolic geometries and tractors. This can either be viewed as a simplified version of the geometric construction based on a choice of metric in a conformal class or a (volume-preserving) connection in a projective class. From that point of view, one uses the realization of tractor bundles and tractor connections in terms of such a choice as well as simplifications of the construction caused by the fact that one does not aim for conformal invariance. Alternatively, it can be viewed as a generalization of the approach of [15] to Riemannian manifolds in a smooth setting. From that point of view, a substantial modification is needed in the construction of the twisted complex, where curvature terms have to be included in order to obtain complexes for non-flat metrics. In addition, the operations based on the Levi-Civita connection do not lead to complexes any more, so alternative descriptions of the cohomology are needed. On the other hand, the actual BGG construction is quite close to the one in [15] (in the setting we consider).

There are some technical difficulties that cannot be avoided, however. On the one hand, considerable input from representation theory is needed, in particular Kostant's algebraic Hodge theory developed in [27]. It should be pointed out that for parabolic geometries, parts of this machinery are automatically present in the tractor connection, while in the approach of this article, they have to be built in by hand. So it will not always be obvious why one proceeds in the chosen way. In order to explicitly describe the bundles showing up in a BGG sequence, one also needs to geometrically interpret Kostant's description of Lie algebra (co)homology from [27]. It is possible, however, to simply accept the results in that direction, a detailed understanding of proofs is not needed (and probably would also not be very helpful). On the other hand, in order to treat different BGG sequences in a uniform fashion, a slightly unusual approach to Riemannian geometry is helpful. We will work quite a lot with bundles induced by representations and natural bundle maps coming from equivariant maps between these representations. However, in each case of interest, these can be brought to an explicit form, and we will discuss this in examples.

# 2 Conformal BGG sequences on Riemannian manifolds

## 2.1 Background from Riemannian geometry

We first recall the relation between representation theory of the orthogonal group $O(n)$ and natural vector bundles on Riemannian $n$-manifolds. A Riemannian metric $g$ on an $n$-manifold $M$ can be equivalently described via the orthonormal frame bundle $p : \mathcal{O}M \to M$, which is a principal fiber bundle with structure group $O(n)$. Given a representation $\mathbb{W}$ of $O(n)$ one can form the associated vector bundle $\mathcal{O}M \times_{O(n)} \mathbb{W}$, which we will denote by $\mathcal{W}M \to M$. This gives a functorial relation between representations of $O(n)$ and vector bundles canonically associated to Riemannian $n$-manifolds, so $O(n)$-equivariant maps between representations induce vector bundle maps between the corresponding bundles. In particular, the standard representation of $O(n)$ on $\mathbb{R}^n$ corresponds to the tangent bundle $TM$, while the adjoint representation on $\mathfrak{o}(n)$ corresponds to the bundle $\mathfrak{o}(TM)$ of skew symmetric endomorphisms of $TM$. The correspondence is compatible with all tensorial constructions.

The Levi-Civita connection of $g$ can be equivalently described as a principal connection on $p : \mathcal{O}M \to M$ which in turn induces a linear connection on each of the associated bundles $\mathcal{W}M$. We will denote all these connections by $\nabla$ and observe that they are compatible with all tensorial operations, which justifies the uniform notation. From above, we know that the bundle maps induced by $\mathfrak{o}(n)$-equivariant maps between representations are parallel for the Levi-Civita connection. Let us elaborate on this in a special case that will be important in what follows. For a representation $\mathbb{W}$ of $O(n)$ we get the infinitesimal action of $\mathfrak{o}(n)$ on $\mathbb{W}$, which defines a bilinear map $\mathfrak{o}(n) \times \mathbb{W} \to \mathbb{W}$. Passing to associated bundles, we obtain a bilinear bundle map $\mathfrak{o}(TM) \times \mathcal{W}M \to \mathcal{W}M$. We will denote both this map and the induced tensorial operation on sections by $\bullet$, so for $\Phi \in \Gamma(\mathfrak{o}(TM))$ and $\sigma \in \Gamma(\mathcal{W}M)$ we get $\Phi \bullet \sigma \in \Gamma(\mathcal{W}M)$.

Denoting all these operations by the symbol $\bullet$ is justified by the fact that they are compatible with constructions for natural vector bundles in a simple way. This comes from the compatibility of the infinitesimal representation with constructions. For example, for $\mathbb{W} = \mathbb{W}_1 \otimes \mathbb{W}_2$, we get $\mathcal{W}M = \mathcal{W}_1M \otimes \mathcal{W}_2M$ and by construction, we get $\Phi \bullet (\sigma_1 \otimes \sigma_2) = (\Phi \bullet \sigma_1) \otimes \sigma_2 + \sigma_1 \otimes (\Phi \bullet \sigma_2)$, and so on. The fact that $\bullet$ is parallel for the Levi-Civita connection is equivalent to the fact that for each vector field $\xi \in \mathfrak{X}(M)$ and any $\sigma \in \Gamma(\mathcal{W}M)$, we obtain

$$\nabla_\xi(\Phi \bullet \sigma) = (\nabla_\xi \Phi) \bullet \sigma + \Phi \bullet (\nabla_\xi \sigma). \tag{2.1}$$

This can be made more explicit for concrete choices. For example, if $\mathbb{W} = \mathbb{R}^n$ and $\mathcal{W}M = TM$, then $\Phi \bullet \eta = \Phi(\eta)$ and (2.1) just boils down to the definition of $\nabla$ on $\mathfrak{o}(TM)$. If $\mathbb{W} = \mathbb{R}^{n*}$, the dual of the standard representation, then $\mathcal{W}M = T^*M$ and by definition of the dual representation, we obtain $\Phi \bullet \alpha = -\alpha \circ \Phi$ for $\alpha \in \Omega^1(M)$, i.e. $(\Phi \bullet \alpha)(\eta) = -\alpha(\Phi(\eta))$. From this formula, one can easily verify (2.1) directly. For higher degree forms and, more generally, for $\binom{0}{k}$-tensor fields, one gets $(\Phi \bullet t)(\eta_1, \dots, \eta_k) = -\sum_i t(\eta_1, \dots, \Phi(\eta_i), \dots, \eta_k)$ and again (2.1) can be easily verified directly from this formula.

An example for the usefulness of the notation is the description of the curvature of the Levi-Civita connection on all natural bundles. The Riemann curvature tensor is the $\binom{1}{3}$-tensor field $R$ defined by

$$R(\xi, \eta)(\zeta) = \nabla_\xi \nabla_\eta \zeta - \nabla_\eta \nabla_\xi \zeta - \nabla_{[\xi,\eta]} \zeta \tag{2.2}$$

for $\xi, \eta, \zeta \in \mathfrak{X}(M)$. The well known symmetries of $R$ imply that it can be viewed as a two-form with values in $\mathfrak{o}(TM)$, i.e. as a section of $\Lambda^2 T^*M \otimes \mathfrak{o}(TM)$. It is well known that for the induced connection on any natural vector bundle the curvature is described via the natural action of this two-form. In our notation, this means that for any natural bundle $\mathcal{W}M$, any $\sigma \in \Gamma(\mathcal{W}M)$ and $\xi, \eta \in \mathfrak{X}(M)$, we get

$$\nabla_\xi \nabla_\eta \sigma - \nabla_\eta \nabla_\xi \sigma - \nabla_{[\xi,\eta]}\sigma = R(\xi, \eta) \bullet \sigma. \tag{2.3}$$

## 2.2 Algebraic setup for conformal BGG's

We consider $\mathbb{R}^{n+2}$, denote the standard basis by $e_0, \dots, e_{n+1}$ and consider the bilinear form $b$ on $\mathbb{R}^{n+2}$ defined by

$$b(x, y) := x_0 y_{n+1} + x_{n+1} y_0 + \textstyle\sum_{i=1}^n x_i y_i.$$

Clearly, this has signature $(1, 1)$ on the plane spanned by $e_0$ and $e_{n+1}$ and equals the standard inner product on the orthogonal subspace spanned by $e_1, \dots, e_n$, so $b$ is Lorentzian. We denote by $G := O(b)$ the orthogonal group of $b$, i.e. the group consisting of all $C \in GL(n + 2, \mathbb{R})$ such that $b(Cx, Cy) = b(x, y)$ for all $x, y \in \mathbb{R}^{n+2}$. There is an obvious subgroup in $O(b)$ consisting of those $C$ for which $Ce_0 = e_0$ and $Ce_{n+1} = e_{n+1}$. Any matrix with this property also maps the subspace spanned by $e_1, \dots, e_n$ to itself and is orthogonal there, so we can view this subgroup as $O(n) \subset G$.

The Lie algebra $\mathfrak{g} := \mathfrak{o}(b)$ of $G$ consists of all matrices $B$ such that $0 = b(Bx, y) + b(x, By)$ for any $x, y \in \mathbb{R}^{n+2}$. It is easy to describe this explicitly, c.f. Section 1.6.3 of [17], as block matrices with blocks of sizes 1, $n$, and 1 of the form

$$\begin{pmatrix} a & Z & 0 \\ X & A & -Z^t \\ 0 & -X^t & -a \end{pmatrix} \text{ with } a \in \mathbb{R}, X \in \mathbb{R}^n, Z \in \mathbb{R}^{n*} \text{ and } A \in \mathfrak{o}(n). \tag{2.4}$$

Of course, the subgroup $O(n)$ corresponds to the subalgebra formed by all matrices with $a = X = Z = 0$. The block form can be viewed as defining a vector space decomposition $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, with the components spanned by $X$, $(a, A)$ and $Z$, respectively, so $\mathfrak{o}(n) \subset \mathfrak{g}_0$. The Lie bracket on $\mathfrak{g}$ is given by the commutator of matrices, which readily implies that this decomposition is compatible with the Lie bracket in the sense that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for all $i$, $j$. Here and in what follows, we agree that $\mathfrak{g}_\ell = \{0\}$ if $\ell \notin \{-1, 0, 1\}$. Such a decomposition is referred to as a $|1|$–*grading* on $\mathfrak{g}$.

In particular, the Lie bracket on $\mathfrak{g}$ restricts to a bilinear map $\mathfrak{g}_0 \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$ which extends the standard representation of $\mathfrak{o}(n) \subset \mathfrak{g}_0$ on $\mathfrak{g}_{-1} = \mathbb{R}^n$. The remaining elements of $\mathfrak{g}_0$ (corresponding to $(a, 0)$ with $a \in \mathbb{R}$) act on $\mathfrak{g}_{-1}$ as $-a\,\mathrm{id}$, which shows that we get an identification of $\mathfrak{g}_0$ with the conformal Lie algebra $\mathfrak{co}(n)$ in this way. In the same way, the restriction $\mathfrak{g}_0 \times \mathfrak{g}_1 \to \mathfrak{g}_1$ identifies $\mathfrak{g}_1$ with the dual of the standard representation of $\mathfrak{co}(n)$.

There is a simple way to realize the $|1|$-grading. Namely, one considers the element$E \in \mathfrak{g}_0$ that corresponds to $a = 1$ and $A = 0$. The adjoint action of $E$ is given by multiplication by $i$ on $\mathfrak{g}_i$ for any $i = -1, 0, 1$. The element $E$ also acts diagonalizable on the standard representation $\mathbb{R}^{n+2}$ of $\mathfrak{o}(b)$ and the eigenspace decomposition there corresponds to the block form of matrices used in (2.4) and the eigenvalues are (from top to bottom) $+1$, $0$, and $-1$. This extends to arbitrary representations and since $\mathfrak{g}$ is semisimple, we consider the infinitesimal representation of $\mathfrak{o}(b)$ on an irreducible representation $\mathbb{V}$ of $O(b)$. It then turns out that the eigenvalues of $E$ form an unbroken string of the form $\lambda, \lambda+1, \dots, \lambda+N$ for some $\lambda \in \mathbb{Z}$ and $N \in \mathbb{N}$. (This easily follows from the fact that all such representations can be obtained from the standard representation via tensorial constructions.) Viewing this as defining a decomposition $\mathbb{V} = \oplus_{i=0}^N \mathbb{V}_i$ we have the fundamental property that for the infinitesimal action, we get $\mathfrak{g}_i \cdot \mathbb{V}_j \subset \mathbb{V}_{i+j}$ (with similar conventions as before).

## 2.3 Passing to geometry

Fix an irreducible representation $\mathbb{V}$ of $O(b)$ with the decomposition $\mathbb{V} = \oplus_{j=0}^N \mathbb{V}_j$ as in §2.2. Then we can restrict to the subgroup $O(n) \subset O(b)$ and one easily concludes that each of the subspace $\mathbb{V}_j$ is invariant under the action of $O(n)$. Hence if we form associated bundles over a Riemannian $n$-manifold $(M, g)$ as in §2.1 we get a decomposition $\mathcal{V}M = \oplus_{j=0}^N \mathcal{V}_j M$. In particular, the components $\mathfrak{g}_i$ of $\mathfrak{g}$ also give rise to associated bundles, which are $TM$ for $i = -1$, $\mathfrak{co}(TM)$ for $i = 0$ and $T^*M$ for $i = 1$. Restricting the infinitesimal representation of $\mathfrak{g}$ on $\mathbb{V}$ (which is $G$-equivariant and hence $O(n)$-equivariant) to the individual components $\mathfrak{g}_i$, we get induced bundles maps as follows:

- An extension $\bullet : \mathfrak{co}(TM) \times \mathcal{V}M \to \mathcal{V}M$ of the map defined in §2.1 such that $\mathfrak{co}(TM) \bullet \mathcal{V}_j M \subset \mathcal{V}_j M$ for any $j$.
- A bilinear bundle map $\bullet : TM \times \mathcal{V}M \to \mathcal{V}M$ such that $TM \bullet \mathcal{V}_j M \subset \mathcal{V}_{j-1} M$.
- A bilinear bundle map $\bullet : T^*M \times \mathcal{V}M \to \mathcal{V}M$ such that $T^*M \bullet \mathcal{V}_j M \subset \mathcal{V}_{j+1} M$.

As before, we will also denote by $\bullet$ the induced tensorial operations on sections. For a section $s$ of $\mathcal{V}M$, we will denote the component in $\mathcal{V}_j M$ by $s_j$ and we will swap between the points of view that $\bullet$ is defined on all of $\mathcal{V}M$ or on the individual components. So for a vector field $\eta \in \mathfrak{X}(M)$ and $s \in \Gamma(\mathcal{V}M)$ we get $\eta \bullet s \in \Gamma(\mathcal{V}M)$ and $(\eta \bullet s)_j = \eta \bullet s_{j+1} \in \Gamma(\mathcal{V}_j M)$. Likewise for $\alpha \in \Omega^1(M)$, we get $(\alpha \bullet s)_j = \alpha \bullet s_{j-1} \in \Gamma(\mathcal{V}_j M)$. As we have noted in §2.1, the fact that these bundle maps come from equivariant maps between the inducing representation implies that for $\xi \in \mathfrak{X}(M)$, we get

$$\nabla_\xi(\eta \bullet s) = (\nabla_\xi \eta) \bullet s + \eta \bullet (\nabla_\xi s) \tag{2.5}$$

and similarly for $\alpha \bullet s$ and the components $\eta \bullet s_j$ and $\alpha \bullet s_j$.

The fact that the infinitesimal representation is compatible with the Lie bracket has several important consequences of us. On the one hand, $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ are abelian subalgebras of $\mathfrak{g}$, which implies that

$$\eta_1 \bullet (\eta_2 \bullet s) = \eta_2 \bullet (\eta_1 \bullet s) \tag{2.6}$$

$$\alpha_1 \bullet (\alpha_2 \bullet s) = \alpha_2 \bullet (\alpha_1 \bullet s). \tag{2.7}$$

On the other hand, the identification of $\mathfrak{g}_0$ with $\mathfrak{co}(\mathfrak{g}_{-1})$ is via the bracket. Thus we see that for $\Phi \in \mathfrak{co}(TM)$ we obtain

$$\Phi \bullet (\eta \bullet s) - \eta \bullet (\Phi \bullet s) = (\Phi(\eta)) \bullet s \tag{2.8}$$

$$\Phi \bullet (\alpha \bullet s) - \alpha \bullet (\Phi \bullet s) = -(\alpha \circ \Phi) \bullet s \tag{2.9}$$

## 2.4 Examples

(1) Let us start with the standard representation $\mathbb{V} = \mathbb{R}^{n+2}$, which we know decomposes as $\mathbb{V}_0 \oplus \mathbb{V}_1 \oplus \mathbb{V}_2$, with the summands spanned by $e_{n+1}$, $\{e_1, \dots, e_n\}$ and $e_0$, respectively. The subalgebra $\mathfrak{o}(n)$ acts trivially on the first and last summand and via the standard representation on $\mathbb{V}_1$. Hence sections of $\mathcal{V}_0 M$ and $\mathcal{V}_2 M$ are just functions, while $\Gamma(\mathcal{V}_1 M) = \mathfrak{X}(M)$. Hence we can write $s \in \Gamma(\mathcal{V}M)$ as a triple $s = (f, \zeta, h)$ for $f, h \in C^\infty(M, \mathbb{R})$ and $\zeta \in \mathfrak{X}(M)$ and in the block form of (2.4) the 0-component $f$ corresponds to the bottom component of the vector. Computing in the block form (2.4) readily shows that

$$\eta \bullet (f, \zeta, h) = (-g(\eta, \zeta), h\eta, 0) \qquad \alpha \bullet (f, \zeta, h) = (0, -f\alpha^\#, \alpha(\zeta)). \tag{2.10}$$

Here we use the usual musical isomorphisms, i.e. $\alpha^\# \in \mathfrak{X}(M)$ is characterized by $g(\alpha^\#, \eta) = \alpha(\eta)$ and the inverse isomorphism is denoted by $\eta \mapsto \eta^\flat$. Notice that from these formulae, the identities (2.6) and (2.7) as well as (2.8) and (2.9) for $\Phi \in \Gamma(\mathfrak{o}(TM))$ are easily verified directly.

One has to be slightly careful with the action of $\mathfrak{co}(TM)$ on $\mathbb{V}$ as obtained in §2.3, though, which is *not* the obvious one: The element $E \in \mathfrak{g}_0$ from §2.2 acts by multiplication by $-1$ on $\mathfrak{g}_{-1}$ and hence corresponds to $-\mathrm{id} \in \mathfrak{co}(TM)$. But it acts by multiplication by $-1$, $0$ and $1$ on $\mathbb{V}_0$, $\mathbb{V}_1$ and $\mathbb{V}_2$, respectively. This corresponds to the well known fact that in the conformal picture, the bundles $\mathcal{V}_0M$ and $\mathcal{V}_2M$ are bundles of densities and not of functions, while $\mathcal{V}_1M$ is a weighted tangent or cotangent bundle. Using this observation, the general versions of (2.8) and (2.9) can be verified directly.

(2) One can now pass to more general representations and bundles via tensorial constructions, but there is a choice of identifications one has to make. For example, consider the irreducible representation $\mathbb{W} := S^2_0\mathbb{V}$ with the subscript indicating trace-freeness with respect to the Lorentzian inner product. The decomposition of $\mathbb{V}$ easily implies that $\mathbb{W} = \mathbb{W}_0 \oplus \dots \oplus \mathbb{W}_4$ with $\mathbb{W}_0 = S^2\mathbb{V}_0$, $\mathbb{W}_1 \cong \mathbb{V}_0 \otimes \mathbb{V}_1$, $\mathbb{W}_2 = (S^2\mathbb{V}_1 \oplus \mathbb{V}_0 \otimes \mathbb{V}_2)_0$, $\mathbb{W}_3 = \mathbb{V}_1 \otimes \mathbb{V}_2$ and $\mathbb{W}_4 = S^2\mathbb{V}_2$. As representations of $O(n)$, these are just $\mathbb{R}$ in degree 0 and 4, $\mathbb{R}^n$ in degree 1 and 3 and $S^2_0\mathbb{R}^n \oplus \mathbb{R}$ in degree 2. Hence a section of $\mathcal{W}M$ can be viewed as $(f_1, \zeta_1, (\Phi, f_2), \zeta_2, f_3)$ with $f_i \in C^\infty(M, \mathbb{R})$, $\zeta_i \in \mathfrak{X}(M)$ and $\Phi$ a trace-free symmetric $\binom{2}{0}$-tensor field. For most components, it is also obvious how to relate them to elements of $S^2_0\mathcal{V}$. Denoting symmetric tensor products by $\odot$, the element with only non-trivial component $f_1$ can be realized as $(f_1, 0, 0) \odot (1, 0, 0)$, and likewise the element with only non-trivial component $f_3$ corresponds to $(0, 0, f_3) \odot (0, 0, 1)$. The elements with only non-trivial component $\zeta_1$ or $\zeta_2$ can be realized as $(0, \zeta_1, 0) \odot (1, 0, 0)$ and $(0, \zeta_2, 0) \odot (0, 0, 1)$, respectively.

For the component with only non-trivial entry $\Phi$ things are still easy. We can write $\Phi = \sum_\ell \xi_\ell \odot \eta_\ell$ such that $\sum_\ell g(\xi_\ell, \eta_\ell) = 0$, and this can be realized as $\sum_\ell (0, \xi_\ell, 0) \odot (0, \eta_\ell, 0)$. Finally, a natural element spanning the trivial subrepresentation in $\mathbb{W}_2$ is $-e_0 \odot e_{n+1} + \frac{1}{n}\sum_{i=1}^n e_i \odot e_i$. Thus in terms of a local orthonormal frame $s_i$, we can realize the section with only non-zero component $f_2$ as $-(f_2, 0, 0) \odot (0, 0, 1) + \sum_{i=1}^n \frac{f_2}{n}(0, s_i, 0) \odot (0, s_i, 0)$. Having set up the identification, one can directly apply the usual rules for a Lie algebra action on a symmetric product to derive from (2.10) the following formulae for $\eta \bullet (f_1, \zeta_1, (\Phi, f_2), \zeta_2, f_3)$ and $\alpha \bullet (f_1, \zeta_1, (\Phi, f_2), \zeta_2, f_3)$:

$$\begin{aligned} &(-g(\eta, \zeta_1), -g(\eta, \Phi) + \tfrac{2-n}{n} f_2\eta, ((\eta \odot \zeta_2)_0, g(\eta, \zeta_2)), 2f_3\eta, 0) \\ &(0, -2f_1\alpha^\#, (-(\alpha^\# \odot \zeta_1)_0, -\alpha(\zeta_1)), i_\alpha\Phi + \tfrac{n+2}{n} f_2\alpha^\#, \alpha(\zeta_2)). \end{aligned} \tag{2.11}$$

Here $i_\alpha\Phi$ is the contraction between $\alpha$ and $\Phi$, so for $\Phi = \sum_\ell \xi_\ell \odot \eta_\ell$, this is given by $\sum_\ell (\alpha(\xi_\ell)\eta_\ell + \alpha(\eta_\ell)\xi_\ell)$. Similarly, $g(\eta, \Phi) = i_{\eta^\flat}\Phi$, where $\eta^\flat \in \Omega^1(M)$ is $g(\eta, \_)$. Finally, in terms of a local orthonormal frame $s_i$, the trace-free part is $(\eta \odot \zeta_2)_0 = \eta \odot \zeta_2 - \frac{1}{n} g(\eta, \zeta_2) \sum_i s_i \odot s_i$. Again the identities (2.6)–(2.9) can then be verified directly, taking into account that the grading element $E$ acts on $\mathbb{W}_0, \dots, \mathbb{W}_4$ by multiplication by $-2, \dots, 2$.

(3) We demonstrate an alternative approach for the example $\mathbb{W} := \Lambda^k\mathbb{V}^*$ for $k = 2, \dots n$. This is close to the presentation of [24] (in a conformal setting). Here we immediately conclude that we get a decomposition of the form $\mathbb{W} = \mathbb{W}_0 \oplus \mathbb{W}_1 \oplus \mathbb{W}_2$ with $\mathbb{W}_0 = \Lambda^{k-1}\mathbb{V}_1^* \wedge \mathbb{V}_2^*$, $\mathbb{W}_1 = \Lambda^k\mathbb{V}_1^* \oplus (\Lambda^{k-2}\mathbb{V}_1^* \wedge \mathbb{V}_0^* \wedge \mathbb{V}_2^*)$ and $\mathbb{W}_2 = \mathbb{V}_0^* \wedge \Lambda^{k-1}\mathbb{V}_1^*$. As representations of $O(n)$, the first and last are isomorphic to $\Lambda^{k-1}\mathbb{R}^{n*}$, the middle one to $\Lambda^k\mathbb{R}^{n*} \oplus \Lambda^{k-2}\mathbb{R}^{n*}$. So sections of $\mathcal{W}M$ can be written as triples $(\varphi_1, (\varphi_2, \varphi_3), \varphi_4)$ with $\varphi_3 \in \Omega^{k-2}(M)$, $\varphi_1, \varphi_4 \in \Omega^{k-1}(M)$ and $\varphi_2 \in \Omega^k(M)$. We fix the identification by requiring that this section maps $k$ sections $(f_i, \zeta_i, h_i)$ of $\mathcal{V}M$ with $i = 1, \dots k$ to

$$\begin{aligned}\textstyle\sum_i(-1)^{i-1}h_i\varphi_1(\zeta_1, \dots, \widehat{\zeta_i}, \dots, \zeta_k) + \varphi_2(\zeta_1, \dots, \zeta_k)\\ + \textstyle\sum_{i<j}(-1)^{i+j}(f_ih_j - f_jh_i)\varphi_3(\zeta_1, \dots, \widehat{\zeta_i}, \dots, \widehat{\zeta_j}, \dots, \zeta_k)\\ + \textstyle\sum_i(-1)^{i-1}f_i\varphi_4(\zeta_1, \dots, \widehat{\zeta_i}, \dots, \zeta_k).\end{aligned}$$

This formula also shows directly how to extract the values of $\varphi_1, \dots, \varphi_4$ on vector fields by plugging appropriate sections of $\mathcal{V}M$ into $(\varphi_1, (\varphi_2, \varphi_3), \varphi_4)$. The usual formulae then show how to convert an action on $(\varphi_1, (\varphi_2, \varphi_3), \varphi_4)$ into an action on those sections, which then by direct computation gives

$$\begin{aligned}\eta \bullet (\varphi_1, (\varphi_2, \varphi_3), \varphi_4) &= (-i_\eta\varphi_2 + \eta^\flat \wedge \varphi_3, (\eta^\flat \wedge \varphi_4, i_\eta\varphi_4), 0)\\ \alpha \bullet (\varphi_1, (\varphi_2, \varphi_3), \varphi_4) &= (0, (-\alpha \wedge \varphi_1, i_{\alpha^\#}\varphi_1), i_{\alpha^\#}\varphi_2 + \alpha \wedge \varphi_3).\end{aligned} \tag{2.12}$$

(4) The last example is $\mathbb{W} = \mathfrak{g}$, the adjoint representation. This is isomorphic to $\Lambda^2\mathbb{V}^*$, but we get some special operations here that we will need later on. Of course, the decomposition has the form $\mathbb{W}_0 \oplus \mathbb{W}_1 \oplus \mathbb{W}_2$ and is just the shifted version of the $|1|$-grading. Thus the most natural identification is $\mathcal{W}_0M = TM$, $\mathcal{W}_1M = \mathfrak{co}(TM)$ and $\mathcal{W}_2M = T^*M$, so sections can be viewed as triples $(\zeta, \Phi, \varphi)$ with $\zeta \in \mathfrak{X}(M)$, $\Phi \in \Gamma(\mathfrak{co}(TM))$ and $\varphi \in \Omega^1(M)$. Note that here $\bullet$ is induced by the Lie bracket of $\mathfrak{g}$, which is also used to identify $\mathfrak{g}_0$ with $\mathfrak{co}(\mathfrak{g}_{-1})$. This readily implies that the $\mathcal{W}_0$-component of $\eta \bullet (\zeta, \Phi, \varphi)$ equals $-\Phi(\eta)$ and the $\mathcal{W}_2$-component of $\alpha \bullet (\zeta, \Phi, \varphi)$ equals $\alpha \circ \Phi$. So we only have to compute the $\mathcal{W}_1M$ components of both operations, and they both come from the bracket $\mathfrak{g}_{-1} \times \mathfrak{g}_1 \to \mathfrak{g}_0$. We denote the corresponding operation $\mathfrak{X}(M) \times \Omega^1(M) \to \Gamma(\mathfrak{co}(TM))$ by $\{\ ,\ \}$ which means that

$$\eta \bullet (\zeta, \Phi, \varphi) = (-\Phi(\eta), \{\eta, \varphi\}, 0) \qquad \alpha \bullet (\zeta, \Phi, \varphi) = (0, -\{\zeta, \alpha\}, \alpha \circ \Phi). \tag{2.13}$$

To compute the operation $\{\ ,\ \}$ explicitly, we just have to take $X \in \mathfrak{g}_{-1}$ and $Z \in \mathfrak{g}_1$ and compute the map $\mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$ that sends $Y$ to $[[X, Z], Y]$. This easily implies that

$$\{\eta, \varphi\}(\xi) = \varphi(\eta)\xi + \varphi(\xi)\eta - g(\xi, \eta)\varphi^\#. \tag{2.14}$$

## 2.5 The twisted connection

To start the BGG construction, we need more ingredients related to the curvature of $g$. For a Riemannian manifold $(M, g)$ of dimension $\geq 3$, let $R$ be the Riemann curvature tensor as defined in (2.2) above. Recall that the *Ricci curvature* Ric is the symmetric $\binom{0}{2}$-tensor field obtained by contraction as

$$\operatorname{Ric}(\eta, \zeta) = \textstyle\sum_i g(R(\xi_i, \eta)(\zeta), \xi_i) \tag{2.15}$$

for a local orthonormal frame $\{\xi_i\}$. The trace Sc of Ric then is the scalar curvature and we denote by $\operatorname{Ric}_0 := \operatorname{Ric} - \frac{1}{n} \operatorname{Sc} g$ its trace-free part. (Vanishing of $\operatorname{Ric}_0$ is the definition of Einstein metrics used in Riemannian geometry.) For some purposes, it is better to use a slight modification of Ric called the *Schouten tensor* $\mathsf{P}$. This can be characterized via $\operatorname{Ric} = (n-2)\mathsf{P} + \operatorname{tr}(\mathsf{P})g$, which easily implies that $\mathsf{P}_0 = \frac{1}{n-2} \operatorname{Ric}_0$ and $2(n-1) \operatorname{tr}(\mathsf{P}) = \operatorname{Sc}$, so $\mathsf{P} = \frac{1}{n-2} \operatorname{Ric}_0 + \frac{1}{2n(n-1)} \operatorname{Sc} g$. Thus $\mathsf{P}$ contains the same information as Ric. The main advantage of $\mathsf{P}$ is that it shows up in a simple formula for the decomposition of the Riemann curvature into its tracefree part, called the *Weyl curvature* $W$ and a trace part, see Section 2.1 of [6]. For our purpose, this can be neatly expressed using the operation $\{\ ,\ \}$ introduced in §2.4 (4) above as

$$R(\xi, \eta)(\zeta) = W(\xi, \eta)(\zeta) + \{\xi, \mathsf{P}(\eta)\}(\zeta) - \{\eta, \mathsf{P}(\xi)\}(\zeta). \tag{2.16}$$

This follows directly from the formula in [6] using (2.14). It is also proved in a slightly more general setting in Section 1.6.6 of [17], which however uses the opposite sign convention for the Schouten tensor. One can actually view (2.16) as the definition of the Weyl curvature, one then has to show that $W$ has the same symmetries as $R$ but in addition lies in the kernel of all contractions. The advantage of the form (2.16) of the decomposition is that it immediately extends to the induced connection on any natural vector bundle, one just has to replace the evaluation on $\zeta$ by the action $\bullet$ on a section $s$.

We need a second curvature quantity, the *Cotton–York tensor* $Y$ of $g$. This is a two-form on $M$ with values in $T^*M$ which is defined by

$$Y(\xi, \eta) := \nabla_\xi \mathsf{P}(\eta) - \nabla_\eta \mathsf{P}(\xi) - \mathsf{P}([\xi, \eta]) \tag{2.17}$$

for $\xi, \eta \in \mathfrak{X}(M)$. Thus $Y$ is the covariant exterior derivative of the Schouten tensor $\mathsf{P}$, see §2.6 below. It is well known, c.f. [6], that for $n = 3$, the Weyl-tensor $W$ always vanishes identically and vanishing of $Y$ is equivalent to conformal flatness of $g$. For $n \geq 4$, it is well known that $W$ vanishes identically if and only if $g$ is conformally flat and that $Y$ can be obtained as the divergence of $W$, so it also vanishes in the conformally flat case.

**Definition 2.1.** Consider an irreducible representation $\mathbb{V}$ of $O(b)$ as in §2.2. Then we define the *twisted connection* $\nabla^{\mathcal{V}}$ on $\mathcal{V}M$ by

$$\nabla^{\mathcal{V}}_{\xi} s := \nabla_{\xi} s + \xi \bullet s - \mathsf{P}(\xi) \bullet s, \tag{2.18}$$

where $\nabla$ is the Levi-Civita connection.

Since the last two terms in (2.18) are tensorial, this indeed is a linear connection on $\mathcal{V}M$. Taking the natural decomposition $\mathbb{V} = \oplus_{i=0}^{N} \mathbb{V}_i$ and the corresponding decomposition $\mathcal{V}M = \oplus \mathcal{V}_i M$ we see that $\nabla_{\xi}$ preserves the summands $\Gamma(\mathcal{V}_i M)$ but $\nabla^{\mathcal{V}}_{\xi}$ does not have this property. Indeed, denoting the components by subscripts, we get

$$(\nabla^{\mathcal{V}}_{\xi} s)_i = \nabla_{\xi} s_i + \xi \bullet s_{i+1} - \mathsf{P}(\xi) \bullet s_{i-1}.$$

Observe that the summand containing the action of $\xi$ corresponds to the operators called $S$ in [15], while the term involving the Schouten tensor is not present there. The main reason for including this term is the following result.

**Theorem 2.2.** For $\xi, \eta \in \mathfrak{X}(M)$ and $s \in \Gamma(\mathcal{V}M)$, the curvature $R^{\mathcal{V}}$ of $\nabla^{\mathcal{V}}$ is given by

$$R^{\mathcal{V}}(\xi, \eta)(s) := W(\xi, \eta) \bullet s + Y(\xi, \eta) \bullet s,$$

where $W$ and $Y$ are the Weyl curvature and the Cotton–York tensor of $g$, respectively. In particular, the connection $\nabla^{\mathcal{V}}$ is flat if and only if the metric $g$ is conformally flat.

*Proof.* We directly use the defining equation

$$R^{\mathcal{V}}(\xi, \eta)(s) = \nabla^{\mathcal{V}}_{\xi} \nabla^{\mathcal{V}}_{\eta} s - \nabla^{\mathcal{V}}_{\eta} \nabla^{\mathcal{V}}_{\xi} s - \nabla^{\mathcal{V}}_{[\xi,\eta]} s$$

for the curvature. Taking $s \in \Gamma(\mathcal{V}M)$ and the components $s_i \in \Gamma(\mathcal{V}_i M)$, we of course get $R^{\mathcal{V}}(\xi, \eta)(s) = \sum_i R^{\mathcal{V}}(\xi, \eta)(s_i)$. From the definition in (2.18) it follows readily that $R^{\mathcal{V}}(\xi, \eta)(s_i)$ may have non-trivial components of degree $i-2$, $i-1$, $i$, $i+1$, and $i+2$ only. By definition, the component in degree $i-2$ equals $\xi \bullet (\eta \bullet s_i) - \eta \bullet (\xi \bullet s_i)$, so this vanishes by (2.6). In the same way, (2.7) implies vanishing of the component of degree $i+2$. For the component in degree $i-1$ we immediately get

$$\nabla_{\xi}(\eta \bullet s_i) + \xi \bullet (\nabla_{\eta} s_i) - \nabla_{\eta}(\xi \bullet s_i) - \eta \bullet (\nabla_{\xi} s_i) - [\xi, \eta] \bullet s_i.$$

By (2.5) the first and fourth term add up to $(\nabla_{\xi}\eta) \bullet s_i$ and likewise the second and third term give $-(\nabla_{\eta}\xi) \bullet s_i$. But torsion freeness of $\nabla$ gives $[\xi, \eta] = \nabla_{\xi}\eta - \nabla_{\eta}\xi$ so the whole expression vanishes. The analysis of the component in degree $i+1$

is very similar, but we have to replace the vector fields acting via $\bullet$ by their image under $\mathsf{P}$. Using the analog of (2.5) for the action of one-forms and the defining equation (2.17), one readily concludes that the component in degree $i+1$ is given by $Y(\xi,\eta)\bullet s_i$.

So it remains to understand the component in degree $i$. Expanding the defining equation, there are terms containing only Levi-Civita derivatives and in view of (2.3), they add up to $R(\xi,\eta)\bullet s_i$. The remaining terms only come from the double derivatives (and not from the derivative in direction of the Lie bracket) and they are given by

$$-\mathsf{P}(\xi)\bullet(\eta\bullet s_i)-\xi\bullet(\mathsf{P}(\eta)\bullet s_i)+\mathsf{P}(\eta)\bullet(\xi\bullet s_i)+\eta\bullet(\mathsf{P}(\xi)\bullet s_i).$$

Since the bundle $\mathcal{V}M$ is induced by a representation of $\mathfrak{g}$, the first term and the last term add up to $\{\eta,\mathsf{P}(\xi)\}\bullet s_i$, while the other two terms add up to $-\{\xi,\mathsf{P}(\eta)\}\bullet s_i$. In view of the extension of (2.16) to associated bundles, this completes the proof. QED

## 2.6 The twisted de Rham sequence

The next step in the construction is standard. Any linear connection on a vector bundle $F$ can be coupled to the exterior derivative to define the so-called covariant exterior derivative on $F$-valued differential forms. Moreover, the composition of two instances of this operator can be explicitly described in terms of the curvature of the initial connection. In particular, starting from a flat connection on $F$, one obtains a differential complex.

The simplest way to implement this is to view $\varphi\in\Omega^k(M,F)$, the space of $F$-valued $k$-forms, as a $k$-linear, alternating map, which associated to $k$ vector fields on $M$ a section of $F$ and is linear over smooth functions in each entry. Given a linear connection $\nabla$ on $F$, the covariant exterior derivative $d^\nabla:\Omega^k(M,F)\to\Omega^{k+1}(M,F)$ is characterized by

$$\begin{aligned}(d^\nabla\varphi)(\xi_0,\dots,\xi_k)=&\textstyle\sum_{i=0}^k(-1)^i\nabla_{\xi_i}\varphi(\xi_0,\dots,\widehat{\xi_i},\dots,\xi_k)\\&+\textstyle\sum_{i<j}(-1)^{i+j}\varphi([\xi_i,\xi_j],\xi_0,\dots,\widehat{\xi_i},\dots,\widehat{\xi_j},\dots\xi_k),\end{aligned}\tag{2.19}$$

where the hats denote omission. Exactly as for the global formula for the exterior derivative, one easily verifies directly that this indeed defines an $F$-valued $k+1$-form.

Applying this to the Levi-Civita connection on any natural bundle $\mathcal{W}M$, we obtain $d^\nabla:\Omega^k(M,\mathcal{W}M)\to\Omega^{k+1}(M,\mathcal{W}M)$. In the case of a bundle $\mathcal{V}M$ induced by a representation $\mathbb{V}$ of $O(b)$ as in §2.3, we can also form $d^{\nabla^\mathcal{V}}:\Omega^k(M,\mathcal{V}M)\to\Omega^{k+1}(M,\mathcal{V}M)$. It is easy to describe the relation of this operator to $d^\nabla$. Via the decomposition $\mathcal{V}M = \oplus_{i=0}^{N} \mathcal{V}_i$ we can decompose any form $\varphi \in \Omega^k(M, \mathcal{V}M)$ into components $\varphi_i \in \Omega^k(M, \mathcal{V}_i M)$. Using this, we formulate

**Proposition 2.3.** For $\varphi \in \Omega^k(M, \mathcal{V}M)$, we get

(1) $d^{\nabla^{\mathcal{V}}}\varphi = d^{\nabla}\varphi + \partial\varphi + \partial^{\mathsf{P}}\varphi$, where

$$\partial\varphi(\xi_0, \dots, \xi_k) = \textstyle\sum_{i=0}^{k} (-1)^i \xi_i \bullet \varphi(\xi_0, \dots, \widehat{\xi_i}, \dots, \xi_k) \tag{2.20}$$

$$\partial^{\mathsf{P}}\varphi(\xi_0, \dots, \xi_k) = \textstyle\sum_{i=0}^{k} (-1)^i \mathsf{P}(\xi_i) \bullet \varphi(\xi_0, \dots, \widehat{\xi_i}, \dots, \xi_k) \tag{2.21}$$

Decomposing into components, we get $(d^{\nabla^{\mathcal{V}}}\varphi)_i = d^{\nabla}\varphi_i + \partial\varphi_{i+1} + \partial^{\mathsf{P}}\varphi_{i-1}$.

(2) $(d^{\nabla^{\mathcal{V}}}(d^{\nabla^{\mathcal{V}}}\varphi))_i$ maps $\xi_0, \dots, \xi_{k+1}$ to

$$\textstyle\sum_{j<\ell} (-1)^{j+\ell} (W(\xi_j, \xi_\ell) \bullet \varphi_i(\xi_0, \dots, \widehat{\xi_j}, \dots, \widehat{\xi_\ell} \dots, \xi_k) \\ + Y(\xi_j, \xi_\ell) \bullet \varphi_{i-1}(\xi_0, \dots, \widehat{\xi_j}, \dots, \widehat{\xi_\ell} \dots, \xi_k)).$$

In particular, $(\Omega^*(M, \mathcal{V}M), d^{\nabla^{\mathcal{V}}})$ is a complex if and only if $g$ is conformally flat.

*Proof.* (1) immediately follows from combining the defining equations (2.18) and (2.19). (2) follows by combining general results on the covariant exterior derivative (see Section 19.13 of [30]) with Theorem 2.2. QED

The notation $\partial$ is chosen here since these operators are directly induced by the standard differential in the complex computing the Lie algebra cohomology of the abelian Lie algebra $\mathfrak{g}_{-1}$ with coefficients in the representation $\mathbb{V}$. This allows for some input from representation theory, which will be very helpful later.

*Remark* 2.4. Directly generalizing parts of [15], one can also define a connection on $\mathcal{V}M$ via $(\xi, s) \mapsto \nabla_\xi s + \xi \bullet s$. The proof of Theorem 2.2 then shows that the curvature of this connection is given by $R(\xi, \eta) \bullet s$, so loosely speaking the action term does not change the curvature. The associated covariant exterior derivative then is explicitly given by $d^{\nabla}\varphi + \partial\varphi$, so the maps $\partial$ correspond to the $S$-operators in [15]. Applying the covariant exterior derivative to $\varphi$ twice, the result sends $\xi_0, \dots, \xi_{k+1}$ to

$$\textstyle\sum_{j<\ell} (-1)^{j+\ell} R(\xi_j, \xi_\ell) \bullet \varphi(\xi_0, \dots, \widehat{\xi_j}, \dots, \widehat{\xi_\ell} \dots, \xi_k).$$

So in the case that the metric $g$ is flat, this makes $\Omega^*(M, \mathcal{W}M)$ into a differential complex, thus providing a direct extension of parts of [15] to this case (in a smooth setting).

For flat metrics, also each $(\Omega^*(M, \mathcal{V}_i M), d^{\nabla})$ is a complex and locally, the sum of the cohomologies of these complexes is isomorphic to the cohomology of this "twisted complex". Locally, one could also derive analogs of the $K$-operators used in [15] via the relation between local parallel sections. In contrast to the case of domains in $\mathbb{R}^n$ as studied in [15], one cannot expect to derive uniform formulae for those operators, though.

## 2.7 The cohomology bundles

We have defined the operation $\partial$ on $\mathcal{V}M$-valued differential forms, but it is evidently tensorial and hence induced by bundle maps $\Lambda^k T^*M \otimes \mathcal{V}M \to \Lambda^{k+1}T^*M \otimes \mathcal{V}M$ for $k = 0, \dots, n-1$, which we denote by the same symbol. By construction $\partial(\Lambda^k T^*M \otimes \mathcal{V}_i M) \subset \Lambda^{k+1}T^*M \otimes \mathcal{V}_{i-1}M$ and Theorem 2.5 readily implies that $\partial \circ \partial = 0$. Alternatively the latter fact can be easily verified directly using (2.6). Hence in each degree $k$, we have natural subbundles $\operatorname{im}(\partial) \subset \ker(\partial) \subset \Lambda^k T^*M \otimes \mathcal{V}M$.

**Definition 2.5.** Consider an irreducible representation $\mathbb{V}$ of $O(b)$ and the corresponding bundle $\mathcal{V}M = \oplus_{i=0}^N \mathcal{V}_i M$. Then for each degree $k = 0, \dots, n$, consider $\operatorname{im}(\partial) \subset \ker(\partial) \subset \Lambda^k T^*M \otimes \mathcal{V}M$ and define the *cohomology bundle* $\mathcal{H}_k^{\mathcal{V}}M := \ker(\partial)/\operatorname{im}(\partial)$.

The terminology here comes from the fact that $\partial$ is induced by the differential in the standard complex computing the cohomology of the (abelian) Lie algebra $\mathfrak{g}_{-1}$ with coefficients in the representation $\mathbb{V}$. Hence the bundle $\mathcal{H}_k^{\mathcal{V}}M$ by construction is the associated bundle corresponding to the cohomology space $H^k(\mathfrak{g}_{-1}, \mathbb{V})$ which naturally carries a representation of $O(n)$ and of $\mathfrak{g}_0 \cong \mathfrak{co}(n)$. For the following developments it will not be really necessary to understand what the cohomology bundles look like, but of course this is needed to deal with examples. It is important to realize that the cohomology bundles are much smaller that the bundles of $\mathcal{V}M$-valued differential forms, and the difference gets more significant the more complicated the representation $\mathbb{V}$ gets.

In simple cases, the explicit form can be determined by direct computations, but this is a point where it becomes increasingly important to use information coming from representation theory. Recall that any finite dimensional representation of $O(n)$ or $\mathfrak{o}(n)$ splits as a direct sum of irreducible representations, which do not contain any non-trivial invariant subspaces. For $G_0 \cong CO(n)$ and $\mathfrak{g}_0 \cong \mathfrak{co}(n)$ the same holds under an additional condition on the representation which is satisfied in all cases arising in the context of this article. A key feature of irreducible representations comes from Schur's lemma. An equivariant map between two irreducible representations is either zero or an isomorphism and in the complex case, this isomorphism is uniquely determined up to a nonzero multiple. In particular, this easily implies that on a $\mathfrak{g}_0$-irreducible subspace in

$\Lambda^k(\mathfrak{g}_{-1})^* \otimes \mathbb{V}$ the grading element $E$ has to act by a scalar multiple of the identity, which implies that it is contained in $\Lambda^k(\mathfrak{g}_{-1})^* \otimes \mathbb{V}_i$ for some index $i$.

Let us illustrate how to use elementary arguments from representation theory in the example that $\mathbb{V}$ is the standard representation $\mathbb{R}^{n+2}$ of $O(b)$. We know that $\mathbb{V} = \mathbb{V}_0 \oplus \mathbb{V}_1 \oplus \mathbb{V}_2$ with $\dim(\mathbb{V}_0) = \dim(\mathbb{V}_2) = 1$ and $\dim(\mathbb{V}_1) = n$. We have also seen above how $\partial$ is compatible with this decomposition. In particular, in degree zero, we get $H^0(\mathfrak{g}_{-1}, \mathbb{V}) = \ker(\partial)$ and $\mathbb{V}_0$ is evidently contained in there. Next, $\mathbb{V}_1 \cong \mathbb{R}^n$ is mapped by $\partial$ to $\mathbb{R}^{n*} \otimes \mathbb{V}_0$ and both the source and the target are irreducible. Verifying directly that this map is non-zero (or using the general information on the structure of cohomology provided below) we conclude that this component of $\partial$ has to be an isomorphism. Finally, on $\mathbb{V}_2$, we obtain the special case $k = 0$ of the sequence

$$\Lambda^k\mathbb{R}^{n*} \otimes \mathbb{V}_2 \xrightarrow{\partial} \Lambda^{k+1}\mathbb{R}^{n*} \otimes \mathbb{R}^n \xrightarrow{\partial} \Lambda^{k+2}\mathbb{R}^{n*} \otimes \mathbb{V}_0 \tag{2.22}$$

which we have to consider in general for $k = 0, \dots n$ to deal with higher degrees. Now the first and last space in this sequence are always irreducible, while for most values of $k$, the middle space splits into three irreducible components. In particular, for $k = 0$, the middle space is isomorphic to $L(\mathbb{R}^n, \mathbb{R}^n)$ and these components correspond to multiples of the identity, symmetric trace-free maps and skew-symmetric maps, respectively. Now one immediately verifies that for $0 \leq k < n$, the first map in the sequence is injective, while the last map is always surjective. Together with the above, this shows that, in degree 0, $\partial$ is injective on $\mathbb{V}_1 \oplus \mathbb{V}_2$, so $H^0(\mathfrak{g}_{-1}, \mathbb{V}) = \mathbb{V}_0$. Next, $\mathbb{R}^{n*} \otimes \mathbb{V}_0 \subset \operatorname{im}(\partial)$ and $\partial$ is injective on $\mathbb{R}^{n*} \otimes \mathbb{V}_2$. Hence $H^1(\mathfrak{g}_{-1}, \mathbb{V})$ comes only from the middle space in (2.22) for $k = 0$, and is isomorphic to tracefree symmetric maps (realized as symmetric maps modulo multiples of the identity). Similarly, one sees that for $k = 1, \dots, n-2$, the cohomology $H^{k+1}(\mathfrak{g}_{-1}, \mathbb{V})$ comes from the middle space in (2.22) and is isomorphic to the intersection of the kernels of the complete alternation and the contraction $\Lambda^{k+1}\mathbb{R}^{n*} \otimes \mathbb{R}^n \to \Lambda^k\mathbb{R}^{n*}$. It is well known that this also is an irreducible representation of $\mathfrak{o}(n)$.

Detailed information on the cohomology spaces for general representations $\mathbb{V}$ can be obtained from more advanced representation theory, which requires substantial background, however. This is based on Kostant's theorem, which was originally proved in [27], see also Section 3.3 of [17] for an exposition. For any complex irreducible representation $\mathbb{V}$ of $\mathfrak{o}(b)$ and each degree $k$, the cohomology $H^k(\mathfrak{g}_{-1}, \mathbb{V})$ splits into a direct sum of irreducible representations. The number of these components is independent of $\mathbb{V}$ and can be described as the cardinality of the certain subset in the Weyl group of $\mathfrak{o}(b)$ (which is a finite group). This subset can be determined algorithmically and knowing this, the highest weights of the corresponding irreducible components in the cohomology can be determined algorithmically from the highest weight of $\mathbb{V}$. The case of a real representation $\mathbb{V}$ can then be dealt with via analyzing the complexification. In either case, this needs substantial input from representation theory of semisimple Lie algebras (description of representations by highest weights, etc.) and thus is beyond the scope of the current article. In what follows, we will not discuss in detail how to apply this theory but just state the results that we need. In particular, the following fundamental facts can be easily deduced from just knowing the subset of the Weyl group. The relevant subsets are discussed in Examples 4.3.7 and 4.3.8 of [5], which uses them for a different propose, however, and thus does not discuss the relation to Lie algebra cohomology.

- If $n$ is odd or $n$ is even and $k \neq n/2$, then $H^k(\mathfrak{g}_{-1}, \mathbb{V})$ is an irreducible representation of $\mathfrak{g}_0$.
- If $n$ is even, then $H^{n/2}(\mathfrak{g}_{-1}, \mathbb{V})$ decomposes into the sum of at most two irreducible representations of $\mathfrak{g}_0$.
- $H^0(\mathfrak{g}_{-1}, \mathbb{V}) \cong \mathbb{V}_0$

Alternatively to the above description as a quotient, one can also realize $\mathcal{H}_k^{\mathcal{V}}M$ as a subbundle of $\Lambda^k T^*M \otimes \mathcal{V}M$. Indeed, it is well known that on any finite dimensional representation of $O(n)$, there is a positive definite inner product that is $O(n)$-invariant. Applying this to each of the representations $\mathbb{V}_i$ and $\Lambda^k \mathbb{R}^{n*}$, we also get an inner product on $\mathbb{V} = \oplus_{i=0}^N \mathbb{V}_i$ and then on $\Lambda^k \mathbb{R}^{n*} \otimes \mathbb{V}$ for each $k = 0, \dots, n$. These inner products in turn induce natural positive definite bundle metrics on each of the bundles $\Lambda^k T^*M \otimes \mathcal{V}M$. Having these at hand, we can form the subbundle $\Upsilon_k := \ker(\partial) \cap \operatorname{im}(\partial)^\perp \subset \Lambda^k T^*M \otimes \mathcal{V}M$ which by construction projects isomorphically onto $\mathcal{H}_k^{\mathcal{V}}M$. Since the grading element $E$ discussed in §2.2 acts by a scalar on each irreducible representation of $\mathfrak{g}_0$, we conclude that for $k \neq \frac{n}{2}$, the subbundle $\Upsilon_k$ is contained in $\Lambda^k T^*M \otimes \mathcal{V}_i M$ for some index $i$.

## 2.8 The Riemannian BGG construction

The BGG construction "compresses" the twisted exterior derivative to higher order operators between the cohomology bundles. We first need appropriate "inverses" to the bundle maps $\partial$. Following [15] we call these $T : \Lambda^k T^*M \otimes \mathcal{V}M \to \Lambda^{k-1} T^*M \otimes \mathcal{V}M$ and use the same symbol for the induced tensorial maps on sections. In terms of the inner products introduced in the end of §2.7 above, $\partial$ induces, in each degree, an isomorphism $\ker(\partial)^\perp \to \operatorname{im}(\partial)$ and we define $T$ to be the inverse of this on $\operatorname{im}(\partial)$ and as zero on $\operatorname{im}(\partial)^\perp$. This readily implies the following properties.

$$\ker(T) = \operatorname{im}(\partial)^\perp \qquad \operatorname{im}(T) = \ker(\partial)^\perp \qquad \Upsilon_k = \ker(\partial) \cap \ker(T) \tag{2.23}$$
$$T \circ T = 0 \qquad T \circ \partial \circ T = T \qquad \partial \circ T \circ \partial = \partial. \tag{2.24}$$

Having these operators at hand, we can proceed similarly as in [15]. We first observe that $T \circ (d^{\nabla^{\mathcal{V}}} - \partial)$ maps each $\Omega^k(M, \mathcal{V}M)$ to itself, but the subspace $\Omega^k(M, \mathcal{V}_i M)$ is mapped to $\Omega^k(M, \oplus_{\ell \geq i+1} \mathcal{V}_\ell M)$. Hence $T \circ (d^{\nabla^{\mathcal{V}}} - \partial)$ is a nilpotent operator. On the other hand, we can compute

$$T \circ d^{\nabla^{\mathcal{V}}} \circ T = T \circ (d^{\nabla^{\mathcal{V}}} - \partial) \circ T + T \circ \partial \circ T = (\operatorname{id} + T \circ (d^{\nabla^{\mathcal{V}}} - \partial)) \circ T.$$

Hence on sections of $\operatorname{im}(T)$, $T \circ d^{\nabla^{\mathcal{V}}}$ coincides with $(\operatorname{id} + T \circ (d^{\nabla^{\mathcal{V}}} - \partial))$ and hence is invertible, with inverse given by $\sum_{i=0}^{\infty} (-1)^i (T \circ (d^{\nabla^{\mathcal{V}}} - \partial))^i$ and the sum is actually finite. Then we define $G := (\sum_{i=0}^{\infty} (-1)^i (T \circ (d^{\nabla^{\mathcal{V}}} - \partial))^i) \circ T$, which obviously implies $\ker(T) \subset \ker(G)$ and $\operatorname{im}(G) \subset \operatorname{im}(T)$.

**Proposition 2.6.** For $\alpha \in \Omega^k(M, \mathcal{V}M)$ with $T(\alpha) = 0$, the form $S(\alpha) := \alpha - G(d^{\nabla^{\mathcal{V}}}(\alpha))$ satisfies $T(S(\alpha)) = 0$, $S(\alpha) - \alpha \in \Gamma(\operatorname{im}(T))$ and $T(d^{\nabla^{\mathcal{V}}}(S(\alpha))) = 0$ and is uniquely determined by theses three properties.

*Proof.* The first two properties of $S(\alpha)$ follow immediately from $\operatorname{im}(G) \subset \operatorname{im}(T) \subset \ker(T)$. For the last property, we just observe that by construction $T \circ d^{\nabla^{\mathcal{V}}} \circ G = T$. If $\varphi \in \Omega^k(M, \mathcal{V}M)$ also satisfies the three properties, then $\varphi - S(\alpha) \in \Gamma(\operatorname{im}(T))$ and $(T \circ d^{\nabla^{\mathcal{V}}})(\varphi - S(\alpha)) = 0$. But as verified above, $T \circ d^{\nabla^{\mathcal{V}}}$ is invertible on $\Gamma(\operatorname{im}(T))$, which implies $\varphi = S(\alpha)$. QED

In particular, we can apply $S$ to $\alpha \in \Gamma(\Upsilon_k)$ to obtain $S(\alpha) \in \Gamma(\ker(T)) \subset \Omega^k(M, \mathcal{V}M)$. Since $S(\alpha) - \alpha \in \Gamma(\operatorname{im}(T))$, we conclude that the component of $S(\alpha)$ in $\Gamma(\Upsilon_k)$ coincides with $\alpha$, whence $S$ is called the *splitting operator*. On the other hand, since $d^{\nabla^{\mathcal{V}}}(S(\alpha)) \in \Gamma(\ker(T))$, we can project it orthogonally to $\Gamma(\Upsilon_{k+1})$ to define $D(\alpha)$ and obtain a differential operator $D = D_k : \Gamma(\Upsilon_k) \to \Gamma(\Upsilon_{k+1})$. These operators are called the *BGG operators* determined by $\mathbb{V}$.

## 2.9 The conformally flat case

If the metric $g$ is conformally flat, then by Proposition 2.3, the twisted de Rham sequence $(\Omega^*(M, \mathcal{V}M), d^{\nabla^{\mathcal{V}}})$ is a complex. This leads to a nice conceptual understanding of the relation to the BGG sequence.

**Theorem 2.7.** If the twisted de Rham sequence $(\Omega^*(M, \mathcal{V}M), d^{\nabla^{\mathcal{V}}})$ is a complex, then also the BGG sequence $(\Gamma(\Upsilon_*), D)$ is a complex and for any open subset $U \subset M$, the two complexes compute the same cohomology on $U$. In particular, $(\Gamma(\Upsilon_*), D)$ is a fine resolution of the sheaf of local parallel sections of $\mathcal{V}M$.

*Proof.* For $\alpha \in \Gamma(\Upsilon_k)$, consider $d^{\nabla^{\mathcal{V}}}(S(\alpha)) \in \Omega^{k+1}(M, \mathcal{V}M)$. By Proposition 2.6, this is a section of $\ker(T)$ and its component in $\Upsilon_{k+1}$ by definition is $D(\alpha)$, so the difference to $D(\alpha)$ is a section of $\operatorname{im}(T)$. But if $d^{\nabla^{\mathcal{V}}} \circ d^{\nabla^{\mathcal{V}}} = 0$, we conclude that it also lies in the kernel of $T \circ d^{\nabla^{\mathcal{V}}}$. Hence the uniqueness part of Proposition 2.6 shows that $d^{\nabla^{\mathcal{V}}}(S(\alpha)) = S(D(\alpha))$. But then $d^{\nabla^{\mathcal{V}}}(S(\alpha)) = 0$ and hence $D(D(\alpha)) = 0$. This shows that $(\Gamma(\Upsilon_*), D)$ is a complex and $S$ is chain map to the twisted de Rham complex and hence there is an induced map in cohomology.

Now suppose that $\varphi \in \Omega^k(M, \mathcal{V}M)$ satisfies $d^{\nabla^{\mathcal{V}}}(\varphi) = 0$ and consider $\tilde{\varphi} := \varphi - d^{\nabla^{\mathcal{V}}}(G(\varphi))$. This is cohomologous to $\varphi$ and we know that $T(d^{\nabla^{\mathcal{V}}}(G(\varphi))) = T(\varphi)$, so $\tilde{\varphi}$ is a section of $\ker(T)$. Denoting by $\alpha$ the component of $\tilde{\varphi}$ in $\Gamma(\Upsilon_k)$, we immediately conclude that $\tilde{\varphi}$ satisfies the properties from Proposition 2.6 and hence $\tilde{\varphi} = S(\alpha)$. Hence the map in cohomology is surjective. On the other hand, suppose that $\alpha \in \Gamma(\Upsilon_k)$ has the property that $S(\alpha) = d^{\nabla^{\mathcal{V}}}(\psi)$ for some $\psi \in \Omega^{k-1}(M, \mathcal{V}M)$. Forming $\tilde{\psi}$ as above and denoting by $\beta$ its component in $\Gamma(\Upsilon_{k-1})$ we get that $\tilde{\psi} = S(\beta)$ and $S(\alpha) = d^{\nabla^{\mathcal{V}}}(S(\beta))$ and hence $\alpha = D(\beta)$. This shows injectivity of the map in cohomology and hence completes the proof of the first part. The fact that one obtains a fine resolution then immediate follows from the corresponding fact for the twisted de Rham sequence proved in Proposition 2.3. QED

This generalizes the setting of [15] for smooth sections to conformally flat Riemannian manifolds. The main difference is that we can still consider the individual "rows" $\Omega^*(M, \mathcal{V}_i M)$ of the twisted complex with $d^{\nabla}$ acting on them, but they are not complexes any more. So for further applications, say in the direction of [16], one would have to start from the full twisted complex, which comes from a flat connection.

## 2.10 The conformally non-flat case

Without the assumption on conformal flatness, one does not obtain complexes and the connection between the twisted sequence and the BGG sequence is less tight. Most applications so far were to the study of the first operator in the BGG sequence, but there certainly is very interesting potential in analyzing the rest of the sequence. The general facts are collected in the following result.

**Theorem 2.8.** Fix a Riemannian manifold $(M, g)$ and an irreducible representation $\mathbb{V}$ of $O(b)$ and consider the corresponding bundle $\mathcal{V}M = \oplus_i \mathcal{V}_i M$ and the connection $\nabla^{\mathcal{V}}$.

(1) Suppose that $s \in \Gamma(\mathcal{V}M)$ satisfies $\nabla^{\mathcal{V}} s = 0$. Then for the component $s_0 \in \Gamma(\mathcal{V}_0 M)$, we get $s = S(s_0)$ and $D(s_0) = 0$. Hence mapping to the $\mathcal{V}_0 M$-component identifies the space of parallel sections of $\mathcal{V}M$ with a linear subspace of the kernel of the first BGG operator.

(2) For $k > 0$, projection to $\Gamma(\mathcal{H}_k^{\mathcal{V}} M)$ identifies $\ker(T) \cap \ker(d^{\nabla^{\mathcal{V}}})$ with a linear subspace of the kernel of $D : \Gamma(\mathcal{H}_k^{\mathcal{V}} M) \to \Gamma(\mathcal{H}_{k+1}^{\mathcal{V}} M)$.

*Proof.* The basic argument is the same for both parts. Suppose that $T(\varphi) = d^{\mathcal{V}} \varphi = 0$ and let $\alpha$ denote the projection of $\varphi$ to $\Gamma(\Upsilon_k)$. Then $\varphi - \alpha \in \Gamma(\operatorname{im}(T))$ and hence Proposition 2.3 implies that $\varphi = S(\alpha)$, which in turn implies that $D(\alpha) = 0$. On the other hand, it also shows that the projection to the component $\Gamma(\Upsilon_k)$ is injective, which completes the argument for (2). In degree zero $T(s) = 0$ is satisfied automatically and $d^{\nabla^{\mathcal{V}}} = \nabla^{\mathcal{V}}$ so (1) follows, too. QED

Following [28], it has become common in the setting of parabolic geometries to call sections $\sigma \in \Gamma(\mathcal{V}_0 M)$, for which $S(\sigma) \in \Gamma(\mathcal{V}M)$ is parallel for $\nabla^{\mathcal{V}}$, *normal solutions* of the first BGG operator. In several cases, these can be explicitly characterized by interesting (tensorial or differential) conditions. For example, starting with the adjoint representation $\mathfrak{o}(b)$, the first BGG operator is the conformal Killing operator on vector fields, so its kernel consists of all conformal Killing fields on $M$. Normal conformal Killing field then turn out to be exactly those, which in addition insert trivially into the Weyl curvature and into the Cotton-York tensor, see [11].

The reason why we have singled out the case $k = 0$ in Theorem 2.8 is because the result in degree zero turns out to be significantly stronger. Indeed, it was shown in [8] (in a more general context) that one can modify the connection $\nabla^{\mathcal{V}}$ in such a way that projection to the component in $\mathcal{V}_0 M$ induces a bijection between parallel sections of this new connection and the kernel of the first BGG operator. So in particular, the kernel of the first BGG operator is of dimension $\leq \dim(\mathbb{V})$ (which is an interesting result in its own right). This actually extends to operators with the same principal part as the first BGG operator. While the construction in [8] does not provide a natural construction of these so-called "prolongation connections", it turned out that there also is an invariant construction in the general setting of parabolic geometries, see [26].

## 2.11 Example

Let us discuss the case $\mathbb{V} = \mathbb{R}^{n+2}$ of the standard representation in detail. From §2.7 we know that $H^0(\mathfrak{g}_{-1}, \mathbb{V}) = \mathbb{V}_0 \cong \mathbb{R}$, $H^1(\mathfrak{g}_{-1}, \mathbb{V}) \cong S_0^2 \mathbb{R}^{n*} \subset \mathfrak{g}_{-1}^* \otimes \mathbb{V}_1$, and that for $2 \leq k < n$, $H^k(\mathfrak{g}_{-1}, \mathbb{V}) \subset \Lambda^k \mathfrak{g}_{-1}^* \otimes \mathbb{V}_1$. To compute the splitting operator in degree zero, we use the notation from Example 2.4 (1), so we write

$s \in \Gamma(\mathcal{V}M)$ as $(f, \eta, h)$ with $f, h \in C^\infty(M, \mathbb{R})$ and $\eta \in \mathfrak{X}(M)$. By definition,

$$\nabla^{\mathcal{V}}_\xi(f, \eta, h) = (df(\xi) - g(\xi, \eta), \nabla_\xi \eta + h\xi + f\mathsf{P}(\xi), dh(\xi) - \mathsf{P}(\xi, \eta)). \tag{2.25}$$

By Proposition 2.6, we can obtain the formula for $S(f)$ by choosing $\eta$ and $h$ in such a way that $T(\nabla^{\mathcal{V}}_\xi(f, \eta, h)) = 0$. The discussion in §2.7 shows that this means that the first component in the right hand side of (2.25) has to vanish, while the second component has to be trace-free. The first condition is equivalent to $\eta = df^\#$. Inserting this into the second component (with varying $\xi$) and taking the trace, we obtain $0 = \Delta f + nh + f\,\mathrm{tr}(\mathsf{P})$ which implies $h = -\frac{1}{n}(\Delta + \mathrm{tr}(\mathsf{P}))f$. Inserting these into (2.25) the middle component becomes the trace-free part of $\nabla^2 f + \mathsf{P}f$ and since this is automatically symmetric, it already lies in $\Upsilon_1$, so we have obtained the formula for the first BGG operator in this case. Thus we obtain

$$S(f) = (f, df^\#, -\tfrac{1}{n}(\Delta + \mathrm{tr}(\mathsf{P}))f) \qquad D(f) = \mathrm{tfp}(\nabla^2 f + \mathsf{P}f), \tag{2.26}$$

where we write tfp to indicate the trace-free part of a $\binom{0}{2}$-tensor field. It turns out that in this case any solution $D$ is normal, i.e. if $D(f) = 0$ then $\nabla^{\mathcal{V}} S(f) = 0$ and that these solutions are related to conformal rescalings of $g$ which are Einstein metrics, see [6].

In degrees $1, \dots, n-1$, both the splitting operators and the BGG operators are easier to get. Sections of $\Lambda^k T^*M \otimes \mathcal{V}_1 M$ can be viewed as $\binom{1}{k}$-tensor fields, which are skew symmetric in the $k$ lower indices. From §2.7 we conclude that $T : \Lambda^k T^*M \otimes \mathcal{V}_1 M \to \Lambda^{k-1} T^*M$ must be a non-zero multiple of the contraction, while $\partial : \Lambda^k T^*M \otimes \mathcal{V}_1 M \to \Lambda^{k+1} T^*M$, up to a non-zero multiple, must be given by lowering the upper index and the completely alternating. We also know that $\Upsilon_k = \ker(T) \cap \ker(\partial)$. For a section $\psi_1 \in \Gamma(\Upsilon_k)$, we know that $\partial\psi_1 = 0$, which together with Proposition 2.3 shows that $d^{\nabla^{\mathcal{V}}}(0, \psi_1, \psi_2)$ has vanishing first component, while the second component is given by $d^\nabla \psi_1 + \partial\psi_2$. The form $S(\psi_1)$ is characterized by the fact that this lies in the kernel of $T$, which says that $\psi_2 = -T(d^\nabla \psi_1)$. So up to a non-zero factor, this is obtained by alternating $\nabla\psi_1$ in the lower indices and then forming the unique contraction.

This also shows that $D(\psi_1)$ is the component of the tracefree part of $d^\nabla \psi_1$ that lies in the kernel of the complete alternation. This can be expressed as

$$d^\nabla \psi_1 - \partial(T(d^\nabla \psi_1)) - T(\partial(d^\nabla \psi_1)),$$

so to obtain a more explicit formula, one only has to make the operations $T$ and $\partial$ explicit. There is a simplification, however. From the definition of $d^{\nabla^{\mathcal{V}}}$ it follows that the component of $d^{\nabla^{\mathcal{V}}} \circ d^{\nabla^{\mathcal{V}}}$ that maps $\Lambda^k T^*M \otimes \mathcal{V}_i$ to $\Lambda^{k+2} T^*M \otimes \mathcal{V}_{k-1}$

is given by $d^\nabla \circ \partial + \partial \circ d^\nabla$. However, this component vanishes by Proposition 2.3, so $\partial(d^\nabla \psi_1) = -d^\nabla(\partial\psi_1) = 0$. Hence $D(\psi_1)$ is the tracefree part of $d^\nabla\psi_1$, i.e.

$$D(\psi_1) = d^\nabla\psi_1 - \partial(T(d^\nabla\psi_1)).$$

While the above description of the splitting operator extends to $k = n-1$, i.e. $S(\psi_1) = (0, \psi_1, -T(d^\nabla\psi_1))$, things are a bit different for the last BGG operator. The point here is that $T : \Lambda^{n-1}T^*M \to \Lambda^n T^*M \otimes TM$ is a linear isomorphism, which is reflected in the fact that $\Upsilon_n \subset \Lambda^n T^*M \otimes \mathcal{V}_2 M$ (and since this is a line bundle, we must have equality). Hence the first two components of $d^{\nabla^\mathcal{V}} S(\psi_1)$ already vanish in this case, so it has to have the form $(0, 0, D(\psi_1))$. Explicitly $D(\psi_1) = -d^\nabla(T(d^\nabla\psi_1)) - \partial^\mathsf{P}(\psi_1)$.

## 2.12 Some general results

Several properties observed in §2.11 above reflect general features that occur in all BGG sequences, in particular, this concerns the first BGG operators. As initially observed in [8], the order $r$ of the first BGG operator can be easily read off from the highest weight of the representation $\mathbb{V}$ that defines the BGG sequence in question. Moreover, it is shown there, that mapping $\mathbb{V}$ to $(\mathbb{V}_0, r)$ actually gives rise to a bijection between irreducible representations of $O(b)$ and pairs consisting of an irreducible representation of $O(n)$ and an integer $r \geq 1$. So an irreducible domain bundle and the order of the first BGG operator can be chosen arbitrarily and then give rise to a unique BGG sequence realizing a first operator with these properties. There also is a universal description of the target bundle of the first BGG operator in representation theory terms, which loosely can be described as that "largest irreducible component" in $S^r_0 T^*M \otimes \mathcal{V}_0 M$. It is also known that there is a unique natural projection from $S^r T^*M \otimes \mathcal{V}_0 M$ onto this component and the principal part of the first BGG operator is obtained by applying this projection to a symmetrized $r$-fold covariant derivative.

In the form of the discussion in higher degrees in §2.11 (i.e. without making the tensorial operations $T$, $\partial$ and $\partial^\mathsf{P}$ explicit), one can derive universal formulae for BGG operators of low order. Several results in that direction (also in higher order) are available in the literature, both in the setting of parabolic geometries, see e.g. [10, 18, 31] and for BGG-like constructions on domains in $\mathbb{R}^n$, see e.g. [4].

The first important observation here is that the order of BGG operators is directly linked to the location of the spaces $\Upsilon_k$ (respectively their components for $n$ even and $k = n/2$). We know in general that $\Upsilon_0 = \mathcal{V}_0 M$ and for the standard representation, we have observed above that $\Upsilon_k \subset \Lambda^k T^*M \otimes \mathcal{V}_1 M$ for $k = 1, \dots, n-1$, while $\Upsilon_n = \Lambda^n T^*M \otimes \mathcal{V}_2 M$. We have also observed there that the first and last operator in the sequence have order two, while all other operators are of order one.

In the cases we consider, it turns out that if a component of $\Upsilon_k$ sits in $\Lambda^k T^*M \otimes \mathcal{V}_i M$ then each component of $\Upsilon_{k+1}$ sits in $\Lambda^{k+1}T^*M \otimes \mathcal{V}_j M$ with $j \geq i$ and (the relevant component of) the BGG operator is of order $j - i + 1$. Using this, we derive universal formulae for BGG operators of order 1 and 2 in our setting, generalizations to higher order are straightforward in principle, but lead to intricate formulae quickly. For simplicity, we ignore the possibility of more than one component in the formulation, the necessary changes are quite obvious.

**Theorem 2.9.** Consider the BGG sequence induced by a representation $\mathbb{V}$ of $O(b)$ and a section $\psi \in \Gamma(\Upsilon_k) \subset \Omega^k(M, \mathcal{V}M)$ and the BGG operator $D : \Gamma(\Upsilon_k) \to \Gamma(\Upsilon_{k+1})$.

(1) If $D$ has order 1, then $D(\psi) = d^\nabla \psi - \partial(T(d^\nabla \psi)))$.

(2) If $D$ has order 2, then

$$D(\psi) = (\mathrm{id} - \partial \circ T - T \circ \partial)\left(-d^\nabla (T(d^\nabla \psi)) - \partial^{\mathsf{P}}(\psi)\right).$$

*Proof.* We know that the is an index $i$ such that $\Upsilon_k \subset \Lambda^k T^*M \otimes \mathcal{V}_i M$ and then for $\psi \in \Gamma(\Upsilon_k)$ we get $S(\psi)_j = 0$ for $j < i$ and $S(\psi)_i = \psi$.

(1) By assumption, $\Upsilon_{k+1} \subset \Lambda^{k+1}T^*M \otimes \mathcal{V}_i M$ and the component $d^{\nabla^{\mathcal{V}}}(S(\psi))_i$ is by definition given by $d^\nabla \psi + \partial(S(\psi)_{i+1})$. As in §2.11, the characterization of the splitting operator implies that $S(\psi)_{i+1} = -T(d^\nabla \psi)$, so $d^{\nabla^{\mathcal{V}}}(S(\psi))_i = d^\nabla \psi - \partial(T(d^\nabla \psi))$ and this lies in $\ker(T)$ by construction. As in §2.11, Proposition 2.3 implies that $\partial \circ d^\nabla = -d^\nabla \circ \partial$ and by assumption $\partial(\psi) = 0$. Hence $d^{\nabla^{\mathcal{V}}}(S(\psi))_i$ lies in $\ker(\partial)$ and hence is a section of $\Upsilon_{k+1}$ which coincides with $D(\psi)$ by definition.

(2) Here $\Upsilon_{k+1} \subset \Lambda^{k+1}T^*M \otimes \mathcal{V}_{i+1}M$ and by definition

$$d^{\nabla^{\mathcal{V}}}(S(\psi))_{i+1} = d^\nabla(S(\psi)_{i+1}) + \partial(S(\psi)_{i+2}) - \partial^{\mathsf{P}}(\psi). \tag{2.27}$$

As in (1), we see that $S(\psi)_{i+1} = -T(d^\nabla \psi)$. We also know that (2.27) has to lie in the kernel of $T$, which easily implies that

$$S(\psi)_{i+2} = T(d^\nabla(T(d^\nabla(\psi)))) + T(\partial^{\mathsf{P}}(\psi)),$$

whence the right hand side of (2.27) can be written as

$$(\mathrm{id} - \partial \circ T)(-d^\nabla(T(d^\nabla(\psi))) - \partial^{\mathsf{P}}(\psi)).$$

To obtain $D(\psi)$, this has to be projected to $\ker(\partial)$, so we have to apply $(\mathrm{id} - T \circ \partial)$ to it. Since $\partial \circ \partial = 0$, this exactly leads to the claimed formula. QED

Together with the above discussion, part (1) of this theorem implies that all conformal Killing operators arise as first BGG operators. Fixing an irreducible representation $\mathbb{V}_0$ of $O(n)$, one takes the irreducible representation $\mathbb{V}$ of $O(b)$ corresponding to $(\mathbb{V}_0, 1)$. The corresponding BGG sequence starts with a first order operators defined on $\mathcal{V}_0M$ and by part (1) of Theorem 2.9 this is simply given by the projection of the covariant derivative to the subbundle $\mathcal{H}_1^{\mathcal{V}}M$. This subbundle corresponds to the maximal irreducible component in $\mathbb{R}^{n*} \otimes \mathbb{V}_0$. For example, if $\mathbb{V}_0 = \mathbb{R}^n \cong \mathbb{R}^{n*}$, then this is $S_0^2\mathbb{R}^{n*}$, while for $\mathbb{V}_0 = S_0^\ell\mathbb{R}^{n*}$ with $\ell \geq 2$, one obtains $S_0^{\ell+1}\mathbb{R}^{n*}$, so these lead to the conformal Killing operator on vector fields and on trace-free symmetric tensor fields. Likewise, putting $\mathbb{V}_0 = \Lambda^\ell\mathbb{R}^{n*}$, one obtains the intersection of the trace-free part with the kernel of the complete alternation in $\Lambda^\ell\mathbb{R}^{n*} \otimes \mathbb{R}^{n*}$. The corresponding operators are often referred to as conformal Killing-Yano operators on differential forms.

## 3 Projective BGG sequences

Apart from conformal geometry, there is a second parabolic geometry that underlies a Riemannian metric, namely a *projective structure*. Basically, this structure on $(M, g)$ is defined by the geodesics of the Levi-Civita connection $\nabla$, viewed as paths (unparametrized curves). Equivalently, it can be described via the class of all linear connections $\hat{\nabla}$ on $TM$, which have the same geodesics as $\nabla$ up to parametrization. Similarly to conformally invariant operators, this leads to the concept of projectively invariant differential operators, which have the same expression in terms of all connections in this projective equivalence class.

Examples of BGG sequences coming from projective differential geometry on domains in $\mathbb{R}^n$ are very important in applications. In particular, there is a projectively invariant version of the Riemannian deformation sequence which in this context is also known as the *Calabi complex* or the *fundamental complex of linear elasticity* and this is heavily used in applied mathematics. Concerning generalizations to the context of Riemannian geometry, there is a drawback of the projective BGG sequences, however. They are complexes only in a projectively flat setting, i.e. in the case that the projective equivalence class of connections contains a flat connection. By a classical result of Beltrami, this condition is very restrictive for Levi-Civita connections, it turns out to be equivalent to constant sectional curvature. Hence complexes are only obtained on space forms in this setting, but the sequences are also available and interesting in projectively non-flat cases.

To cover more cases in which complexes are obtained, we use a more general setting in this part of the article that we will describe next. Even in this more general setting, things are strictly parallel to the discussion in §2 and in particular, we use analogous notation to there throughout the discussion.

### 3.1 Background and algebraic setup for projective BGG sequences

Rather than starting from a Riemannian manifold $(M, g)$ we will start from a pair $(M, \nabla)$ where $M$ is an orientable manifold of fixed dimension $n$ and $\nabla$ is a torsion-free linear connection on the tangent bundle $TM$. In addition, we assume that $\nabla$ preserves a volume form and we fix such a form $\nu \in \Omega^n(M)$, so this is parallel for the induced connection on $\Lambda^n T^*M$ and it gives $M$ an orientation. Analogous to the orthonormal frame bundle discussed in §2.1, we then have a volume preserving frame bundle $\mathcal{SL}M$ with structure group $SL(n, \mathbb{R})$ and $\nabla$ is induced by a principal connection on that bundle. Hence representations of $SL(n, \mathbb{R})$ give rise to associated bundles and $SL(n, \mathbb{R})$-equivariant maps induce bundle maps that are parallel for the appropriate induced connection. As in §2.1 the standard representation $\mathbb{R}^n$ of $SL(n, \mathbb{R})$ induces $TM$ and the dual representation $\mathbb{R}^{n*}$ induces $T^*M$. Via tensorial constructions this leads to all types of tensor bundles (possibly with additional symmetry properties). The adjoint representation induces the bundle $\mathfrak{sl}(TM)$ of trace-free endomorphisms of $TM$, the curvature of $\nabla$ can be viewed as an element of $\Omega^2(M, \mathfrak{sl}(TM))$, and the analog of formula (2.3) holds for all induced connections. Note that outside of the Riemannian setting, one has to carefully distinguish between $TM$ and $T^*M$ and correspondingly between tensor fields of different types.

The algebraic setup for projective BGG sequences starts from the Lie group $G := SL(n+1, \mathbb{R})$ and we denote the standard basis of $\mathbb{R}^{n+1}$ as $e_0, e_1, \dots, e_n$. Then there is an obvious inclusion $SL(n, \mathbb{R}) \hookrightarrow G$ as those maps that fix $e_0$ and preserve the subspace spanned by $e_1, \dots, e_n$. The Lie algebra $\mathfrak{g}$ of $G$ consists of all trace-free matrices of size $(n+1) \times (n+1)$ and we write such matrices in a block form with blocks of size 1 and $n$ as

$$\begin{pmatrix} a & Z \\ X & A \end{pmatrix} \text{ with } a \in \mathbb{R}, X \in \mathbb{R}^n, Z \in \mathbb{R}^{n*} \text{ and } a + \operatorname{tr}(A) = 0. \tag{3.1}$$

Of course, the Lie subalgebra of $SL(n, \mathbb{R}) \subset G$ corresponds to the matrices with $a = X = Z = 0$ (and hence $\operatorname{tr}(A) = 0$). Exactly as in §2.2, the block form defines a $|1|$-grading of $\mathfrak{g}$, with components spanned by $X$, $(a, A)$ and $Z$, respectively. Also, the bracket defines an extension to $\mathfrak{g}_0$ of the standard representation of $\mathfrak{sl}(n, \mathbb{R})$ on $\mathbb{R}^n \cong \mathfrak{g}_{-1}$ and its dual representation on $\mathbb{R}^{n*} \cong \mathfrak{g}_1$. A complement to $\mathfrak{sl}(n, \mathbb{R})$ in $\mathfrak{g}_0$ is spanned by the element $E$ corresponding to $a = \frac{n}{n+1}$ and $A = \frac{-1}{n+1}\mathbb{I}$, which acts as a grading element. This also shows that

$\mathfrak{g}_0 \cong \mathfrak{gl}(n, \mathbb{R})$, so the corresponding associated bundle can be identified with $\mathfrak{gl}(TM) \cong T^*M \otimes TM$.

The block form and hence the $|1|$-grading of $\mathfrak{g}$ is determined by the decomposition $\mathbb{R}^{n+1} = \mathbb{R}^n \oplus \mathbb{R}$ with the summands spanned by $e_1, \dots, e_n$ and by $e_0$, respectively. Since any irreducible representation $\mathbb{V}$ of $G$ can be obtained from $\mathbb{R}^{n+1}$ via tensorial constructions, we as in §2.2 obtain a grading $\mathbb{V} = \oplus_{j=0}^N \mathbb{V}_j$, such that $\mathfrak{g}_i \cdot \mathbb{V}_j \subset \mathbb{V}_{i+j}$. As in §2.3, $\mathbb{V}$ gives rise to an associated bundle of $\mathcal{SL}M$, which we denote by $\mathcal{V}M = \oplus_{j=0}^N \mathcal{V}_j M$. We also obtain bundle maps $\bullet$ as there, for which the analogs of equations (2.5)–(2.9) hold. In the interpretations of the latter equations, one only has to be careful to use the right extensions of representations of $\mathfrak{sl}(n, \mathbb{R})$ to $\mathfrak{g}_0 \cong \mathfrak{gl}(n, \mathbb{R})$.

## 3.2 Examples

(1) For the standard representation $\mathbb{V} = \mathbb{R}^{n+1}$ of $G$ we know that $\mathbb{V} = \mathbb{V}_0 \oplus \mathbb{V}_1$ and as representations of $\mathfrak{sl}(n, \mathbb{R})$ this equals $\mathbb{R}^n \oplus \mathbb{R}$. Hence sections can be written as pairs $(\eta, f)$ with $\eta \in \mathfrak{X}(M)$ and $f \in C^\infty(M, \mathbb{R})$. The bundle maps $\bullet : TM \times \mathcal{V}M \to \mathcal{V}M$ and $\bullet : T^*M \times \mathcal{V}M \to \mathcal{V}M$ are characterized by $\xi \bullet (\eta, f) = (f\xi, 0)$ and $\alpha \bullet (\eta, f) = (0, \alpha(\eta))$. Here the analogs of (2.6) and (2.7) are satisfied trivially since both sides are 0. To interpret the analogs of equations (2.8) and (2.9) on all of $\mathfrak{g}_0$, one has to take into account that the grading element $E$ which corresponds to $-\operatorname{id}_{TM}$ acts via $(\eta, f) \mapsto (\frac{-1}{n+1}\eta, \frac{n}{n+1} f)$.

For the dual representation $\mathbb{V}^*$, one similarly obtains $\mathbb{V}^* = \mathbb{R} \oplus \mathbb{R}^{n*}$, so sections can be written as pairs $(f, \varphi)$ with $f \in C^\infty(M, \mathbb{R})$ and $\varphi \in \Omega^1(M)$. The definition of the dual action readily leads to $\xi \bullet (f, \varphi) = (-\varphi(\xi), 0)$ and $\alpha \bullet (f, \varphi) = (0, -f\alpha)$.

(2) Starting from these two basic examples, one can apply tensorial constructions as in §2.4 to pass to general irreducible representations of $G$. Let us discuss the example $\Lambda^2\mathbb{V}^* = \mathbb{R}^{n*} \oplus \Lambda^2\mathbb{R}^{n*}$ which is relevant for elasticity. Sections of the corresponding bundle can be written as $(\varphi, \psi)$ with $\varphi \in \Omega^1(M)$ and $\psi \in \Omega^2(M)$, and we can realize $(\varphi, 0)$ as $(0, \varphi) \wedge (1, 0)$ and $(0, \psi_1 \wedge \psi_2)$ as $(0, \psi_1) \wedge (0, \psi_2)$. Using this, one easily verifies that $\xi \bullet (\varphi, \psi) = (-\psi(\xi, \_), 0)$ and $\alpha \bullet (\varphi, \psi) = (0, \alpha \wedge \varphi)$.

Likewise, we can consider $S^2\mathbb{V}^* = \mathbb{R} \oplus \mathbb{R}^{n*} \oplus S^2\mathbb{R}^{n*}$, so sections can be written as triples $(f, \varphi, \Phi)$, where $f \in C^\infty(M, \mathbb{R})$, $\varphi \in \Omega^1(M)$ and $\Phi$ is a symmetric $\binom{0}{2}$-tensor field. Via the realizations $(f, 0, 0) = (f, 0) \odot (1, 0)$, $(0, \varphi, 0) = (0, \varphi) \odot (1, 0)$ and $(0, 0, \varphi_1 \odot \varphi_2) = (0, \varphi_1) \odot (0, \varphi_2)$ one easily computes that

$$\xi \bullet (f, \varphi, \Phi) = (-\varphi(\xi), -\Phi(\xi, \_), 0) \qquad \alpha \bullet (f, \varphi, \Phi) = (0, -2f\alpha, -\alpha \odot \varphi).$$

(3) Let us finally look at the adjoint representation $\mathbb{W} := \mathfrak{g}$ in a similar spirit as in the conformal case. We already know that $\mathcal{W}M = TM \oplus \mathfrak{gl}(TM) \oplus T^*M$

so we write sections as $(\zeta, \Phi, \varphi)$ with $\zeta \in \mathfrak{X}(M)$, $\varphi \in \Omega^1(M)$ and $\Phi$ a $\binom{1}{1}$-tensor field that we interpret as an endomorphism of $TM$. As in the conformal case, the main step towards understanding this is to determine the bilinear map $\{\ ,\ \} : \mathfrak{X}(M) \times \Omega^1(M) \to \mathfrak{gl}(TM)$ induced by the bracket $\mathfrak{g}_{-1} \times \mathfrak{g}_1 \to \mathfrak{g}_0$. To determine this, we again have to compute $[[X, Z], Y] \in \mathfrak{g}_{-1}$ for $X, Y \in \mathfrak{g}_{-1}$ and $Z \in \mathfrak{g}_1$, which easily implies that

$$\{\eta, \varphi\}(\xi) = \varphi(\xi)\eta - \varphi(\eta)\xi. \tag{3.2}$$

In terms of this operation, we then get

$$\eta \bullet (\zeta, \Phi, \varphi) = (-\Phi(\eta), \{\eta, \varphi\}, 0) \qquad \alpha \bullet (\zeta, \Phi, \varphi) = (0, -\{\zeta, \alpha\}, \alpha \circ \Phi).$$

### 3.3 The twisted de Rham sequence

Using the operations we have just introduced, the discussion of curvature in the projective setting looks formally almost identical to the discussion in the conformal case. The definition of curvature is exactly as in (2.2) and the definition of Ricci-curvature just needs an equivalent reformulation of (2.15). On takes a local frame $\{\xi_i\}$ for $TM$ and the dual coframe $\sigma_i$ for $T^*M$ (i.e. $\sigma_i(\xi_j) = \delta_{ij}$ and defines

$$\mathrm{Ric}(\eta, \zeta) = \textstyle\sum_i \sigma_i(R(\xi_i, \eta)(\zeta)). \tag{3.3}$$

This is equivalent to (2.15) since a local frame $\{\xi_i\}$ it orthonormal for $g$ if and only if the dual coframe is $\{\xi_i^\flat\}$. For general linear connections on $TM$, the Ricci-curvature is not symmetric. Symmetry of Ric is equivalent to the fact that $\nabla$ preserves a volume form and hence is satisfied in the cases we consider. The general definition of the projective Schouten tensor mixes the symmetric and the skew symmetric part of Ric with different factors, see Section 3.1 of [6]. In the volume preserving case, this boils down to $\mathsf{P} = \frac{1}{n-1}\mathrm{Ric}$. The projective version of the Cotton-York tensor $Y$ is then again defined by formula (2.17), so $Y = d^\nabla \mathsf{P} \in \Omega^2(M, T^*M)$.

The choice of the definition of $\mathsf{P}$ is motivated by the analog of formula (2.16) using the operations associated to $\mathfrak{sl}(n+1, \mathbb{R})$, i.e.

$$R(\xi, \eta)(\zeta) = W(\xi, \eta)(\zeta) + \{\xi, \mathsf{P}(\eta)\}(\zeta) - \{\eta, \mathsf{P}(\xi)\}(\zeta).$$

Using this to define the *projective Weyl curvature* $W \in \Omega^2(M, \mathrm{End}(TM))$ one verifies that $W$ has the same symmetries as $R$ and in addition any possible contraction of $W$ vanishes (so in particular it has values in $\mathfrak{sl}(TM)$). Thus we obtain the decomposition of $R$ into a trace-free part and a trace part in the same form as in the conformal case. Also the relation of the curvature quantities to projective flatness is parallel to the conformal case (with a shift in dimension): For $n \geq 3$, $\nabla$ is projectively flat if and only if $W$ vanishes identically and this implies $Y \equiv 0$. If $n = 2$, $W$ always vanishes identically and $\nabla$ is projectively flat if and only if $Y$ vanishes identically. Having all this at hand, we can proceed formally in exactly the same way as in the conformal case.

**Definition 3.1.** Consider an irreducible representation $\mathbb{V}$ of $SL(n+1,\mathbb{R})$ as in §3.1. Then using the operations $\bullet$ from there, we define the *twisted connection* $\nabla^{\mathcal{V}}$ on $\mathcal{V}M$ by

$$\nabla^{\mathcal{V}}_{\xi} s := \nabla_{\xi} s + \xi \bullet s - \mathsf{P}(\xi) \bullet s. \tag{3.4}$$

The proof of Theorem 2.2 only uses the formal properties of the operations and hence also implies the following.

**Theorem 3.2.** For $\xi, \eta \in \mathfrak{X}(M)$ and $s \in \Gamma(\mathcal{V}M)$, the curvature $R^{\mathcal{V}}$ of $\nabla^{\mathcal{V}}$ is given by

$$R^{\mathcal{V}}(\xi,\eta)(s) = W(\xi,\eta) \bullet s + Y(\xi,\eta) \bullet s,$$

where $W$ and $Y$ are the (projective) Weyl curvature and the Cotton–York tensor of $\nabla$, respectively. In particular, the connection $\nabla^{\mathcal{V}}$ is flat if and only if $\nabla$ is projectively flat.

As before, we get the covariant exterior derivatives $d^{\nabla}$ and $d^{\nabla^{\mathcal{V}}}$ that act on $\Omega^*(M,\mathcal{V}M)$, which both raise the form-degree by one. The relation between the two operations is formally exactly as in Proposition 2.3 but using the operations associated to $\mathfrak{sl}(n+1,\mathbb{R})$. Hence we again get a twisted de Rham sequence associated to each irreducible representation $\mathbb{V}$ of $SL(n+1,\mathbb{R})$ and this is a complex if and only if $\nabla$ is projectively flat.

## 3.4 Cohomology bundles and BGG construction

The definition of the cohomology bundles $\mathcal{H}^{\mathcal{V}}_k M$ as a quotient works exactly as in the conformal case. The operators $\partial$ on $\mathcal{V}M$-valued forms are induced by bundle maps $\partial : \Lambda^k T^*M \otimes \mathcal{V}M \to \Lambda^{k+1}T^*M \otimes \mathcal{V}M$ such that $\partial \circ \partial = 0$. As in §2.7, we obtain natural subbundles $\operatorname{im}(\partial) \subset \ker(\partial) \subset \Lambda^k T^*M \otimes \mathcal{V}M$ and we define $\mathcal{H}^{\mathcal{V}}_k M$ as the quotient $\ker(\partial)/\operatorname{im}(\partial)$. By construction, this bundle is induced by the representation $H^k(\mathfrak{g}_{-1},\mathbb{V})$ of $SL(n,\mathbb{R})$.

Both the elementary arguments and the deeper use of representation theory discussed in §2.7 have analogs in the projective case. In particular, Kostant's theorem implies that for $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{R})$ each of the cohomology spaces $H^k(\mathfrak{g}_{-1},\mathbb{V})$ is an irreducible representation of $\mathfrak{g}_0$ and, as before, $H^0(\mathfrak{g}_{-1},\mathbb{V}) \cong \mathbb{V}_0$.

There also is an interpretation via subbundles of $\Lambda^k T^*M \otimes \mathcal{V}M$ and for Levi-Civita connections they can be described exactly as in §2.7 via inner products. For more general connections, there also is a distinguished natural subbundle

$\Upsilon_k \subset \ker(\partial) \subset \Lambda^k T^*M \otimes \mathcal{V}M$ which projects isomorphically onto $\mathcal{H}_k^{\mathcal{V}}M$, but one has to use an alternative description. In fact, one can directly define $\mathfrak{g}_0$-equivariant maps $\partial^* : \Lambda^k T^*M \otimes \mathcal{V}M \to \Lambda^{k-1}T^*M \otimes \mathcal{V}M$ for each $k$, the so-called *Kostant codifferential.* As the notation suggests, they are adjoint to $\partial$ with respect to an inner product of Lie theoretic origin, so they satisfy $\partial^* \circ \partial^* = 0$, and one puts $\Upsilon_k := \ker(\partial) \cap \ker(\partial^*)$. The explicit formula for $\partial^*$ is not needed for our purposes, it can be found (in a more general setting) in [27] or section 3.3 of [17]. In simple situations, it is easy to directly find the representations $\mathrm{im}(\partial^*) \subset \ker(\partial^*)$ in each degree from elementary representation theory arguments, see §3.5 below for examples.

The adjointness between $\partial$ and $\partial^*$ implies that $\partial$ restricts to a linear isomorphism $\mathrm{im}(\partial^*) \to \mathrm{im}(\partial)$ and that $\mathrm{im}(\partial)$ is complementary to $\ker(\partial^*)$. Now one defines bundle maps $T : \Lambda^k T^*M \otimes \mathcal{V}M \to \Lambda^{k-1}T^*M \otimes \mathcal{V}M$ as the inverse of $\partial$ on $\mathrm{im}(\partial)$ and as zero on $\ker(\partial^*)$. Doing this, it is clear that the equalities in (2.24) also hold here and we also get $\Upsilon_k = \ker(\partial) \cap \ker(T)$. The first two equalities in (2.23) have an analog here, namely that $\Lambda^k T^*M \otimes \mathcal{V}M$ can be written as $\ker(T) \oplus \mathrm{im}(\partial)$ or as $\mathrm{im}(T) \oplus \ker(\partial)$.

At this point, the rest of the BGG construction can be carried out in the current setting without changes. The operators $G$ and $S$ can be defined by exactly the same formula as in §2.8 and Proposition 2.6 holds. This allows us to define the BGG operators as in the conformal case. In the projectively flat case, the relation between the twisted de Rham sequence and the BGG sequences is exactly as in Theorem 2.7. Without assuming projective flatness, we get the obvious analog of Theorem 2.8.

## 3.5 General results and examples

With rather obvious changes, the general results from §2.12 extend to the projective setting. One has a bijection between irreducible representations $\mathbb{V}$ of $SL(n+1,\mathbb{R})$ and pairs $(\mathbb{V}_0, r)$ of an irreducible representation of $SL(n,\mathbb{R})$ and and integer $r \geq 1$, which equals the order of the first BGG operator in the sequence determined by $\mathbb{V}$. The bundle $\mathcal{H}_1^{\mathcal{V}}M$ now corresponds to the maximal irreducible component in $S^r T^*M \otimes \mathcal{V}_0 M$.

With the projective versions of all operations involved, the explicit formula for BGG operators of order 1 and 2 from Theorem 2.9 hold in the projective setting without changes. Parallel to the discussion in §2.12, this shows that one obtains the Killing operator on 1-forms and, more generally, on $\binom{0}{\ell}$-tensor fields and the Killing-Yano operators on differential forms as first BGG operators in this setting.

To discuss some explicit examples, consider the standard representation $\mathbb{V} := \mathbb{R}^{n+1}$ of $SL(n+1,\mathbb{R})$. This decomposes as $\mathbb{V}_0 \oplus \mathbb{V}_1 = \mathbb{R}^n \oplus \mathbb{R}$. Here we get

$\partial : \Lambda^k\mathbb{R}^{n*} \to \Lambda^{k+1}\mathbb{R}^{n*} \otimes \mathbb{R}^n$ for $k = 0, \dots, n-1$ which is a non-zero multiple of the inclusion of the trace-part, so it is injective for all $k$ and bijective for $k = n-1$. This means that $\Upsilon_k$ is the kernel of the contraction in $\Lambda^k T^*M \otimes TM$ while $\Upsilon_n = \Lambda^n T^*M$ (and $\Upsilon_0 = \mathcal{V}_0 M = TM$). Consequently, only the last BGG operator is of second order here, while all other BGG operators have order one.

Sections of $\Upsilon_k$ for $k = 1, \dots, n-1$ can be interpreted as $\binom{1}{k}$-tensor fields $\psi$, which are completely alternating in the lower indices and lie in the kernel of the unique (up to sign) contraction available in this situation. Since there are just two components in the decomposition of $\mathbb{V}$, the splitting operators can be read off the proof of Theorem 2.9. Up to a non-zero constant, $T : \Lambda^\ell T^*M \otimes TM \to \Lambda^{\ell-1}T^*M$ is the unique contraction. Since $T(\psi) = 0$ we conclude that $T(d^\nabla\psi)$ coincides (up to that factor) with unique nonzero contraction of $\nabla\psi$, which is the natural analog of the divergence in this situation. This has to be put into the other component to obtain $S(\psi)$.

In particular, one gets the usual divergences on vector fields for $k = 0$ and the first BGG operator maps $\eta \in \mathfrak{X}(M)$ to the trace-free part of the $\binom{1}{1}$-tensor field $\nabla\eta$. Explicitly, $D(\eta) = \nabla\eta - \frac{1}{n}\operatorname{div}(\eta)\operatorname{id}$, where $\operatorname{div}(\eta)$ denotes the divergence of $\eta$. There is a similar formula for the next BGG operators, which is easy to derive. For the last BGG operator, the target is one-dimensional and on the relevant space both $T$ and $\partial$ are zero, so the universal formula in part (2) of Theorem 2.9 simplifies to a linear combination of $d\operatorname{div}(\psi)$ and $\partial^{\mathsf{P}}(\psi)$ for $\psi \in \Gamma(\Upsilon_{n-1}) \subset \Omega^{n-1}(M, TM)$. Observe that $\partial^{\mathsf{P}}(\psi)$ up to a multiple is just the alternation of $(\xi_1, \dots, \xi_n) \mapsto \mathsf{P}(\xi_1)(\psi(\xi_2, \dots, \xi_n))$.

For the dual $\mathbb{W} = \mathbb{V}^*$ of the standard representation, we get $\mathbb{W} = \mathbb{W}_0 \oplus \mathbb{W}_1 = \mathbb{R} \oplus \mathbb{R}^{n*}$. The maps $\partial : \Lambda^k\mathbb{R}^{n*} \otimes \mathbb{R}^{n*} \to \Lambda^{k+1}\mathbb{R}^{n*}$ are non-zero multiples of the alternation, and hence are bijective for $k = 0$ and surjective for all $k = 0, \dots, n-1$. Hence sections of $\Upsilon_0$ are just smooth functions, $\Upsilon_k \subset \Lambda^k\mathbb{R}^{n*} \otimes \mathbb{R}^{n*}$ is the kernel of the complete alternation for $k = 1, \dots, n-1$ and $\Upsilon_n \cong \mathbb{R}^{n*}$. Hence the first BGG operator is of second order, while all other BGG operators are of first order. In degree zero, one gets $S(f) = (f, df)$ while in higher degrees the splitting operators are just inclusions. The first BGG operator is given by $D(f) = \nabla^2 f + \mathsf{P}f$, and this is related to Ricci-flat connections that are projectively equivalent to $\nabla$, see [6]. For $k \geq 1$, sections of $\Upsilon_k$ can be viewed as $\binom{0}{k+1}$-tensor fields that are alternating in the first $k$ entries, but lie in the kernel of the complete alternation. For such a tensor field $\psi$, $D\psi$ then is obtained by forming $d^\nabla\psi$, i.e. alternating $\nabla\psi$ in the first $k+1$ entries and then projecting to the kernel of the complete alternation.

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