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On some Lusternick–Schnirelmann type invariants

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Abstract. In this paper, we show that the invariant $\mathrm{R}_0(X)$, introduced in [15], coincides with $cat_0(X)$ for any rationally elliptic space X. Additionally, we define, for any space X over an arbitrary field $\mathbb{K}$, an *Ext-version* homotpy invariant $\mathrm{L}_{\mathbb{K}}(X)$ of the Ginsburg invariant $l_{\mathbb{K}}(X)$. Then, we establish the equality between $\mathrm{L}_0(X) := \mathrm{L}_{\mathbb{Q}}(X)$ and $l_0(X)$ in the case where X is rationally elliptic.

Keywords: LS-category, Toomer and Ginsburg invariants, Milnor-Moore and Eilenberg-Moore spectral sequences.

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Introduction

In [16], the author introduced, for any topological space X, what is subsequently called the *Toomer invariant* $e_{\mathbb{K}}(X)$, over a given field $\mathbb{K}$. This invariant is defined as the least integer p for which the term $E_\infty^{p,q}$ in the Milnor-Moore spectral sequence:

$$Ext^{p,q}_{H_*(\Omega X;\mathbb{K})}(\mathbb{K},\mathbb{K}) \Longrightarrow H^{p+q}(X;\mathbb{K}) \tag{1}$$

is non-zero. The author then showed that $e_{\mathbb{K}}(X)$ provides a lower bound for the *Lusternik-Schnirelmann category* (LS category for short), denoted $cat(X)$ [11] and defined as the smallest number (plus one, for normalization) of open sets that are contractile in X and form a cover of X. That is,

$$e_{\mathbb{K}}(X) \leq cat(X)$$

for any field $\mathbb{K}$. The study of this invariant has led to interesting results in practice. Among these, we cite Y. Félix and S. Halperin's analogous inequality $e_0(X) \leq cat_0(X)$ [3, Theorem 4.7], where $e_0(X) := e(X_0)$ and $cat_0(X) :=$

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$cat(X_0)$, with X_0 being the rationalization space of X. Several specialists in rational homotopy theory worked then hard to specify the largest class of spaces for which these two invariants are equal. Referring to [8], we know that this class includes all Poincaré duality spaces over $\mathbb{Q}$ (see §2). In particular, it includes all rationally elliptic spaces, which are characterized by their rational cohomology $H^*(X,\mathbb{Q})$ and rational homotopy $\pi_*(X)\otimes\mathbb{Q}$, both of which are finite-dimensional. These results are achieved using Sullivan's approach to rational homotopy theory, which associates to every simply connected space X (having the homotopy type of a finite-type CW-complex) a unique (up to isomorphism) *minimal Sullivan algebra* $(\Lambda V, d)$. This algebra satisfies $H^*(X,\mathbb{Q}) \cong H(\Lambda V, d)$ and $V \cong Hom_{\mathbb{Z}}(\pi_*(X),\mathbb{Q})$. Here, $(\Lambda V, d)$ is a commutative differential graded algebra (cdga for short) with specific properties (see §2).

In what follows, unless otherwise stated, X will denote a simply connected finite-type CW-complex, and $(\Lambda V, d)$ will denote its minimal Sullivan model (or simply its "minimal model").

Refeering to [3] (see also [8]), $cat_0(X) = cat(\Lambda V, d)$ where the later is defined as the least integer m such that the projection:

$$p : (\Lambda V, d) \to (\frac{\Lambda V}{\Lambda^{\geq m+1} V}, \bar{d})$$

has a retraction as a morphism of cdga's, while $e_0(X) := e(\Lambda V, d)$ is the least integer m such that p is injective in cohomology. An equivalente definition of $e(\Lambda V, d)$ is given using the isomorphism, established in [3], between (1) and the so-called hte *Ext-Milnor-Moore* spectral sequence [8, Proposition 9.1] (see §2):

$$E_2^{p,q} = H^{p,q}(\Lambda V, d_2) \Rightarrow H^{p+q}(\Lambda V, d). \tag{2}$$

as follows:

$$e(\Lambda V, d) = \sup\{p \mid \text{such that } E_\infty^{p,q} \neq 0\}. \tag{3}$$

Here, $E_\infty^{p,q}$ is the infty term of (2).

Next, consider the (rational) *Ext-Eilenberg-Moore spectral sequence* introduced in [13] (see §2) :

$$Ext^{p,q}_{(\Lambda V, d_2)}(\mathbb{Q}, (\Lambda V, d_2)) \Longrightarrow Ext^{p+q}_{(\Lambda V, d)}(\mathbb{Q}, (\Lambda V, d)). \tag{4}$$

In [15], the second author introduced the invariant $\mathrm{R}(\Lambda V, d)$ defined similarly to $e(\Lambda V, d)$, as follows:

$$\mathrm{R}(\Lambda V, d) = \sup\{p \mid \text{such that } E_\infty^{p,q} \neq 0\}, \tag{5}$$

where $E_\infty^{p,q}$ stands here for the infty term of (4). He then constructed an isomorphism between the spectral sequence (4) and the *topological Eilenberg-Moore* *spectral sequence* [15, Theorem 1.2]:

$$Ext^{p,q}_{H_*(\Omega X;\mathbb{Q})}(\mathbb{Q}, H_*(\Omega X;\mathbb{Q})) \Longrightarrow Ext^{p+q}_{C_*(\Omega X,\mathbb{Q})}(\mathbb{Q}, C_*(\Omega X,\mathbb{Q})) \quad (6)$$

by which he introduced a new (topological) homotpy invariant:

$$\mathrm{R}_0(X) := \mathrm{R}(\Lambda V, d).$$

The first main result of this paper extends (in the rational case) the result of [15, Theorem 1.1] as follows :

Theorem 1. *Let X be a rationally elliptic space. Then :*

$$\mathrm{R}_0(X) = e_0(X) = cat_0(X).$$

Our second result is related to rational Ginsburg's lower bound invariant $l_0(X)$ of $cat_0(X)$ [9]. This invariant is defined as the greatest integer $t \geq 2$ such that the differential δ_t of the $t-th$ page $(E_t^{*,*}, \delta_t)$ of (1) is non-zero. In other words, $l_0(X) = t$ if and only if $E_\infty^{*,*} = E_{t+1}^{*,*}$. Clearly, one may similarly define $l(\Lambda V, d)$ in terms of (2) and immediately conclude, thanks to the isomorphism between (1) and (2) [3, Proposition 9.1], that $l_0(X) = l(\Lambda V, d)$.

Likewise to $\mathrm{R}_0(X)$ and $\mathrm{R}(\Lambda V, d)$, we introduce, in this paper, the *Ext-Ginsburg invariant* $\mathrm{L}_0(X)$ (resp. $\mathrm{L}(\Lambda V, d)$) as the greatest integer $t \geq 2$ such that the differential δ_t in the $t-th$ term, $(E_t^{*,*}, \delta_t)$ of (6) (resp. d_t in the $t-th$ term $(E_t^{*,*}, d_t)$ of (4)) is non-zero. Again, by the isomorphism between (6) and (4) [15, Theorem 1.2], we have $\mathrm{L}_0(X) = \mathrm{L}(\Lambda V, d)$

As a second result, we prove the following

Theorem 2. *Let X be a rationally elliptic space. Then,* $\mathrm{L}_0(X) = l_0(X)$.

1 Preliminaries

In this section, unless otherwise stated, our ground field $\mathbb{K}$ is of characteristic zero. The main references are [8] and [6].

1.1 Sullivan model

Let $V = \bigoplus_{i=0}^{+\infty} V^i$ be a graded $\mathbb{K}$-vector space, TV the graded tensor algebra on V, and $\Lambda V = TV/I$ the free commutative graded algebra, where I is the graded ideal generated by homogeneous elements of the form $v\otimes w-(-1)^{|v||w|}w\otimes v$ for $v, w \in V$. Here and henceforth, $\mid v \mid$ denotes the degree of v.

A *Sullivan algebra* is a free commutative differential graded algebra of the form $(\Lambda V, d)$, with V has a well-ordered basis $\{v_\alpha\}$ such that $dv_\alpha \in \Lambda V_{<\alpha}$, with

$V_{<\alpha}$ being the subspace of V generated by $\{v_\beta,\, \alpha < \beta\}$. Such an algebra is said *minimal* if the differential satisfies :

$$Imd \subseteq \Lambda^+ V.\Lambda^+ V.$$

When $V^0 = \mathbb{K}$ and $V^1 = 0$ (i.e. $(\Lambda V, d)$ is 1-connected), this is equivalent to :

$$d(V) \subseteq \Lambda^{\geq 2} V = \bigoplus_{i \geq 2} \Lambda^i V.$$

A *(minimal) Sullivan model* for a commutative differential graded algebra (A, d) is a quasi-isomorphism (i.e. a morphism inducing an isomorphism in cohomology)

$$m : (\Lambda V, d) \xrightarrow{\cong} (A, d)$$

from a (minimal) Sullivan algebra $(\Lambda V, d)$. It is well known [8, Proposition 12,2] that any cdga (A, d) with $H^0(A) = \mathbb{K}$ and $H^1(A) = 0$ has a minimal Sullivan model. In particular, for a path-connected topological space X, a Sullivan model for X over $\mathbb{Q}$ is a Sullivan model for the singular cochain complex $C^*(X; \mathbb{Q})$:

$$m : (\Lambda V, d) \xrightarrow{\cong} C^*(X; \mathbb{K}).$$

Recall that X is said to be *rationally elliptic* if its rational cohomology and rational homotopy are both finite-dimensional. In terms of its minimal Sullivan model $(\Lambda V, d)$, this means that $\dim H(\Lambda V, d) < \infty$ and $\dim V < \infty$.

Recall also that a finite-dimensional graded algebra A is said to be *Poincaré duality algebra* over $\mathbb{K}$ if it is commutative and satisfies the following conditions:

(1) $A^m \cong \mathbb{K}\omega$ for some integer $m \geq 0$.

(2) For all p, the bilinear form $A^p \otimes A^{m-p} \to A^m \cong \mathbb{K}$ is a non-degenerate.

The generating element ω of degree m is called the *fundamental class* of A. As a particular case of interest for our purposes, the cohomology $H^*(X, \mathbb{Q})$ of any rationally elliptic space X is a Poincaré duality algebra over $\mathbb{Q}$ [8]. We then say that X is a *Poincaré duality space* over $\mathbb{Q}$. The fundamental class, denoted ω, of $H^*(X, \mathbb{Q})$ is called the *fundamental class* of X, and its degree, denoted N, is called the *formal dimension* of X (or of $(\Lambda V, d)$).

1.2 The evaluation map.

Let (A, d) be an augmented differential graded algebra (dga) over a field $\mathbb{K}$, and $\alpha : (P, d) \xrightarrow{\cong} (\mathbb{K}; 0)$ be a minimal semifree resolution of K [8]. This defines the differential graded complex $(\mathcal{A}, \mathcal{D})$, where $\mathcal{A} = Hom_{(A,d)}((P,d),(A,d))$ and $\mathcal{D}$, its differential, is given by :

$$\mathcal{D}(f) = f \circ d - (-1)^{|f|} d \circ f.$$

This induces, the map of complexes :

$$\begin{array}{rlcl} ev: & (Hom_{(A,d)}((P,d),(A,d)),D) & \longrightarrow & (A,d) \\ & f & \longmapsto & f(p) \end{array}$$

where $p \in P$ is a cocycle representing $1_{\mathbb{K}}$.

The *evaluation map* of (A,d) is :

$$ev_{(A,d)} := H(ev) : Ext_{(A,d)}(\mathbb{K},(A,d)) \longrightarrow H(A,d).$$

Here, $Ext_{(A,d)}(\mathbb{K},(A,d)) = H(\mathcal{A},\mathcal{D})$, where Ext is the Eilenberg-Moore differential functor [6].

It is well known that $ev_{(A,d)}$ is independent of the choice of (P,d) and p [13, 14]. In particular, for any topological path-connected space X, $ev_{C^*(X;\mathbb{K})}$ is called the *evaluation map of* X over $\mathbb{K}$. It is an homotopy invariant of X. Another important invariant is the $\mathbb{K}$-*formal dimension of* X [6, §5], denoted $fd(X,\mathbb{K})$, and defined as the greatest integer $r \in \mathbb{Z}$ such that $[Ext_{C^*(X;\mathbb{K})}(\mathbb{K}, C^*(X;\mathbb{K}))]^r \neq 0$ or $-\infty$ if $Ext_{C^*(X;\mathbb{K})}(\mathbb{K}, C^*(X;\mathbb{K})) = 0$. By [6, §Remark 1.3], $fd(\Lambda V, d) = fd(X,\mathbb{Q})$ for any minimal Sullivan model of X.

A Sullivan algebra $(\Lambda V, d)$ is called a *Gorenstein algebra* if its dimension fulfills $\dim Ext_{(\Lambda V,d)}(\mathbb{K},(\Lambda V,d)) = 1$. It is well known that if $\dim V < \infty$, then $(\Lambda V, d)$ is a Gorenstein algebra [6]. Moreover, $ev_{(\Lambda V,d)} \neq 0$ if and only if $\dim H(\Lambda V, d) < \infty$ [13]. Thus, when $\dim V < \infty$, $(\Lambda V, d)$ is elliptic if and only if $ev_{(\Lambda V,d)} \neq 0$. The space X is said a *Gorenstein space* over $\mathbb{Q}$ if its model $(\Lambda V, d)$ is.

1.3 Spectral sequences

Our main references for spectral sequences are [12] and [8].

A spectral sequence consists in a sequence of differential bigraded vector spaces $E_r^{*,*}$, together with a sequence of isomorphisms $\sigma_r : E_{r+1}^{*,*} \xrightarrow{\cong} H(E_r^{*,*})$. That is, for any integer r, we have $E_r^{*,*} = \oplus E_r^{p,q}$, and there is a linear map $d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$, of degree $+1$ and bidegree $(r, 1-r)$, such that $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$. Thus, for any $r \geq 0$, $(E_r^{*,*}, d_r^{*,*})$ is a cochain complex.

A spectral sequence $(E_r^{*,*}, d_r^{*,*})$ is said *convergent* if, for each pair (p,q), there exists an integer $r(p,q)$ such that $E_r^{p,q} = E_{r(p,q)}^{p,q}$ holds for all $r \geq r(p,q)$. We denote $E_\infty^{p,q} = E_{r(p,q)}^{p,q}$.

A filtration FA on a (commutative) differential graded algebra (dga) (A,d) is defined as a family of graded subalgebras $\{F^pA\}$, for $p\in\mathbb{Z}$, linked by the inclusions :

$$\cdots\subseteq F^{p+1}A\subseteq F^pA\subseteq F^{p-1}A\subseteq\cdots\subseteq A,$$

such that $dF^pA\subseteq F^pA$ for all integers $p\in\mathbb{Z}$. Such an algebra is said a filtered algebra and denoted (A,FA,d). Its filtration induces naturally a filtration on $H(A,d)$ given by :

$$F^pH(A,d)=Im(H(F^pA)\longrightarrow H(A)),$$

and a spectral sequence $(E_r^{*,*},d_r)$ [8]. This spectral sequence is said convergent to $H(A,d)$ if $E_\infty^{p,q}\cong G^{p,q}H(A,d)=F^pH(A)/F^{p+1}H(A)$. If this is the case, we write :

$$E_2^{p,q}\Longrightarrow H^{p+q}(A,d).$$

The filtration FA on a dga (A,d) is said to be *bounded* if, for each p, there exist two integers $s(p)$ and $n(p)$ so that :

$$0=F^{s(p)}A\subseteq F^{s(p)-1}A\subseteq\cdots\subseteq F^{n(p)}A\subseteq A.$$

In such situation, the associated spectral sequence to (A,FA,d) converges to $H(A,d)$ [12]. Its zero-term is $(E_0^{p,q}(A),d_0)=(G^{p,q}A,Gd)$, and its first term is $(E_1^{p,q}(A),d_1)=(H^{p,q}(GA,Gd),d_1)$.

Below, we provide examples of spectral sequences that will be used in the next section.

Consider a minimal Sullivan algebra $(\Lambda V,d)$ and express the differential as $d=d_k+d_{k+1}+\dots$ (where $k\geq 2$ by minimality). Note that d_k is also a differential.

We filter ΛV as follows :

$$F^p(\Lambda V)=\Lambda^{\geq p}V,\ \ p\geq 0. \tag{7}$$

This is an increasing and bounded filtration, hence, it defines the *algebraic Milnor-Moore spectral sequence*:

$$E_k^{p,q}(\Lambda V)=H^{p,q}(\Lambda V,d_k)\Longrightarrow H^{p+q}(\Lambda V,d). \tag{8}$$

Next, consider the dga $A=(Hom_{(\Lambda V,d)}((\Lambda V\otimes\Lambda sV,\delta),(\Lambda V,d)),D)$, where $(\Lambda V\otimes\Lambda sV,\delta)$ is a $(\Lambda V,d)$-semifree resolution of $\mathbb{K}$ [8]. The differential δ extends that of ΛV by $\delta(sv)=v-sdv$ for all $v\in V$ [4]. We filter A by :

$$F^p(A)=\{f\in A,\mid f(\Lambda sV)\subseteq\Lambda^{\geq p}V\},\ \ \ p\geq 0. \tag{9}$$

This filtration gives rise to the *algebraic Eilenberg-Moore spectral sequence* [14] (see also [15]) :

$$E_k^{p,q}(A)=Ext_{(\Lambda V,d_k)}^{p,q}(\mathbb{K},(\Lambda V,d_k))\Longrightarrow Ext_{(\Lambda V,d)}^{p+q}(\mathbb{K},(\Lambda V,d)). \tag{10}$$

Remark 1. As defined, it is clear that the evaluation map $ev_{(\Lambda V,d)}$ preserves the filtration (9) and (7). In particular, it induces a morphism between the spectral sequences (4) and (2). That is, there is a sequence of differential graded vector spaces, denoted $E_i^{p,q}(ev) : E_i^{p,q}(A) \to E_i^{p,q}(\Lambda V)$ for $k \leq i \leq \infty$, between the corresponding pages.

In a similar way, we consider the *minimal free model* (TW, d) of the (commutative) dga (A, d) [5] and again express the differential as $d = d_k + d_{k+1} + \dots$ (where $k \geq 2$ by minimality). Filtering TW by $F^p(TW) = T^{\geq p}(W)$ defines [5, Proposition A.8] the following spectral sequence :

$$E_k^{p,q}(TW) = Ext^{p,q}_{H(TW,d)}(\mathbb{K}, \mathbb{K}) \Longrightarrow Ext^{p+q}_{(TW,d)}(\mathbb{K}, \mathbb{K}) = H(TW, d). \tag{11}$$

When (TW, d) designate the Adams-Hlton model of X [5], this spectral sequence is isomorphic to (1) (cf. [6, Remark 1.3]). As for Sullivan models, an easy calculation shows that indeed $Ext^{p,q}_{H(TW,d)}(\mathbb{K}, \mathbb{K})$.

Next, consider the dga $B = (Hom_{TW,d}((TW \otimes (\mathbb{K} \oplus sW), \delta), (TW, d)), D)$, where $(TW \otimes (\mathbb{K} \oplus sW), \delta)$ serves as a (TW, d)-semifree resolution of $\mathbb{K}$ [8]. The differential δ which extends that of TW and is specifically given by $\delta(sv) = v - sdv$ for all $v \in W$ (see [4, 8]). Additionally, filtering B by

$$F^p(B) = \{f \in B, \mid f(\mathbb{K} \oplus sW) \subseteq T^{\geq p}W\};\ p \geq 0,$$

we obtain the following spectral sequence, introduced in [1] :

$$E_k^{p,q}(B) = Ext^{p,q}_{(TW,d_k)}(\mathbb{K}, (TW, d_k)) \Longrightarrow Ext^{p+q}_{(TW,d)}(\mathbb{K}, (TW, d)). \tag{12}$$

Once again using the Adams-Hlton model of X, we see that (12) is isomorphic to (6).

2 Proofs of the main results

Recall from the introduction that the R_0-invariant introduced in [15] for any simply connected finite-type CW-complex X is defined (with notations of §2) in terms of (4) and (6) as follows :

$$\mathrm{R}_0(X) = \mathrm{R}(\Lambda V, d) = \sup\{m \mid E_\infty^{m,*}(A) \neq 0\}.$$

This is indeed inspired from the definitin of the Toomer's invariant of X, which is defined in terms of (1) and (2) as follows:

$$e_0(X) = e(\Lambda V, d) := \sup\{m \mid E_\infty^{m,*}(\Lambda V) \neq 0\}.$$

It is well known that when X is elliptic (and hence a Poincaré duality space), $e(\Lambda V, d)$ can be equivalently defined in terms of the fundamental class ω as follows:

$$e(\Lambda V, d) = \sup\{m \mid \omega \text{ can be represented by a cocycle in } \Lambda^{\geq m} V\}. \quad (13)$$

Referring to [15, Remark 3.4], when $\dim V < \infty$ (i.e. when $(\Lambda V, d)$ is a Gorenstein cdga), an equivalent definition of $\mathrm{R}(\Lambda V, d)$ is similarly given in terms of the generating class Ω of $Ext^*_{(\Lambda V,d)}(\mathbb{K}, (\Lambda V, d))$ as follows :

$$\mathrm{R}(\Lambda V, d) = sup\{m \mid \Omega \text{ can be represented by a cocycle in } F^m(A)\}, \quad (14)$$

where $A = Hom_{(\Lambda V,d)}((\Lambda V \otimes \Lambda(sV), d), (\Lambda V, d))$.

We now recall and prove our first main result.

Theorem 3. *(Theorem 1) Let X be a rationally elliptic space. Then :*

$$\mathrm{R}_0(X) = e_0(X) = cat_0(X).$$

Proof. First, note that since X is a Poincaré duality space over $\mathbb{Q}$, it follows from [7] that $e_0(X) = cat_0(X)$. Let $(\Lambda V, d)$ be a minimal Sullivan model of X. By hypothesis, $\dim V < \infty$, and both $(\Lambda V, d_k)$ and $(\Lambda V, d)$ are Gorenstein algebras of the same formal dimension N, as established in [6, Theorem 5.2]. Therefore, the first term of the spectral sequence (4), $E_k^{*,*}(A) = Ext^*_{(\Lambda V,d_k)}(\mathbb{K}, (\Lambda V, d_k))$, is concentrated in a fixed bidegree (p, q) with $p+q = N$. By the convergence of (4), we have $\dim E_\infty^{p,q}(A) = 1$. It follows that $\mathrm{R}(\Lambda V, d)$ is precisely the integer p. For the remainder of the proof, we identify $E_\infty^{p,q}(A)$ with $Ext^N_{(\Lambda V,d)}(\mathbb{K}, (\Lambda V, d))$ and denote by $\Omega_0 := [f_0]$ (resp. $[\Omega]$) the unique generating class of $E_k^{p,q}(A) = Ext^N_{(\Lambda V,d_k)}(\mathbb{Q}, (\Lambda V, d_k))$ (resp. of $Ext^N_{(\Lambda V,d)}(\mathbb{K}, (\Lambda V, d))$).

Case 1 : Assume $ev_{(\Lambda V,d_k)} \neq 0$. Then $(\Lambda V, d_k)$ is an elliptic cdga [14]. By homogeneity of d_k, its fundamental class $\omega_0 = ev_{(\Lambda V,d_k)}[f_0] = [f_0(1)]$, where $f_0(1) \in (\Lambda^p V)^{p+q}$. Note that the cocycle representing $1_{\mathbb{Q}} \in \Lambda V \otimes \Lambda(sV)$ is necessarily $1 =: 1_{\mathbb{Q}} \in V^0 = \mathbb{Q}$. Therefore, $e(\Lambda V, d_k) = p$. By [10, Theorem 5], we conclude that $e(\Lambda V, d) = p = \mathrm{R}(\Lambda V, d)$.

Case 2 : Assume $ev_{(\Lambda V,d_k)} = 0$. In this case, $(\Lambda V, d_k)$ is not elliptic, but, since $\dim V < \infty$ we still have $\dim H^N(\Lambda V, d_k) < \infty$. By [2, Theorem 2], there is a unique basis element in $H^N(\Lambda V, d_k)$ that survives to the term $E_\infty^{*,*}(\Lambda V)$. Since $(\Lambda V, d)$ is elliptic (and hence a Poincaré duality algebra) with formal dimension N, we may identify the one-dimensional vector spaces $E_\infty^{*,*}(\Lambda V)$ and $H^N(\Lambda V, d)$. Let ω denotes the generating element resulting from that identification and put, using (13), $e(\Lambda V, d) = p'$ and (p', q'), the associated bidegree i.e. such that $p' + q' = N$. It results that $E_\infty^{*,*}(\Lambda V) = E_\infty^{p',q'}(\Lambda V)$. Once again, since $(\Lambda V, d)$

is elliptic, the evaluation map $ev_{(\Lambda V,d)} : Ext^{*,*}_{(\Lambda V,d)}(\mathbb{Q},(\Lambda V,d)) \to H^N(\Lambda V,d)$ is non-zero. By Remark 1, the induced map $E^{*,*}_\infty(ev) : E^{*,*}_\infty(A) \to E^{p',q'}_\infty(\Lambda V)$ is also non-zero. It follows that $E^{*,*}_\infty(A) = E^{p',q'}_\infty(A)$, and consequently, $\mathrm{R}(\Lambda V,d) = p' = e(\Lambda V,d)$.

QED

Next, we introduce the *Ext-Ginsburg invariant* $\mathrm{L}_0(X)$ (cf. §1), defined in the same spirit of the Ginsburg's invariant $l_0(X)$, which was originally defined using the spectral sequence (1) [9]. By the isomorphism between (1) and (2) [3, Proposition 9.1], $l_0(X)$ has the following algebraic characterization :

$$l(\Lambda V,d) = sup\{j \geq 0 \mid d_j \neq 0\} = min\{m \mid E^{*,*}_{m+1}(\Lambda V) = E^{*,*}_\infty(\Lambda V)\}$$

where d_j denotes the differential on the $j-th$ page in the spectral sequence (2).

For our more general setting, we define :

Definition 1. For a field $\mathbb{K}$ of arbitrary characteristic and a simply connected finite type CW-complex X, the *the Ext-Ginsburg invariant* of X over $\mathbb{K}$ is defined as :

$$\mathrm{L}_\mathbb{K}(X) = sup\{j \mid \delta_j \neq 0\},$$

where δ_j denotes the differential of the j^{th} page of the spectral sequence (6) with $\mathbb{K}$ replacing $\mathbb{Q}$. Equivalently,

$$\mathrm{L}_\mathbb{K}(X) = min\{m \mid E^{*,*}_{m+1} = E^{*,*}_\infty\}.$$

When $\mathbb{K} = \mathbb{Q}$, using the minimal Sullivan model $(\Lambda V,d)$ of X, we similarly define :

$$\mathrm{L}(\Lambda V,d) = sup\{j \mid d_j \neq 0\} = min\{m \mid E^{*,*}_{m+1}(A) = E^{*,*}_\infty(A)\},$$

where δ_j denotes the differential on the $j-th$ page of the spectral sequence (4). By the isomorphism between (6) and (4) established in [15, Theorem 1.1], we obtain the algebraic characterization of $\mathrm{L}_\mathbb{Q}(X)$ which we denote $\mathrm{L}_0(X)$ as follows:

$$\mathrm{L}_0(X) = \mathrm{L}(\Lambda V,d).$$

We now reccal and prove our second main result.

Theorem 4. *(Theorem 1.2) Let X be a rationally elliptic space. Then:*

$$\mathrm{L}_0(X) = l_0(X).$$

Proof. Since $(\Lambda V, d)$ is elliptic and a Gorenstein algebra, the evaluation map $ev_{(\Lambda V,d)}$ is non-zero. Therefore, it induces a non-zero isomorphism of one-dimensional spaces $E_\infty^{p,q}(ev) : E_\infty^{p,q}(A) \to E_\infty^{p,q}(\Lambda V)$ for some unique bidegree (p, q). By Remark 1 and the definition of $\mathrm{L}_0(X)$, this isomorphism is precisely

$$E^{p,q}_{\mathrm{L}_0(X)+1}(ev) : E^{p,q}_{\mathrm{L}_0(X)+1}(A) \to E^{p,q}_{\mathrm{L}_0(X)+1}(\Lambda V).$$

Thus, $E_\infty^{p,q}(\Lambda V) = E^{p,q}_{\mathrm{L}_0(X)+1}(\Lambda V)$. By the definition of $l_0(X)$, we conclude that $l_0(X) \leq \mathrm{L}_0(X)$.

On the other hand, by the identifications made above, $H^N(\Lambda V, d) = E_\infty^{p,q}(\Lambda V) = E^{p,q}_{l_0(X)+1}(\Lambda V)$ is generated by the fundamental class $\omega \in H^N(\Lambda V, d)$. However, $\omega = [f(1)] = ev_{(\Lambda V,d)}([f])$ for some $[f] \in Ext^N_{(\Lambda V,d)}(\mathbb{Q}, (\Lambda V, d)) = E_\infty^{p,q}(A)$. By Remark 1, $[f] \in E^{p,q}_{l_0(X)+1}(A)$, so $E_\infty^{p,q}(A) = E^{p,q}_{l_0(X)+1}(A)$. Consequently, by the definition of $\mathrm{L}_0(X)$, we obtain $\mathrm{L}_0(X) \leq l_0(X)$. QED

Remark 2. Using the spectral sequence (11) (resp. (12)), we may define $l_0(X)$ (resp. $\mathrm{L}_0(X)$) in terms of the minimal free model (TW, d) of $C^*(X, \mathbb{Q})$. This model is obtained [5] as the dual bar construction $\Omega(TV, d)$ of the Adams-Hilton model $(TV, d) \overset{\simeq}{\to} C_*(\Omega(X), \mathbb{Q})$ of X. Recall that in $(T(V), d)$, each V_i has a basis indexed by the $i + 1$-cells of X.

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