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Horocycle trajectories on a Hadamard Kähler manifold

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Abstract. In this paper we first give an estimate on angles between initial vectors of geodesics and trajectories of trajectory-harps, which are variations of geodesics associated with trajectories for Kähler magnetic fields. As an application we investigate unbounded trajectories on a Hadamard Kähler manifolds having single limit points.

Keywords: Kähler magnetic fields, Hadamard manifolds, ideal boundary, horocycle, trajectory-harps, string-elevation

MSC 2020 classification: primary 53C22, secondary 53B35

Introduction

In order to study Kähler manifolds from the Riemannian geometric point of view, the second author investigates trajectories for Kähler magnetic fields, which are constant multiples of Kähler forms. For magnetic fields on a Riemannian manifold, which are closed 2-forms, see [11], for example. He considers that they play a similar role as of geodesics on general Riemannian manifolds. At each point of a trajectory for a Kähler magnetic field, its tangent and acceleration vectors span a complex line, hence it is expected that studying trajectories gives more information than studying only geodesics on Kähler manifolds. Corresponding to geodesic triangles considered in Toponogov's comparison theorem, the second author studied variations of geodesics associated with trajectories which are called trajectory-harps. In [2, 4], he gave estimates on lengths of geodesics segments of trajectory harps and on angles between geodesics and

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trajectories by comparing them with those of trajectory-harps on complex space forms.

In this paper, we first add one more estimate on quantities of trajectory-harps. We study angles between initial vectors of trajectories and those of geodesics. Though it seems that these angles are closely related with angles between geodesics and trajectories at non-initial points. But as trajectory-harps are variations of geodesics whose initial points stick to the initial point of trajectories, the second author could not give estimates in previous papers, and considered "zenith angles" in [3] instead. As an application of our new estimate, we study asymptotic behaviors of unbounded trajectories on a Hadamard Kähler manifold, a simply connected complete Kähler manifold of nonpositive curvature. When absolute values of sectional curvatures of planes tangent to a trajectory-harp are not less than the square of geodesic curvature of the trajectory half-line, it has limit point in the ideal boundary. In this paper, we pay attention to trajectories which are unbounded in both directions and have single limit points in the ideal boundary. Under an assumption on sectional curvatures on trajectory-harps associated with such trajectories, we show that they are parts of totally geodesic complex lines. Concerning our result, in [1], we studied such trajectories on symmetric spaces of non-compact type, and showed that the range of geodesic curvatures of such trajectories indicate the ranks of underlying symmetric spaces. Our result is a partial extension to trajectories on general Hadamard Kähler manifolds.

1 Tarjectory-harps

Let $(M, \langle\ ,\ \rangle, J)$ be a complete Kähler manifold with complex structure J and Riemannian metric $\langle\ ,\ \rangle$. We take a Kähler magnetic field $\mathbb{B}_\kappa = \kappa\mathbb{B}_J$ ($\kappa \in \mathbb{R}$), which is a constant multiple of the Kähler form $\mathbb{B}_J$ on M. We say that a smooth curve γ parameterized by its arclength is a *trajectory* for $\mathbb{B}_\kappa$ if it satisfies the differential equation $\nabla_{\dot\gamma}\dot\gamma = \kappa J\dot\gamma$. Since M is complete, it is defined on a whole real line $\mathbb{R}$. We suppose that a trajectory $\gamma : [0,T] \to M$ satisfies $\gamma(t) \neq \gamma(0)$ for $0 < t \leq T$. Precisely speaking, as we consider a part of a trajectory, we need to say that γ is a trajectory segment or trajectory half-line according as T is finite or T is infinite. But for the sake of simplicity, we usually call it a trajectory. We say that a smooth variation $\alpha_\gamma : [0,T] \times \mathbb{R} \to M$ of geodesics is a *trajectory-harp* associated with γ if it satisfies the following conditions:

i) $\alpha_\gamma(t, 0) = \gamma(0)$ for every $t \in [0,T]$,

ii) the curve $s \mapsto \alpha_\gamma(0, s)$ is the geodesic of initial vector $\dot\gamma(0)$,

iii) when $t > 0$, the curve $s \mapsto \alpha_\gamma(t,s)$ is the geodesic of unit speed joining $\gamma(0)$ and $\gamma(t)$.

We can define a trajectory-harp if the image $\gamma([0,T))$ of the trajectory lies in the ball centered at $\gamma(0)$ whose radius is the minimum value of conjugate values at $\gamma(0)$. We give some terminologies on trajectory-harps. We call the geodesic segment joining $\gamma(0)$ and $\gamma(t)$ in a trajectory-harp the *string* at $\gamma(t)$, and call γ the *arch* of this trajectory-harp. We denote by $\ell_\gamma(t)$ the length of the string at $\gamma(t)$, and call it the *string-length* at $\gamma(t)$. We set $\delta_\gamma(t) = \langle \frac{\partial \alpha_\gamma}{\partial s}\big(t, \ell_\gamma(t)\big), \dot\gamma(t)\rangle$ and call it the *string-cosine* at $\gamma(t)$. It is known that the derivative of the string-length coincides with the string-cosine, i.e. $\frac{d}{dt}\ell_\gamma = \delta_\gamma$ (see [2]). Given t_0 with $0 < t_0 \leq T$, we put $\mathcal{HP}_\gamma(t_0) = \big\{\alpha_\gamma(t,s) \bigm| 0 \leq t \leq t_0,\ 0 \leq s \leq \ell_\gamma(t)\big\}$, and call it the harp-body at t_0.

On a complex space form $\mathbb{C}M^n(c)$ of constant holomorphic sectional curvature c, which is a complex projective space $\mathbb{C}P^n(c)$, a complex Euclidean space $\mathbb{C}^n$ and a complex hyperbolic space $\mathbb{C}H^n(c)$ acording as c is positive, zero and negative, for each trajectory $\gamma : [0, 2\pi/\sqrt{\kappa^2+c}\,) \to \mathbb{C}M^n(c)$ for $\mathbb{B}_\kappa$, we can define its trajectory-harp. Here, we treat $2\pi/\sqrt{\kappa^2+c}$ as infinity when $\kappa^2 + c \leq 0$. We use such a convention through out of this paper. If we take two trajectories γ_1, γ_2 for $\mathbb{B}_\kappa$ on $\mathbb{C}M^n(c)$, then their trajectory-harps are congruent to each other, that is, there is an isometry φ of $\mathbb{C}M^n(c)$ satisfying $\alpha_{\gamma_2}(t,s) = \varphi\circ\alpha_{\gamma_1}(t,s)$. Therefore, string-lengths and string-cosines for trajectory-harps associated with trajectories for $\mathbb{B}_\kappa$ on $\mathbb{C}M^n(c)$ do not depend on trajectories. We hence denote them as $\ell_\kappa(t;c)$ and $\delta_\kappa(t;c)$. The string-length $\ell_\kappa(t;c)$ is given by the following relation (see [2]): When $c > 0$, it satisfies

$$\sqrt{\kappa^2+c}\ \sin\big(\sqrt{c}\,\ell_\kappa(t;c)/2\big) = \sqrt{c}\,\sin\big(\sqrt{\kappa^2+c}\ t/2\big),$$

when $c = 0$, it satisfies

$$|\kappa|\,\ell_\kappa(t;0) = 2\sin\big(|\kappa|t/2\big),$$

and when $c < 0$, it satisfies

$$\begin{aligned}
&\sinh\big(\sqrt{|c|}\,\ell_\kappa(t;c)/2\big)\\
&\quad = \begin{cases}
\sqrt{|c|/(|c|-\kappa^2)}\ \sinh\big(\sqrt{|c|-\kappa^2}\ t/2\big), & \text{when } |\kappa| < \sqrt{|c|},\\
\sqrt{|c|}\ t/2, & \text{when } |\kappa| = \sqrt{|c|},\\
\sqrt{|c|/(\kappa^2+c)}\ \sin\big(\sqrt{\kappa^2+c}\ t/2\big), & \text{when } |\kappa| > \sqrt{|c|}.
\end{cases}
\end{aligned}$$

The string-cosine $\delta_\kappa(t;c)$ is given by

$$\delta_\kappa(t;c)=\begin{cases}\dfrac{\sqrt{|c|-\kappa^2}\,\cosh\bigl(\sqrt{|c|-\kappa^2}\,t/2\bigr)}{\sqrt{|c|\cosh^2\bigl(\sqrt{|c|-\kappa^2}\,t/2\bigr)-\kappa^2}}, & \text{when } \kappa^2+c<0,\\[2ex] \dfrac{2}{\sqrt{|c|t^2+4}}, & \text{when } \kappa^2+c=0,\\[2ex] \dfrac{\sqrt{\kappa^2+c}\,\cos\bigl(\sqrt{\kappa^2+c}\,t/2\bigr)}{\sqrt{\kappa^2+c\cos^2\bigl(\sqrt{\kappa^2+c}\,t/2\bigr)}}, & \text{when } \kappa^2+c>0.\end{cases}$$

Since we have $\frac{d}{dt}\ell_\kappa(t;c)=\delta_\kappa(t;c)$, we see that $\ell_\kappa(\cdot;c)$ is monotone increasing in the interval $[0,\pi/\sqrt{\kappa^2+c}\,]$. Clearly, we have $\ell_{-\kappa}(t;c)=\ell_\kappa(t;c)$ and $\delta_{-\kappa}(t;c)=\delta_\kappa(t;c)$. When $\kappa_1>\kappa_2>0$, we have $\ell_{\kappa_1}(t;c)<\ell_{\kappa_2}(t;c)$ for $t\in\bigl(0,\pi/\sqrt{\kappa_1^2+c}\,\bigr]$.

For trajectory-harps on a general Kähler manifold M, we can compare string-lengths and string-cosines and those on complex space forms. For a trajectory-harp α_γ associated with a trajectory $\gamma:[0,T]\to M$ for $\mathbb{B}_\kappa$ and a constant c, we define $T_\gamma(c)$ so that $T_\gamma(c)=\min\{t_*\}$ if there is t_* satisfying $0<t_*\le T$ and $\ell_\gamma(t_*)=\ell_\kappa\bigl(\pi/\sqrt{\kappa^2+c}\,;c\bigr)$, and set $T_\gamma(c)=T$ in other case. We denote by $\tau_\kappa(\cdot;c):\bigl[0,\ell_\kappa(\pi/\sqrt{\kappa^2+c}\,;c)\bigr]\to\mathbb{R}$ the inverse function of the function $\ell_\kappa(\cdot;c):\bigl[0,\pi/\sqrt{\kappa^2+c}\,\bigr]\to\mathbb{R}$. When $\kappa_1>\kappa_2>0$, we have $\delta_{\kappa_1}\bigl(\tau_{\kappa_1}(s;c);c\bigr)<\delta_{\kappa_2}\bigl(\tau_{\kappa_2}(s;c);c\bigr)$ for every $s\in\bigl(0,\ell_{\kappa_1}\bigl(\pi/\sqrt{\kappa_1^2+c}\,;c\bigr)\bigr]$.

Proposition 1 ([2, 4]). *Let $\gamma:[0,T]\to M$ be a trajectory for a non-trivial Kähler magnetic field $\mathbb{B}_\kappa$ on a complete Kähler manifold M. We suppose that sectional curvatures of planes tangent to the harp-body $\mathcal{HP}_\gamma(T)$ of the associated trajectory-harp α_γ are not greater than a constant c. We then have the following*:

(1) $\ell_\gamma(t)\ge\ell_\kappa(t;c)$ *for* $0\le t\le\min\{T,\pi/\sqrt{\kappa^2+c}\,\}$;

(2) $\delta_\gamma(t)\ge\delta_\kappa\bigl(\tau_\kappa(\ell_\gamma(t);c);c\bigr)$ *for* $0\le t\le T_\gamma(c)$;

(3) *If $\ell_\gamma(t_0)=\ell_\kappa(t_0;c)$ at some t_0 with $0<t_0\le\min\{T,\pi/\sqrt{\kappa^2+c}\,\}$, then the harp-body $\mathcal{HP}_\gamma(t_0)$ is totally geodesic, totally complex and of constant sectional curvature c.*

2 Horocycle trajectories

Let M be a Hadamard Kähler manifold, which is a complete simply connected Kähler manifold of nonpositive sectional curvature. Since M is a Hadamard manifold, we can define its ideal boundary ∂M as the set of all asymptotic classes of geodesic rays. With the cone topology on $\overline{M} = M \cup \partial M$, this $\overline{M}$ is a compactification of M (see [7]). There are many interesting results joining geometries of ideal boundaries and of Hadamard manifolds (see [5, 8], for example). We say that a smooth curve γ parameterized by its arclength is unbounded in both directions if both of the sets $\gamma([0,\infty))$ and $\gamma((-\infty,0])$ are unbounded in M. We set

$$\gamma(\infty) = \lim_{t\to\infty} \gamma(t), \;\; \gamma(-\infty) = \lim_{t\to-\infty} \gamma(t) \in \partial M$$

if they exist. We call them the *limit points* of γ. Such a curve γ is said to be a *horocycle* if its limit points coincide (i.e. $\gamma(\infty) = \gamma(-\infty)$) and if it satisfies $\langle \dot\gamma(t), \dot\sigma(0)\rangle = 0$ for each geodesic σ satisfying $\sigma(0) = \gamma(t)$ and $\sigma(\infty) = \gamma(\infty)$. In particular, if we can take σ so that its initial vector is parallel to $J\dot\gamma(t)$ for each t, we call it a holomorphic horocycle. On $\mathbb{C}H^n(c)$, we know that

(1) a trajectory for $\mathbb{B}_\kappa$ is unbounded in both directions if and only if $|\kappa| \le \sqrt{|c|}$;

(2) when $|\kappa| < \sqrt{|c|}$, it has two distinct points at infinity;

(3) when $|\kappa| = \sqrt{|c|}$, it is a holomorphic horocycle.

We take a trajectory half-line $\gamma : [0,\infty) \to M$ for $\mathbb{B}_\kappa$. We suppose that we can define a trajectory-harp associated with γ and that sectional curvatures of planes tangent to its harp-body $\mathcal{HP}_\gamma(\infty)$ are not greater than $-\kappa^2$. We here make mention on this assumption a bit. If we suppose that we can define a trajectory-harp associated with a trajectory-segment $\gamma|_{[0,T]}$ and sectional curvatures of planes tangent to $\mathcal{HP}_\gamma(T)$ are not greater than $-\kappa^2$, then we have $\lim_{t\uparrow T} \delta_\gamma(t)$ is strictly positive by Proposition 1. In particular, we can define a trajectory harp associated with $\gamma|_{[0,T+\ell_\gamma(T)]}$ by joining $\gamma(0)$ and $\gamma(t)$ by a unique geodesic of unit speed, because we have $\gamma(t) \ne \gamma(0)$ for every t with $0 < t \le T + \ell_\gamma(T)$. Thus, we can calculate sectional curvatures of planes tangent to its harp-body. Therefore, our assumption is essentially made on sectional curvatures. Hence, for the sake of simplicity, we just say that sectional curvatures of planes tangent to its harp-body are not greater than $-\kappa^2$. Under this assumption, by Proposition 1, we find that the trajectory half-line γ is unbounded. Moreover, by studying initial vectors $\left\{\frac{\partial\alpha_\gamma}{\partial s}(t,0) \;\middle|\; t \ge 0\right\}$ of strings, we find that γ has its limit points $\gamma(\infty)$ in the ideal boundary ∂M, and that the initial vector of the geodesic joining $\gamma(0)$ and $\gamma(\infty)$ is $\lim_{t\to\infty} \frac{\partial\alpha_\gamma}{\partial s}(t,0)$ (see [3, 9]). We call this geodesic the *limit string* of this trajectory-harp.

We now study horocycle trajectories. For a smooth curve γ, we denote its reversed curve $t \mapsto \gamma(-t)$ by γ^{-1}. When γ is a trajectory for $\mathbb{B}_\kappa$, then its reversed curve γ^{-1} is a trajectory for $\mathbb{B}_{-\kappa}$. When we can define trajectory-harps associated with $\gamma|_{[0,\infty)}$ and with $\gamma^{-1}|_{[0,\infty)}$, we put $\mathcal{HP}_\gamma = \mathcal{HP}_{\gamma|_{[0,\infty)}}(\infty) \cup \mathcal{HP}_{\gamma^{-1}|_{[0,\infty)}}(\infty)$ and call it the total harp-body. If sectional curvatures of planes tangent to the total harp-body $\mathcal{HP}_\gamma$ are not greater than $-\kappa^2$, then it has limit points $\gamma(\infty)$ and $\gamma(-\infty) = \gamma^{-1}(\infty)$. We here study the case that they coincide with each other.

Theorem 1. *Let γ be a trajectory for $\mathbb{B}_\kappa$ on a Hadamard Kähler manifold M. Suppose that sectional curvatures of planes tangent to the total harp-body $\mathcal{HP}_\gamma$ are not greater than $-\kappa^2$. If the limit points of γ coincide with each other, then we have the following*:

(1) *γ is a holomorphic horocycle*;

(2) *the total harp-body $\mathcal{HP}_\gamma$ is totally geodesic holomorphic and of constant sectional curvature $-\kappa^2$.*

For a unit tangent vector $v \in U_pM$ at a point $p \in M$, we set $S_v = \{e^{\sqrt{-1}\theta}v \mid \theta \in \mathbb{R}\}$. We say that v satisfies the κ-HC-condition if for every $w \in S_v$ the trajectory γ_w for $\mathbb{B}_\kappa$ whose initial vector $\dot\gamma_w(0)$ is w satisfies $\gamma_w(\infty) = \gamma_w(-\infty)$ and sectional curvatures of planes tangent to its total harp-body $\mathcal{H}_{\gamma_w}$ are not greater than $-\kappa^2$.

Corollary 1. *Let M be a Hadamard Kähler manifold. If there exist a positive κ and a unit tangent vector which satisfies the κ-HC-condition, then M contains a totally geodesic, totally complex $\mathbb{C}H^1(-\kappa^2)$.*

3 String-elevations of trajectory-harps

In this section, we study angles between initial vectors of strings and arches of trajectory-harps. Let $\alpha_\gamma : [0,T] \times \mathbb{R} \to M$ be a trajectory-harp associated with a trajectory $\gamma : [0,T] \to M$ on a Kähler manifold M. We set $\eta_\gamma(t) = \big\langle \frac{\partial \alpha_\gamma}{\partial s}(t,0), \dot\gamma(0)\big\rangle$, and call it the *string-elevation* of this trajectory-harp. For each t_0 $(0 < t_0 \le T)$, if we take the curve $\sigma(t) = \gamma(t_0 - t)$, it is a trajectory for $\mathbb{B}_{-\kappa}$ and the curve $s \mapsto \alpha_\gamma\big(t_0, \ell_\gamma(t_0) - s\big)$ is a string of the corresponding trajectory-harp α_σ. We hence have $\eta_\gamma(t_0) = \delta_\sigma(t_0)$. But we should note that their harp-bodies $\mathcal{HP}_\gamma(t_0)$ and $\mathcal{HP}_\sigma(t_0)$ may not coincide with each other. Therefore, we prepare an estimate on string-elevations in this section.

Lemma 1. *The string-elevation η_γ for a trajectory γ for $\mathbb{B}_\kappa$ satisfies $\eta_\gamma(0) = 1$, $\lim_{t\downarrow 0} \eta'_\gamma(t) = 0$, $\lim_{t\downarrow 0} \eta''_\gamma(t) = -\kappa^2/4$.*

Proof. We note that we can extend the trajectory-harp smoothly as $\alpha_\gamma : (-\epsilon, T] \times \mathbb{R} \to M$ with some positive ϵ. We hence extend the string-elevation as a function $\eta_\gamma : (-\epsilon, T] \to \mathbb{R}$. The first equality is trivial because $\frac{\partial\alpha_\gamma}{\partial s}(0,0) = \dot\gamma(0)$ by definition. Since α_γ is a variation of geodesics of unit speed, the Jacobi field $s \mapsto \frac{\partial\alpha_\gamma}{\partial t}(t,s)$ along the geodesic $s \mapsto \alpha_\gamma(t,s)$ satisfies $\big\langle\big(\nabla_{\frac{\partial\alpha_\gamma}{\partial s}}\frac{\partial\alpha_\gamma}{\partial t}\big)(t,s), \frac{\partial\alpha_\gamma}{\partial s}(t,s)\big\rangle = 0$. In particular, we have $\eta'_\gamma(0) = 0$.

We shall check that $\eta''_\gamma(0) = \big\langle\big(\nabla_{\frac{\partial\alpha_\gamma}{\partial t}}\nabla_{\frac{\partial\alpha_\gamma}{\partial t}}\frac{\partial\alpha_\gamma}{\partial s}\big)(0,0), \frac{\partial\alpha_\gamma}{\partial s}(0,0)\big\rangle$ coincides with $-\kappa^2/4$. As we have $\gamma(t) = \alpha_\gamma\big(t, \ell_\gamma(t)\big)$ and $\ell'_\gamma(t) = \delta_\gamma(t)$, we get

$$\begin{aligned}\dot\gamma(t) &= \frac{\partial\alpha_\gamma}{\partial t}\big(t,\ell_\gamma(t)\big) + \delta_\gamma(t)\frac{\partial\alpha_\gamma}{\partial s}\big(t,\ell_\gamma(t)\big),\\ \nabla_{\dot\gamma}\dot\gamma(t) &= \Big(\nabla_{\frac{\partial\alpha_\gamma}{\partial t}}\frac{\partial\alpha_\gamma}{\partial t}\big(t,\ell_\gamma(t)\big)\Big) + 2\delta_\gamma(t)\Big(\nabla_{\frac{\partial\alpha_\gamma}{\partial t}}\frac{\partial\alpha_\gamma}{\partial s}\Big)\big(t,\ell_\gamma(t)\big)\\ &\qquad + \delta'_\gamma(t)\frac{\partial\alpha_\gamma}{\partial s}\big(t,\ell_\gamma(t)\big),\end{aligned}$$

because $s \mapsto \alpha(t,s)$ is a geodesic for each t. Since $\alpha_\gamma(t,0) = \gamma(0)$ we have $\big(\nabla_{\frac{\partial\alpha_\gamma}{\partial t}}\frac{\partial\alpha_\gamma}{\partial t}\big)(0,0) = 0$. Thus, by using $\nabla_{\dot\gamma}\dot\gamma = \kappa J\dot\gamma$ and $\delta_\gamma(0) = 1$, $\delta'_\gamma(0) = 0$ (see Lemma 1 in [2]), we find $2\big(\nabla_{\frac{\partial\alpha_\gamma}{\partial t}}\frac{\partial\alpha_\gamma}{\partial s}\big)(0,0) = \kappa J\dot\gamma(0)$. Since we have

$$\begin{aligned}0 &= \frac{d}{dt}\Big\langle\Big(\nabla_{\frac{\partial\alpha_\gamma}{\partial s}}\frac{\partial\alpha_\gamma}{\partial t}\Big)(t,s), \frac{\partial\alpha_\gamma}{\partial s}(t,s)\Big\rangle\\ &= \Big\|\Big(\nabla_{\frac{\partial\alpha_\gamma}{\partial t}}\frac{\partial\alpha_\gamma}{\partial s}\Big)(t,s)\Big\|^2 + \Big\langle\Big(\nabla_{\frac{\partial\alpha_\gamma}{\partial t}}\nabla_{\frac{\partial\alpha_\gamma}{\partial t}}\frac{\partial\alpha_\gamma}{\partial s}\Big)(t,s), \frac{\partial\alpha_\gamma}{\partial s}(t,s)\Big\rangle,\end{aligned}$$

we obtain $\eta''_\gamma(0) = -\kappa^2\|J\dot\gamma(0)\|^2/4 = -\kappa^2/4$. QED

Theorem 2. *Let $\gamma : [0,T] \to M$ be a trajectory for a Kähler magnetic field $\mathbb{B}_\kappa$ on a complete Kähler manifold M. Suppose that we can define a trajectory-harp $\alpha_\gamma : [0,T] \times \mathbb{R} \to M$ associated with γ and that sectional curvatures of planes tangent to its harp body $\mathcal{HP}(T)$ are not greater than a constant c. Then, we have*

$$\eta_\gamma(t) \geq \delta_\kappa\big(\tau_\kappa(\ell_\gamma(t);c);c\big) \quad \text{and} \quad \eta'_\gamma(t) \geq \frac{d}{dt}\delta_\kappa\big(\tau_\kappa(\ell_\gamma(t);c);c\big)$$

for $0 \leq t \leq T_\gamma(c)$. If $\eta_\gamma(t_0) = \delta_\kappa\big(\tau_\kappa(\ell_\gamma(t_0);c);c\big)$ holds at some t_0 with $0 < t_0 \leq T_\gamma(c)$, then the harp-body $\mathcal{HP}(t_0)$ is totally geodesic, totally complex and of constant sectional curvature c.

Proof. We study the differential $\eta'_\gamma(t) = \big\langle\big(\nabla_{\frac{\partial\alpha_\gamma}{\partial s}}\frac{\partial\alpha_\gamma}{\partial t}\big)(t,0), \dot\gamma(0)\big\rangle$. We take positive $\hat\kappa$ with $|\kappa| < \hat\kappa$. By Lemma 1, for a sufficiently small positive ϵ, we have $\eta_\gamma(t) > \delta_{\hat\kappa}(t : c)$ and $\delta'_{\hat\kappa}(t : c) < \eta'_\gamma(t) < 0$ for $0 < t < \epsilon$. We study the first inequality on η_γ. We denote by $T_{\hat\kappa}$ the maximum positive number satisfying $T_{\hat\kappa} \le T_\gamma(c)$ and that

$$\ell_\gamma(t) \le \ell_{\hat\kappa}\big(\pi/\sqrt{\hat\kappa^2+c}\,;c\big), \quad \eta_\gamma(t) \ge \delta_{\hat\kappa}\big(\tau_{\hat\kappa}\big(\ell_\gamma(t);c\big);c\big)$$

hold for $0 \le t \le T_{\hat\kappa}$. When $T_{\hat\kappa} = T$ for all $\hat\kappa$ with $|\kappa| < \hat\kappa < |\kappa| + \epsilon'$ for some sufficiently small positive ϵ', we can get the first inequality. Hence we study the case $T_{\hat\kappa} < T$. Let $\hat\gamma : \big[0, \pi/\sqrt{\hat\kappa^2+c}\,\big] \to \mathbb{C}M^1(c)$ be a trajectory for $\mathbb{B}_{\hat\kappa}$ on $\mathbb{C}M^1(c)$. We denote by $\hat\alpha_{\hat\gamma}$ its associated trajectory-harp. We note that $\frac{\partial\alpha_\gamma}{\partial t}(t,s)$ is a Jacobi field along the geodesic $s \mapsto \alpha_\gamma(t,s)$ and that $\big\|\frac{\partial\alpha_\gamma}{\partial t}\big(t,\ell_\gamma(t)\big)\big\| = \sqrt{1-\delta_\gamma(t)^2}$, because $\dot\gamma(t) = \frac{\partial\alpha_\gamma}{\partial t}\big(t,\ell_\gamma(t)\big) + \delta_\gamma(t)\frac{\partial\alpha_\gamma}{\partial s}\big(t,\ell_\gamma(t)\big)$. Since $\frac{\partial\hat\alpha_{\hat\gamma}}{\partial t}$ also has the same properties and $\delta_\mu\big(\tau_\mu(s:c):c\big)$ is monotone decreasing with respect to μ, by Proposition 1, we have for each t with $0 < t \le T_{\hat\kappa}$ the following:

$$\begin{aligned}\Big\|\frac{\partial\alpha_\gamma}{\partial t}\big(t,\ell_\gamma(t)\big)\Big\| &= \sqrt{1-\delta_\gamma(t)^2} \le \sqrt{1-\delta_\kappa\big(\tau_\kappa(\ell_\gamma(t);c);c\big)^2} \\ &< \sqrt{1-\delta_{\hat\kappa}\big(\tau_{\hat\kappa}(\ell_\gamma(t);c);c\big)^2} = \Big\|\frac{\partial\hat\alpha_{\hat\kappa}}{\partial t}\big(\tau_{\hat\kappa}(\ell_\gamma(t);c),\ell_\gamma(t)\big)\Big\|.\end{aligned}$$

As we have $\frac{\partial\alpha_\gamma}{\partial t}(t,0) = 0$ and $\frac{\partial\hat\alpha_{\hat\gamma}}{\partial t}(t,0) = 0$ because $\alpha_\gamma(t,0) = \gamma(0)$ and $\hat\alpha_{\hat\gamma}(t,0) = \hat\gamma(0)$, by the assumption on sectional curvatures of planes tangent to $\mathcal{HP}_\gamma(T)$, Rauch's comparison theorem on Jacobi fields shows that

$$\Big\|\Big(\nabla_{\frac{\partial\alpha_\gamma}{\partial s}}\frac{\partial\alpha_\gamma}{\partial t}\Big)(t,0)\Big\| < \Big\|\Big(\nabla_{\frac{\partial\hat\alpha_{\hat\gamma}}{\partial s}}\frac{\partial\hat\alpha_{\hat\gamma}}{\partial t}\Big)\big(\tau_{\hat\kappa}(\ell_\gamma(t);c),0\big)\Big\|$$

(see [6], for example). Since $\frac{\partial\alpha_\gamma}{\partial t}$ is orthogonal to $\frac{\partial\alpha_\gamma}{\partial s}$, for $0 \le t \le T_{\hat\kappa}$, we have

$$\begin{aligned}\eta_\gamma'(t) &= \Big\langle\Big(\nabla_{\frac{\partial\alpha_\gamma}{\partial s}}\frac{\partial\alpha_\gamma}{\partial t}\Big)(t,0), \dot\gamma(0)\Big\rangle \\ &\ge -\sqrt{1-\eta_\gamma(t)^2}\ \Big\|\Big(\nabla_{\frac{\partial\alpha_\gamma}{\partial s}}\frac{\partial\alpha_\gamma}{\partial t}\Big)(t,0)\Big\| \\ &> -\sqrt{1-\delta_{\hat\kappa}\big(\tau_{\hat\kappa}(\ell_\gamma(t);c);c\big)^2}\ \Big\|\Big(\nabla_{\frac{\partial\hat\alpha_{\hat\gamma}}{\partial s}}\frac{\partial\hat\alpha_{\hat\gamma}}{\partial t}\Big)\big(\tau_{\hat\kappa}(\ell_\gamma(t);c),0\big)\Big\| \\ &= \delta_{\hat\kappa}'\big(\tau_{\hat\kappa}(\ell_\gamma(t);c),c\big).\end{aligned}$$

If we suppose $\ell_\gamma(T_{\hat\kappa}) < \ell_{\hat\kappa}\big(\pi/\sqrt{\hat\kappa^2+c};c\big)$, then $\delta_{\hat\kappa}'\big(\tau_{\hat\kappa}(\ell_\gamma(T_{\hat\kappa});c);c\big) < 0$ and $\eta_\gamma(T_{\hat\kappa}) = \delta_{\hat\kappa}\big(\tau_{\hat\kappa}\big(\ell_\gamma(T_{\hat\kappa});c\big);c\big)$ by definition of $T_{\hat\kappa}$. Hence, the above inequality and Proposition 1 lead us to

$$\begin{aligned}\frac{d}{dt}\delta_{\hat\kappa}\big(\tau_{\hat\kappa}(\ell_\gamma(t);c);c\big) &= \delta_{\hat\kappa}'\big(\tau_{\hat\kappa}(\ell_\gamma(t);c);c\big) \times \frac{\delta_\gamma(t)}{\delta_{\hat\kappa}\big(\tau_{\hat\kappa}(\ell_\gamma(t);c);c\big)} \\ &< \delta_{\hat\kappa}'\big(\tau_{\hat\kappa}(\ell_\gamma(t);c);c\big) < \eta_\gamma'(t).\end{aligned}$$

Therefore, we obtain that $\ell_\gamma(t) < \ell_{\hat\kappa}\big(\pi/\sqrt{\hat\kappa^2+c};c\big)$ and $\eta_\gamma(t) \ge \delta_{\hat\kappa}\big(\tau_{\hat\kappa}\big(\ell_\gamma(t);c\big);c\big)$ beyond $T_{\hat\kappa}$. This contradictories to the definition of $T_{\hat\kappa}$. We hence find that $\ell_\gamma(T_{\hat\kappa}) = \ell_{\hat\kappa}\big(\pi/\sqrt{\hat\kappa^2+c};c\big)$ holds. Letting $\hat\kappa \downarrow |\kappa|$, we have $\lim_{\hat\kappa\downarrow\kappa} T_{\hat\kappa} = T_\gamma(c)$, and hence obtain $\eta_\gamma(t) \ge \delta_\kappa\big(\tau_\kappa(\ell_\gamma(t);c);c\big)$ for $0 \le t \le T_\gamma(c)$.

Once we get the inequality $\eta_\gamma(t) \ge \delta_\kappa\big(\tau_\kappa(\ell_\gamma(t);c);c\big)$ for $0 \le t \le T_\gamma(c)$, the above argument guarantees $\eta'_\gamma(t) \ge \frac{d}{dt}\delta_\kappa\big(\tau_\kappa(\ell_\gamma(t);c),c\big)$ for $0 \le t \le T_\gamma(c)$ and that the equality holds at some t_1 $(0 < t_1 \le T_\gamma(c))$ if and only if $\eta_\gamma(t_1) = \delta_\kappa\big(\tau_\kappa(\ell_\gamma(t_1);c),c\big) = \delta_\gamma(t_1)$.

Now we consider the case $\eta_\gamma(t_0) = \delta_\kappa\big(\tau_\kappa(\ell_\gamma(t_0);c),c\big)$ holds at some t_0 $(0 < t_0 \le T_\gamma(c))$. As we have

$$\eta_\gamma(t_0) = 1 + \int_0^{t_0} \eta'_\gamma(t)\,dt,$$

$$\delta_\kappa\big(\tau_\kappa(\ell_\gamma(t_0);c),c\big) = 1 + \int_0^{t_0} \frac{d}{dt}\delta_\kappa\big(\tau_\kappa(\ell_\gamma(t);c),c\big)\,dt,$$

we find that $\eta'_\gamma(t) = \frac{d}{dt}\delta_\kappa\big(\tau_\kappa(\ell_\gamma(t);c),c\big)$ for $0 \le t \le t_0$, which guarantees $\delta_\gamma(t) = \delta_\kappa\big(\tau_\kappa(\ell_\gamma(t);c),c\big)$ for all $0 \le t \le t_0$. Thus we get the conclusion with the aid of Proposition 1. *QED*

Given a trajectory-harp $\alpha_\gamma : [0,T]\times\mathbb{R} \to M$, we call the length of the curve $[0,t_0] \ni t \mapsto \frac{\partial\alpha_\gamma}{\partial s}(t,0) \in U_{\gamma(0)}M$ in the unit tangent space at $\gamma(0)$ the *zenith angle* of γ at t_0 (see [2]). In [9], we showed the existence of limit points of unbounded trajectories by using zenith angles of trajectory-harps associated to them. Trivially, the above angle is not smaller than $\cos^{-1}\eta_\gamma(t_0)$. We can study limit points of unbounded trajectories also by using string-elevations.

We now apply Theorem 2 to prove Theorem 1.

Proof of Theoerm 1. We take trajectory half-lines $\gamma : [0,\infty) \to M$ for $\mathbb{B}_\kappa$ and $\gamma^{-1} : [0,\infty) \to M$ for $\mathbb{B}_{-\kappa}$. Then their limit-strings are the geodesic σ of unit speed satisfying $\sigma(0) = \gamma(0)$ and $\sigma(\infty) = \gamma(\infty) = \gamma(-\infty)$. We therefore have

$$\angle\big(\dot\sigma(0),\dot\gamma(0)\big) = \lim_{t\to\infty}\cos^{-1}\eta_\gamma(t) \le \lim_{t\to\infty}\cos^{-1}\delta_\kappa\big(\tau_\kappa(\ell_\gamma(t);c);c\big) = \frac{\pi}{2},$$

$$\angle\big(\dot\sigma(0),-\dot\gamma(0)\big) = \lim_{t\to\infty}\cos^{-1}\eta_{\gamma^{-1}}(t) \le \lim_{t\to\infty}\cos^{-1}\delta_\kappa\big(\tau_\kappa(\ell_{\gamma^{-1}}(t);c);c\big) = \frac{\pi}{2}$$

with $c = -\kappa^2$ by Theorem 2. As we have

$$\angle\big(\dot\sigma(0),\dot\gamma(0)\big) + \angle\big(\dot\sigma(0),-\dot\gamma(0)\big) \ge \angle\big(\dot\gamma(0),-\dot\gamma(0)\big) = \pi,$$

we obtain

$$\lim_{t\to\infty}\cos^{-1}\eta_\gamma(t)=\lim_{t\to\infty}\cos^{-1}\delta_\kappa\big(\tau_\kappa(\ell_\gamma(t);c);c\big)=\frac{\pi}{2},$$
$$\lim_{t\to\infty}\cos^{-1}\eta_{\gamma^{-1}}(t)=\lim_{t\to\infty}\cos^{-1}\delta_\kappa\big(\tau_\kappa(\ell_{\gamma^{-1}}(t);c);c\big)=\frac{\pi}{2},$$

which guarantees

$$\eta_\gamma'(t)=\frac{d}{dt}\delta_\kappa\big(\tau_\kappa(\ell_\gamma(t);c);c\big)\quad\text{and}\quad \eta_{\gamma^{-1}}'(t)=\frac{d}{dt}\delta_\kappa\big(\tau_\kappa(\ell_{\gamma^{-1}}(t);c);c\big)$$

for $t\geq 0$. Thus, the total harp-body $\mathcal{HP}_\gamma=\mathcal{HP}_\gamma(\infty)\cup\mathcal{HP}_{\gamma^{-1}}(\infty)$ is totally geodesic, totally complex and of constant sectional curvature c. In particular, we find that $\dot\sigma(0)=\mathrm{sgn}(\kappa)J\dot\gamma(0)$, where $\mathrm{sgn}(\kappa)$ denotes the signature of κ, and that for an arbitrary t_0, the geodesic σ_{t_0} joining $\gamma(t)$ and $\gamma(\infty)$ lies on $\mathcal{HP}_\gamma$. Since it is the limit-string of the trajectory-harps associated with $\gamma(t+t_0):[0,\infty)\to M$ and $\gamma(t_0-t):[0,\infty)\to M$, and every string of these trajectory-harps lies on $\mathcal{HP}_\gamma$, by the same argument as above, we find that σ_{t_0} and γ cross orthogonally at $\gamma(t_0)$. This completes the proof. QED

We here mention the case of bounded trajectories.

Remark 1. Let γ be a trajectory for a Kähler magnetic field $\mathbb{B}_\kappa$ on a Kähler manifold whose sectional curvatures are not greater than a nonnegative constant c. If $\gamma\big(\pi/\sqrt{\kappa^2+c}\,\big)=\gamma\big(-\pi/\sqrt{\kappa^2+c}\,\big)$, then it is closed and $\mathcal{HP}_\gamma\big(\pi/\sqrt{\kappa^2+c}\,\big)\cup\mathcal{HP}_{\gamma^{-1}}\big(\pi/\sqrt{\kappa^2+c}\,\big)$ is totally geodesic, and sectional curvatures of planes tangent to this body are c.

Remark 2. Let c be a negative constant, and γ be a trajectory for a Kähler magnetic field $\mathbb{B}_\kappa$ with $\kappa^2>|c|$. If $\gamma\big(\pi/\sqrt{\kappa^2+c}\,\big)=\gamma\big(-\pi/\sqrt{\kappa^2+c}\,\big)$ and sectional curvatures of planes tangent to the harp-body $\mathcal{HP}_\gamma\big(\pi/\sqrt{\kappa^2+c}\,\big)\cup\mathcal{HP}_{\gamma^{-1}}\big(\pi/\sqrt{\kappa^2+c}\,\big)$ are not greater than c, then it is closed and the harp-body is totally geodesic, and sectional curvatures of planes tangent to this body are c.

4 Horocycle trajectories on products of complex hyperbolic spaces

Let $\mathcal{S}_p$ denote the subset of the unit tangent space U_pM of a Hadamard Kähler manifold M which consists of unit tangent vectors satisfying κ-HC condition. It is natural to consider that if $\mathcal{S}_p$ contains a standard sphere S^k at some point $p \in M$, then M contains a totally geodesic $\mathbb{C}H^k(-\kappa^2)$ as a factor. Unfortunately, to answer this problem, we need some more study. In this section, we study trajectories on a product of complex hyperbolic spaces.

Let γ be a trajectory for $\mathbb{B}_\kappa$ on a product of complex hyperbolic spaces and a complex Euclidean space $M = \mathbb{C}^m \times \mathbb{C}H^{n_1}(c_1) \times \cdots \times \mathbb{C}H^{n_r}(c_r)$. We express γ as $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_r)$, where γ_0 lies on $\mathbb{C}^m$ and γ_i lies on $\mathbb{C}H^{n_i}(c_i)$ for $i = 1, \dots, r$. As was pointed out in [1], it is a horocycle trajectory if and only if $\|\dot\gamma_i\| \le |\kappa|/\sqrt{|c_i|}$ for every $i \ge 1$ and $\|\dot\gamma_{i_0}\| = |\kappa|/\sqrt{|c_{i_0}|}$ for some i_0.

When γ is a horocycle trajectory, if we denote by $i_1 < i_2 < \cdots < i_k$ all the indices of components satisfying $\|\dot\gamma_{i_j}\| = |\kappa|/\sqrt{|c_{i_j}|}$, then the point at infinity of γ coincide with one of the points at infinity of the geodesic of initial vector $(w_0, \dots, w_r)$ with $w_i = J\dot\gamma_i(0)/\big(\sum_j \|\dot\gamma_{i_j}\|\big)$ when $i = i_j$ and $w_i = 0$ when $i \ne i_j$ for all j. Thus we have the following.

Proposition 2. *Let $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_r)$ be a trajectory for $\mathbb{B}_\kappa$ $(\kappa \ne 0)$ on a product $M = \mathbb{C}^m \times \mathbb{C}H^{n_1}(c_1) \times \cdots \times \mathbb{C}H^{n_r}(c_r)$. It is a holomorphic horocycle trajectory if and only if we have indices $i_1, \dots, i_k$ $(1 \le i_1 < i_2 < \cdots < i_k \le r)$ satisfying $\|\dot\gamma_{i_j}\| = |\kappa|/\sqrt{|c_{i_j}|}$ for $j = 1, \dots, k$ and $\|\dot\gamma_i\| = 0$ for $i \ne i_j$. In particular, we have $\kappa^2\big(\frac{1}{|c_{i_1}|} + \cdots + \frac{1}{|c_{i_k}|}\big) = 1$.*

We directly compute the string-length and the string-cosine for a holomorphic horocycle trajectory $\gamma = (\gamma_1, \dots, \gamma_r)$ for a non-trivial $\mathbb{B}_\kappa$ on $M = \mathbb{C}H^{n_1}(c_1) \times \cdots \times \mathbb{C}H^{n_r}(c_r)$ whose velocity vector satisfies $\|\dot\gamma_i(t)\| = |\kappa|/\sqrt{|c_i|}$ for $i = 1, \dots, r$. By putting $a_i = |\kappa|/\sqrt{|c_i|}$, the curve $\tilde\gamma_i$ defined by $\tilde\gamma_i(u) = \gamma_i(u/a_i)$ is a horocycle trajectory for $\mathbb{B}_{\mathrm{sgn}(\kappa)\sqrt{|c_i|}}$ on $\mathbb{C}H^{n_i}(c_i)$. Hence, the string length $\tilde\ell_i$ of $\tilde\gamma_i$ is given by

$$\tilde\ell_i(u) = \ell_{\sqrt{|c_i|}}(u; c_i) = \big(2/\sqrt{|c_i|}\,\big)\sinh^{-1}\big(\sqrt{|c_i|}\,u/2\big).$$

Since we have $|\kappa| = 1\Big/\sqrt{\frac{1}{|c_1|} + \cdots + \frac{1}{|c_r|}}$, we obtain

$$\begin{aligned}
\ell_\gamma(t) &= \sqrt{\{\tilde\ell_1(a_1 t)\}^2 + \cdots + \{\tilde\ell_r(a_r t)\}^2} \\
&= 2\sqrt{\frac{1}{|c_1|} + \cdots + \frac{1}{|c_r|}}\ \sinh^{-1}\left(\frac{t}{2\sqrt{\frac{t}{|c_1|} + \cdots + \frac{1}{|c_r|}}}\right) \\
&= \ell_{\frac{1}{\sqrt{\frac{1}{|c_1|}+\dots+\frac{1}{|c_r|}}}}\left(t; \frac{1}{\frac{1}{|c_1|} + \dots + \frac{1}{|c_r|}}\right).
\end{aligned}$$

We denote by $\tilde\alpha_i$ and $\tilde\delta_i$ the trajectory-harp and the string-cosine associated with $\tilde\gamma_i$. Then the trajectory-harp $\alpha_\gamma = (\alpha_1, \dots, \alpha_r)$ associated with γ is given by

$$\alpha_\gamma(t;s) = \Big(\tilde{\alpha}_1\Big(a_1 t, \frac{\tilde{\ell}_1(a_1 t)}{\ell_\gamma(t)} s\Big), \dots, \tilde{\alpha}_r\Big(a_r t, \frac{\tilde{\ell}_r(a_r t)}{\ell_\gamma(t)} s\Big)\Big).$$

Since we have

$$\tilde{\delta}_i(u) = \Big\langle \dot{\tilde{\gamma}}_i(u), \frac{\partial \tilde{\alpha}_i}{\partial \nu}\big(u, \tilde{\ell}_i(u)\big)\Big\rangle = \frac{2}{\sqrt{|c_i| u^2 + 4}},$$

we obtain

$$\begin{aligned}
\delta_\gamma(t) &= \sum_{i=1}^{r} \Big\langle \dot{\gamma}_i(t), \frac{\partial \alpha_i}{\partial s}\big(t, \ell_i(t)\big)\Big\rangle = \sum_{i=1}^{r} a_i \frac{\tilde{\ell}_i(a_i t)}{\ell(t)}\, \tilde{\delta}_i(a_i t) \\
&= \frac{2\kappa^2}{\sqrt{\kappa^2 t^2 + 4}} \sum_{i=1}^{r} \frac{1}{|c_i|} = \frac{2}{\sqrt{\kappa^2 t^2 + 4}} = \frac{2}{\sqrt{\dfrac{t^2}{\frac{1}{|c_1|} + \dots + \frac{1}{|c_r|}} + 4}} \\
&= \delta_{\frac{1}{\sqrt{\frac{1}{|c_1|} + \dots + \frac{1}{|c_r|}}}}\Big(t; \frac{1}{\frac{1}{|c_1|} + \dots + \frac{1}{|c_r|}}\Big),
\end{aligned}$$

and $\eta_\gamma(t) = \delta_\gamma(t)$.

5 An estimate of string-elevations from above

In section 4, we gave an estimate of string-elevations from below. We now give a corresponding estimate from above under the condition that sectional curvatures are bounded from below.

Let $\gamma : [0, T] \to M$ be a trajectory for $\mathbb{B}_\kappa$ on a Kähler manifold M. We say that the trajectory-harp $\alpha_\gamma : [0, T] \times \mathbb{R} \to M$ associated with γ is holomorphic at the origin if the initial tangent vector $\frac{\partial \alpha_\gamma}{\partial s}(t, 0)$ of each string is contained in the complex subspace of $T_{\gamma(0)}M$ spanned by $\dot{\gamma}(0)$ for $0 \leq t \leq T$. Similarly, we say α_γ is holomorphic at the arch if $\frac{\partial \alpha_\gamma}{\partial s}\big(t, \ell_\gamma(t)\big)$ is contained in the complex subspace of $T_{\gamma(t)}M$ spanned by $\dot{\gamma}(t)$ for $0 \leq t \leq T$. It is likely that one of these conditions shows the other. But for now, we cannot say any more about their relationship. In [10], Shi and the second author gave estimates on string-lengths and string-cosines. We set

$$\begin{aligned}
R_\gamma &= \sup\{t \mid \delta_\gamma(\tau) > 0 \text{ for } 0 \leq \tau < t\}, \\
C_\gamma &= \sup\{t \mid \ell_\gamma(\tau) \leq c^H_{\gamma(0)} \text{ for } 0 \leq \tau < t\}.
\end{aligned}$$

Here, $c^H_{\gamma(0)}$ denotes the minimum conjugate value of $\gamma(0)$ along geodesics tangent to the harp-body $\mathcal{HP}_\gamma(T)$. Since we have $\ell_\gamma(t) \leq t$, when sectional curvatures of planes tangent to $\mathcal{HP}_\gamma(T)$ are not less than a constant c, then $C_\gamma \geq \pi/\sqrt{c}$.

Proposition 3 ([10]). *Let $\gamma : [0,T] \to M$ be a trajectory for a non-trivial Kähler magnetic field $\mathbb{B}_\kappa$ on a complete Kähler manifold M. We suppose that its corresponding trajectory-harp is holomorphic at the arch and that sectional curvatures of planes tangent to the harp-body $\mathcal{HP}_\gamma(T)$ of the associated trajectory-harp α_γ are not less than a constant c. We then have the following*:

(1) $\ell_\gamma(t) \le \ell_\kappa(t;c)$ *and* $\delta_\gamma(t) \le \delta_\kappa\big(\tau_\kappa(\ell_\gamma(t);c);c\big)$ *for* $0 \le t \le \min\{R_\gamma, C_\gamma\}$;

(2) *If* $\ell_\gamma(t_0) = \ell_\kappa(t_0;c)$ *at some* t_0 *with* $0 < t_0 \le \min\{R_\gamma, C_\gamma\}$, *then the harp-body* $\mathcal{HP}_\gamma(t_0)$ *is totally geodesic, totally complex and of constant curvature* c.

By using this result we give an estimate of string-elevations from above.

Theorem 3. *Let $\gamma : [0,T] \to M$ be a trajectory for $\mathbb{B}_\kappa$. Suppose that we can define a trajectory-harp $\alpha_\gamma : [0,T] \times \mathbb{R} \to M$ associated with γ. If α_γ is holomorphic at the origin and at the arch, and if sectional curvatures of planes tangent to its harp body $\mathcal{HP}(T)$ are not less than a constant c, then, we have*

$$\eta_\gamma(t) \le \delta_\kappa\big(\tau_\kappa(\ell_\gamma(t);c);c\big) \quad \text{and} \quad \eta_\gamma'(t) \le \frac{d}{dt}\delta_\kappa\big(\tau_\kappa(\ell_\gamma(t);c);c\big)$$

for $0 \le t \le \min\{R_\gamma, C_\gamma\}$. *If* $\eta_\gamma(t_0) = \delta_\kappa\big(\tau_\kappa(\ell_\gamma(t_0);c);c\big)$ *holds at some* t_0, *then the harp-body* $\mathcal{HP}(t_0)$ *is totally geodesic, totally complex and of constant sectional curvature* c.

Proof. Our proof goes through almost the same as of Theorem 2 by taking positive $\hat{\kappa}$ so that $\hat{\kappa} < |\kappa|$. Since the trajectory-harp α_γ is holomorphic at the origin, we find that $\nabla_{\frac{\partial\alpha_\gamma}{\partial s}}\frac{\partial\alpha_\gamma}{\partial s}(t,0)$ is contained in the complex subspace spanned by $\dot\gamma(0)$. Therefore we have

$$\begin{aligned}
\eta_\gamma'(t) &= -\sqrt{1-\eta_\gamma(t)^2}\ \Big\|\Big(\nabla_{\frac{\partial\alpha_\gamma}{\partial s}}\frac{\partial\alpha_\gamma}{\partial t}\Big)(t,0)\Big\| \\
&< -\sqrt{1-\delta_{\hat\kappa}\big(\tau_{\hat\kappa}(\ell_\gamma(t);c);c\big)^2}\ \Big\|\Big(\nabla_{\frac{\partial\hat\alpha_{\hat\gamma}}{\partial s}}\frac{\partial\hat\alpha_{\hat\gamma}}{\partial t}\Big)\big(\tau_{\hat\kappa}(\ell_\gamma(t);c),0\big)\Big\| \\
&= \delta_{\hat\kappa}'\big(\tau_{\hat\kappa}(\ell_\gamma(t);c),c\big).
\end{aligned}$$

By using this, we can get our result by the same argument as of the proof of Theorem 2. QED

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