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An Analysis of Quasilinear Elliptic Systems with L^{∞}-Type Data

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Abstract. The present study establishes the existence and uniqueness of a solution of weak energy for a boundary value problem within a smooth, bounded, open domain Ω in $\mathbb{R}^n$ where $n \geq 3$. The problem is defined by the following equation:

$$\begin{cases} -\operatorname{div}\left[a(z,\upsilon,D\upsilon)\right] + |\upsilon|^{p(z)-2}\upsilon = f & \text{in } \Omega, \\ \upsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

where the function f is constrained to lie within the space $L^{\infty}(\Omega;\mathbb{R}^m)$. The proof of existence relies on the utilization of the concept of Young measures.

Keywords: Quasilinear elliptic systems, weak energy solution, Young measure, $p(z)$-variable exponents.

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1 Introduction and presentation of findings

Let Ω be an open bounded subset of $\mathbb{R}^n$, with the smooth boundary $\partial\Omega$. The concept of Young measure was introduced by Norbert Hungerbühler, who examined it in [23] concerning the Dirichlet problem for the quasilinear elliptic system :

$$(\mathcal{P}_f) \begin{cases} -\operatorname{div}\left[\sigma(z,\upsilon(z),D\upsilon(z))\right] = f & \text{in } \Omega, \\ \upsilon(z) = 0 & \text{on } \partial\Omega, \end{cases}$$

for arbitrary right hand side f belongs to $W^{-1,p'}(\Omega)$. They have proved the existence of a weak solution under classical regularity, growth and coercivity conditions but with only very mild monotonicity assumptions and Young measures to achieve the result, (see also [14, 15]). Georg Dolzmann, Norbert Hungerbühler and Stefan Müller have studied in [14] the existence of a solution υ for the nonlinear elliptic system :

$$(\mathcal{P}_\mu)\ \begin{cases} -\operatorname{div}[\sigma(z,\upsilon,D\upsilon)]=\mu & \text{in } \mathcal{D}'(\Omega),\\ \upsilon=0 & \text{on } \partial\Omega,\end{cases}$$

where μ is Radon measure on Ω with finite mass. Shulin Zhou [31] proposed the following sign condition:

$$a_i(z,\upsilon,\hbar)\cdot\hbar_i\geq 0 \quad \text{for } i=1,\dots,m,$$

as an alternative to the angle condition:

$$a(z,\upsilon,\hbar):N\hbar\geq 0,$$

which was assumed in [23]. This condition was utilized to establish the existence and regularity of solutions to $(\mathcal{P}_f)$ with $f=\mu\in\mathcal{M}(\Omega;\mathbb{R}^m)$. For additional findings, we suggest consulting [12, 6, 24, 25, 4] and [16, 5, 3, 2, 1, 17, 10] where the utilization of Young measures is explored for a range of quasilinear systems.

The aim of this paper is to study the existence of weak energy solutions of the boundary value problems for quasilinear elliptic systems of the form

$$(\mathcal{P})\ \begin{cases} -\operatorname{div}[a(z,\upsilon,D\upsilon)]+|\upsilon|^{p(z)-2}\upsilon=f & \text{in } \Omega,\\ \upsilon=0 & \text{on } \partial\Omega,\end{cases}$$

where Ω is a bounded open domain in $\mathbb{R}^n (n\geq 3)$ with a smooth boundary $\partial\Omega$ and f belongs to $L^\infty(\Omega;\mathbb{R}^m)$. Here $\upsilon:\Omega\to\mathbb{R}^m,\ m\in\mathbb{N}^*$, is a vector-valued function and $D\upsilon$ is the Jacobian matrix of υ given by

$$D\upsilon(z)=(D_1\upsilon(z),D_2\upsilon(z),\dots,D_n\upsilon(z)) \quad \text{with} \quad D_i=\partial/\partial_i(z_i).$$

We denote by $\mathbb{M}^{m\times n}$ the real space of all $m\times n$ matrices equipped with the inner product $\hbar:\eta=\sum_{i,j}\hbar_{ij}\eta_{ij}$ for all $\hbar,\eta\in\mathbb{M}^{m\times n}$.

Let us consider a function denoted as $a:\Omega\times\mathbb{R}^m\times\mathbb{M}^{m\times n}\to\mathbb{M}^{m\times n}$, which is a Carathéodory function. This implies that for every $(s,\hbar)\in\mathbb{R}^m\times\mathbb{M}^{m\times n}$, the function $z\mapsto a(z,w,\hbar)$ is measurable, and for almost every $z\in\Omega$, the function $(w,\hbar)\mapsto a(z,w,\hbar)$ is continuous. Moreover, the function $\hbar\mapsto a(z,\upsilon,\hbar)$

is continuously differentiable, satisfying the following conditions in conjunction with a convex and C^1-mapping denoted as $\mathcal{A} : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$, we have

$$a(z, \upsilon, \hbar) = \frac{\partial}{\partial \hbar}\mathcal{A}(z, \upsilon, \hbar) \tag{1.1}$$

and

$$\mathcal{A}(z, \upsilon, 0) = 0, \tag{1.2}$$

for almost every $z \in \Omega$ and all $\upsilon \in \mathbb{R}^m$.
Moreover, we assume that there exist $0 \leq \alpha_1(z) \in L^{p'(z)}(\Omega)$

$$|a(z, w, \hbar)| \leq \alpha_1(z) + |w|^{p(z)-1} + |\hbar|^{p(z)-1}, \tag{1.3}$$

for almost every $z \in \Omega$ and for every $(w, \hbar) \in \mathbb{R}^m \times \mathbb{M}^{m\times n}$. In addition, the mapping $\hbar \to a(z, w, \hbar)$ is monotone, i.e.,

$$(a(z, w, \hbar) - a(z, w, \eta)) : (\hbar - \eta) > 0, \quad \forall \hbar, \eta \in \mathbb{M}^{m\times n}. \tag{1.4}$$

Finally, the following inequality holds:

$$|\hbar|^{p(z)} \leq a(z, w, \hbar) : \hbar \leq p(z)\mathcal{A}(z, w, \hbar). \tag{1.5}$$

In this paper, the source term in $(\mathcal{P})$ is assumed to be in $L^\infty(\Omega; \mathbb{R}^m)$ and a to satisfy conditions (1.1)-(1.5). The primary goal is to demonstrate the existence and uniqueness of a weak energy solution using the concept of Young measure and energy functionals. Furthermore, a is assumed to be the third argument derivative of another function $\mathcal{A}$. This assumption is essential to establish an energy functional for the problem and subsequently minimize it to obtain a weak solution. The key outcome of the paper lies in validating the adequate assumptions for such minimization. An exemplary instance falling within the scope of our assumptions (1.1) -(1.5) is illustrated by the subsequent $p(z)$-Laplacian problem:

$$\mathcal{A}(z, \upsilon, \hbar) = \frac{1}{p(z)}|\hbar|^{p(z)}, \quad a(z, \upsilon, \hbar) = |\hbar|^{p(z)-2}\hbar.$$

The rest of this document is organized as follows: Section 2 provides a concise overview of Young measures, while Section 3 is dedicated to articulating the existence result and its proof.

2 Mathematical Preliminaries

We will review the necessary notations, definitions, and properties of our function spaces, which can be found in references such as [11, 21, 27, 29, 30], as well as provide an overview of Young measures, as explained in references such as [9, 13, 26]. For any open bounded subset Ω of $\mathbb{R}^n$, where $n \geq 2$, we define $C_+(\overline{\Omega}) = \{p(z);\ p(z) \in C(\overline{\Omega}),\ p(z) > 1 \text{ for any } z \in \Omega\}$. For every $p \in C_+(\bar{\Omega})$, we denote

$$p^- = \inf_{z\in\Omega} p(z) \text{ and } p^+ = \sup_{z\in\Omega} p(z).$$

The Sobolev space $W^{1,p(z)}(\Omega;\mathbb{R}^m)$ consists of all functions υ in the Lebesgue space

$$L^{p(z)}(\Omega;\mathbb{R}^m) = \left\{\upsilon : \Omega \to \mathbb{R}^m \text{ measurable} : \int_\Omega |\upsilon(z)|^{p(z)}\, \mathrm{d}z < \infty\right\},$$

such that $D\upsilon \in L^{p(z)}(\Omega;\mathbb{M}^{m\times n})$. The space $L^{p(z)}(\Omega;\mathbb{R}^m)$ is endowed with the norm

$$\|\upsilon\|_{L^{p(z)}(\Omega;\mathbb{R}^m)} := \|\upsilon\|_{p(z)} = \inf\left\{\beta > 0, \int_\Omega \left|\frac{\upsilon(z)}{\beta}\right|^{p(z)} \mathrm{d}z \leq 1\right\},$$

it is a Banach space. Moreover, it is reflexive if and only if $1 < p^- \leq p^+ < \infty$. Its dual is defined by $L^{p'(z)}(\Omega;\mathbb{R}^m)$ where $\frac{1}{p(z)} + \frac{1}{p'(z)} = 1$. For any $\upsilon \in L^{p(z)}(\Omega;\mathbb{R}^m)$ and $w \in L^{p'(z)}(\Omega;\mathbb{R}^m)$, the generalized Hölder inequality

$$\left|\int_\Omega \upsilon w\, \mathrm{d}z\right| \leq \left(\frac{1}{p^-} + \frac{1}{p^+}\right) \|\upsilon\|_{p(z)} \|w\|_{p'(z)},$$

holds true. The space $W^{1,p(z)}(\Omega;\mathbb{R}^m)$ is endowed with the norm

$$\|\upsilon\|_{1,p(z)} = \|\upsilon\|_{p(z)} + \|D\upsilon\|_{p(z)}.$$

Proposition 1. *([28, 18]) We denote* $\sigma(\upsilon) = \int_\Omega |\upsilon|^{p(z)} dz, \forall \upsilon \in L^{p(z)}(\Omega;\mathbb{R}^m)$. *If* $\upsilon_k, \upsilon \in L^{p(z)}(\Omega;\mathbb{R}^m)$ *and* $p^+ < \infty$*, then*

i) $\|\upsilon\|_{p(z)} < 1 (= 1; > 1) \Leftrightarrow \sigma(\upsilon) < 1 (= 1; > 1)$.

ii) $\|\upsilon\|_{p(z)} > 1 \Rightarrow \|\upsilon\|_{p(z)}^{p^-} \leq \sigma(\upsilon) \leq \|\upsilon\|_{p(z)}^{p^+}$; $\|\upsilon\|_{p(z)} < 1 \Rightarrow \|\upsilon\|_{p(z)}^{p^+} \leq \sigma(\upsilon) \leq \|\upsilon\|_{p(z)}^{p^-}$.

iii) $\|\upsilon_k\|_{p(z)} \to 0 \Leftrightarrow \sigma(\upsilon_k) \to 0$; $\|\upsilon_k\|_{p(z)} \to +\infty \Leftrightarrow \sigma(\upsilon_k) \to +\infty$.

Let us define $W_0^{1,p(z)}(\Omega;\mathbb{R}^m)$ as the closure of $C_0^\infty(\Omega;\mathbb{R}^m)$ in $W^{1,p(z)}(\Omega;\mathbb{R}^m)$, and its dual space as $W^{-1,p'(z)}(\Omega;\mathbb{R}^m)$. Here, we denote

$$p^*(z) := \begin{cases} \frac{np(z)}{n-p(z)} & \text{if } p(z) < n \\ \infty & \text{if } p(z) \geq n. \end{cases}$$

As stated in the introduction, we use the tool of Young measures to prove the existence result. This concept of Young measures is a nice tool to understand and control difficulties that arises when weak convergence does not behave as one desires with respect to nonlinear functionals and operators. For convenience of the readers not familiar with this concept, we give an overview needed in this paper. See [8, 17, 22] for more details.

By $C_0(\mathbb{R}^m)$ we denote the set of functions $g \in C(\mathbb{R}^m)$ satisfying the condition $\lim_{|\lambda|\to\infty} g(\lambda) = 0$. Its dual can be identified with the space of signed Radon measures with finite mass denoted by $\mathcal{M}(\mathbb{R}^m)$. The related duality pairing is given by

$$\langle \nu, g\rangle = \int_{\mathbb{R}^m} g(\lambda)d\nu(\lambda) \quad \text{for } \nu : \Omega \to \mathcal{M}(\mathbb{R}^m)$$

Lemma 1. *(See p. 19 in [19]) Assume that the sequence $\{\omega_j\}_{j\geq 1}$ is bounded in $L^\infty(\Omega;\mathbb{R}^m)$. Then there exist a subsequence $\{\omega_k\}_{k\geq 1} \subset \{\omega_j\}_{j\geq 1}$ and a Borel probability measure σ_z on $\mathbb{R}^m$ for a.e. $z \in \Omega$ such that for each $g \in C(\mathbb{R}^m)$, we have*

$$g(\omega_k) \rightharpoonup^* \bar{g} \text{ weakly in } L^\infty(\Omega),$$

where $\bar{g}(z) := \int_{\mathbb{R}^m} g(\gamma)d\sigma_z(\gamma)$ for a.e. $z \in \Omega$. We call $\{\sigma_z\}_{z\in\Omega}$ the family of Young measure associated with $\{\omega_k\}_{k\geq 1}$.

Lemma 2. *([22]) If $|\Omega| < \infty$ then*
$\omega_k \to \omega$ in measure $\Leftrightarrow$ $\sigma_z = \delta_{\omega(z)}$ for a.e. $z \in \Omega$.

Lemma 3. *([5],[13]) Suppose $\Omega \subset \mathbb{R}^n$ is a Lebesgue measurable set (which may not necessarily be bounded) and $w_j : \Omega \to \mathbb{R}^m$ is a sequence of Lebesgue measurable functions, where $j = 1, 2, \ldots$. Then, there exists a subsequence w_k and a family denoted as σ_z of non-negative Radon measures on $\mathbb{R}^n$, such that*

(i) $\|\sigma_z\|_{\mathcal{M}} := \int_{\mathbb{R}^m} d\sigma_z(\gamma) \leq 1$ *for almost every* $z \in \Omega$.

(ii) $g(w_k) \rightharpoonup^* \bar{g}$ *weakly in* $L^\infty(\Omega)$ *for any* $g \in C_0(\mathbb{R}^m)$, *where* $\bar{g} = \langle \sigma_z, g\rangle$ *and* $C_0(\mathbb{R}^m) = \{g \in C(\mathbb{R}^m) : \lim_{|w|\to\infty} |g(w)| = 0\}$.

(iii) If for any $R > 0$

$$\lim_{\ell\to\infty} \sup_{k\in\mathbb{N}} |\{z \in \Omega \cap B_R(0) : |w_k(z)| \geq \ell\}| = 0,$$

then $\|\sigma_z\|_{\mathcal{M}} = 1$ *for almost every* $z \in \Omega$*, and for any measurable* $\Omega' \subset \Omega$*, we have* $g(w_k) \rightharpoonup \bar{g} = \langle \sigma_z, g \rangle$ *weakly in* $L^1(\Omega')$ *for continuous* g *provided the sequence* $g(w_k)$ *is weakly precompact in* $L^1(\Omega')$*.*

Lemma 3 is the fundamental theorem of the Young measure, and the following Fatou-type lemma can be seen as its application and it is useful for us.

Lemma 4. *([5],[13]) Let* $\Psi : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$ *be a Carathéodory function and* $\omega_k : \Omega \to \mathbb{R}^m$ *a sequence of measurable functions such that* $D\omega_k$ *generates the Young measure* σ_z*, with* $\|\sigma_z\|_{\mathcal{M}(\mathbb{M}^{m\times n})} = 1$ *for almost every* $z \in \Omega$*, then*

$$\liminf_{k\to\infty} \int_\Omega \Psi(z, \omega_k, D\omega_k)\ \mathrm{d}z \geq \int_\Omega \int_{\mathbb{M}^{m\times n}} \Psi(z, \omega, \gamma) d\sigma_z(\gamma)\ \mathrm{d}z,$$

provided that the negative part $\Psi^-(z, D\omega_k(z))$ *is equiintegrable.*

3 Existence and uniqueness of weak energy solution for $f \in L^\infty(\Omega; \mathbb{R}^m)$

Before we proceed with the main finding of this study, let's provide the following definition for weak energy solutions of $(\mathcal{P})$.

Definition 1. A weak energy solution of $(\mathcal{P})$ is a function $\upsilon \in W_0^{1,p(z)}(\Omega; \mathbb{R}^m)$ such that

$$\int_\Omega (a(z, \upsilon, D\upsilon) : D\varphi)\ \mathrm{d}z + \int_\Omega |\upsilon|^{p(z)-2}\upsilon\varphi\ \mathrm{d}x = \int_\Omega f(z)\varphi\ \mathrm{d}z,$$

for all $\varphi \in W_0^{1,p(z)}(\Omega; \mathbb{R}^m)$.

The primary outcome is as follows.

Theorem 1. *Assume* $f \in L^\infty(\Omega; \mathbb{R}^m)$ *and* (1.1)-(1.5) *hold. Then there exists a unique weak energy solution of* $(\mathcal{P})$*.*

3.1 Existence

Let us define the energy functional $\mathcal{V} : W_0^{1,p(z)}(\Omega; \mathbb{R}^m) \to \mathbb{R}$ by

$$\mathcal{V}(\upsilon) = \int_\Omega \mathcal{A}(z, \upsilon, D\upsilon)\ \mathrm{d}z + \int_\Omega \frac{1}{p(z)}|\upsilon|^{p(z)}\ \mathrm{d}z - \int_\Omega f\upsilon\ \mathrm{d}z.$$

Proposition 2. *The functional* $\mathcal{V}$ *is well-defined on* $W_0^{1,p(z)}(\Omega;\mathbb{R}^m)$ *and* $\mathcal{V}\in C^1\left(W_0^{1,p(z)}(\Omega;\mathbb{R}^m),\mathbb{R}\right)$ *with the derivative given by*

$$\langle \mathcal{V}'(\upsilon),\varphi\rangle=\int_\Omega (a(z,\upsilon,D\upsilon):D\varphi)\ \mathrm{d}z+\int_\Omega |\upsilon|^{p(z)-2}\upsilon\varphi\ \mathrm{d}z-\int_\Omega f\varphi\ \mathrm{d}z,$$

for all $\varphi\in W_0^{1,p(z)}(\Omega;\mathbb{R}^m)$.

Proof. For any $z\in\Omega,\upsilon\in W_0^{1,p(z)}(\Omega;\mathbb{R}^m)$ and $\hbar\in\mathbb{M}^{m\times n}$, we have

$$\mathcal{A}(z,\upsilon,\hbar)=\int_0^1\frac{d}{dt}\mathcal{A}(z,\upsilon,t\hbar)\ \mathrm{d}t=\int_0^1 a(z,\upsilon,t\hbar):\hbar\ \mathrm{d}t.$$

Using the assumption (1.3), we obtain

$$\begin{aligned}\mathcal{A}(z,\upsilon,\hbar)&\le\int_0^1\left(\alpha_1(z)+|\upsilon|^{p(z)-1}+t^{p(z)-1}|\hbar|^{p(z)-1}\right)|\hbar|\ \mathrm{d}t \\ &\le \alpha_1(z)|\hbar|+|\upsilon|^{p(z)-1}|\hbar|+\frac{1}{p(z)}|\hbar|^{p(z)}.\end{aligned}\tag{3.6}$$

By applying the Hölder inequality, we can deduce that

$$0\le\int_\Omega|\mathcal{A}(z,\upsilon,D\upsilon)|\ \mathrm{d}z\le\|\alpha_1\|_{p'(z)}\|D\upsilon\|_{p(z)}+\|\upsilon\|_{p(z)}^{p(z)-1}\|D\upsilon\|_{p(z)}+\frac{1}{p(z)}\|D\upsilon\|_{p(z)}^{p(z)}$$

and

$$\int_\Omega|f\upsilon|\ \mathrm{d}x\le\|f\|_{p'(z)}\|\upsilon\|_{p(z)}.$$

From this, we can deduce that $\mathcal{V}$ is well-defined on $W_0^{1,p(z)}(\Omega;\mathbb{R}^m)$.
Let us fix $z\in\Omega$ and $0<|r|<1$. By the mean value theorem, there exists $\theta\in[0,1]$ such that

$$\left\{\begin{array}{ll}|a(z,\upsilon,D\upsilon+\theta D\varphi)||D\varphi| & =\frac{|\mathcal{A}(z,\upsilon,D\upsilon+rD\varphi)-\mathcal{A}(z,\upsilon,D\upsilon)||D\varphi|}{|r|}\\ & \le\left(\alpha_1(z)+|\upsilon|^{p(z)-1}+|D\upsilon+\theta rD\varphi|^{p(z)-1}\right)|D\varphi|\\ & \le\Big(\alpha_1(z)+|\upsilon|^{p(z)-1}+2^{p(z)-2}\big(|D\upsilon|^{p(z)-1}\\ & +(\theta r)^{p(z)-1}|D\varphi|^{p(z)-1}\big)\Big)|D\varphi|.\end{array}\right.$$

Using Hölder's inequality, we obtain

$$\int_\Omega\alpha_1(z)|D\varphi|\ \mathrm{d}z\le\|\alpha_1\|_{p'(z)}\|D\varphi\|_{p(z)},$$

then

$$\int_\Omega |Dv|^{p(z)-1}|D\varphi| \, \mathrm{d}z \le \left(\int_\Omega |Dv|^{(p(z)-1)p'(z)} \, \mathrm{d}z + 1 \right) \|D\varphi\|_{p(z)}$$
$$\le \left(\|D\varphi\|_{p(z)}^{p^-} + \|D\varphi\|_{p(z)}^{p^-} + 1 \right) \|D\varphi\|_{p(z)}$$

and

$$\int_\Omega |D\varphi|^{p(z)-1}|D\varphi| \, \mathrm{d}z \le \int_\Omega |D\varphi|^{p(z)} \, \mathrm{d}z \le \|D\varphi\|_{p(z)}^{p^-} + \|D\varphi\|_{p(z)}^{p^+}.$$

From these inequalities, we infer that

$$\left(\alpha_1(z) + |v|^{p(z)-1} + 2^{p(z)-2} \left(|Dv|^{p(z)-1} + (\theta r)^{p(z)-1} |D\varphi|^{p(z)-1} \right) \right) |D\varphi| \in L^1(\Omega).$$

Thanks to the Lebesgue theorem, it can be inferred that

$$\langle \mathcal{V}'(v), \varphi \rangle = \int_\Omega a(z, v, Dv) : D\varphi \, \mathrm{d}z + \int_\Omega |v|^{p(z)-2} v\varphi \, \mathrm{d}z - \int_\Omega f\varphi \, \mathrm{d}z.$$

Let's assume that $v_k \to v$ in $W_0^{1,p(z)}(\Omega;\mathbb{R}^m)$. Consequently, $(v_k)_k$ forms a bounded sequence in $W_0^{1,p(z)}(\Omega;\mathbb{R}^m)$. As mentioned in Lemma 1 there exists a Young measure ν_z generated by Dv_k in $L^{p(z)}(\Omega;\mathbb{M}^{m\times n})$ satisfying the conditions outlined in Lemma 3. By utilizing (1.4) and [5, Lemma 5.3], we can deduce that

$$\begin{cases} 0 & \le (a(z,v,\gamma) - a(z,v,Dv+\tau\hbar)) : (\gamma - Dv - \tau\hbar) \\ & = a(z,v,Dv) : (\gamma - Dv) \\ & -a(z,v,\gamma) : \tau\hbar - a(z,v,Dv+\tau\hbar) : (\gamma - Dv - \tau\hbar), \end{cases}$$

which gives

$$-a(z,v,\gamma) : \tau\hbar \ge -a(z,v,Dv) : (\gamma - Dv) + a(z,v,Dv+\tau\hbar) : (\gamma - Dv - \tau\hbar),$$

for every $\gamma, \hbar \in \mathbb{M}^{m\times n}$ and $\tau \in \mathbb{R}$. We have $\hbar \mapsto a(z,v,\hbar)$ is continuously differentiable, hence we can write

$$\begin{cases} a(z,v,Dv+\tau\hbar) : (\gamma - Dv - \tau\hbar) \\ = a(z,v,Dv+\tau\hbar) : (\gamma - Dv) - a(z,u,Dv+\tau\hbar) : \tau\hbar \\ = a(z,v,Dv) : (\gamma - Dv) + \tau((\zeta a(z,v,Dv)\hbar) : (\gamma - Dv) - a(z,v,Dv) : \hbar) + o(\tau), \end{cases}$$

where ζ is the derivative of a with respect to its third variable. Therefore,

$$-a(z,v,\gamma) : \tau\hbar \ge \tau((\zeta a(z,v,Dv)\hbar) : (\gamma - Dv) - a(z,v,Dv) : \hbar) + o(\tau),$$

which implies, since τ is arbitrary in $\mathbb{R}$, that

$$a(z,\upsilon,\gamma):\hbar = a(z,\upsilon,D\upsilon):\hbar + (\zeta a(z,\upsilon,D\upsilon)\hbar):(D\upsilon-\gamma), \tag{3.7}$$

on the support of σ_z. Since $(a(z,\upsilon_k,D\upsilon_k))_k$ is equiintegrable by (1.3) and $(\upsilon_k)_k$ is bounded in $W_0^{1,p(z)}(\Omega;\mathbb{R}^m)$, it follows that its weak L^1-limit $\bar{a}$ is given by

$$\begin{cases} \bar{a}(z) & := \displaystyle\int_{\mathbb{M}^{m\times n}} a(z,\upsilon,\gamma)d\sigma_z(\gamma) \\ & \overset{(3.2)}{=} a(z,\upsilon,D\upsilon)\underbrace{\displaystyle\int_{\operatorname{supp}\sigma_z} d\nu_z(\gamma)}_{:=1} + (\zeta a(z,\upsilon,D\upsilon))^t \underbrace{\displaystyle\int_{\operatorname{supp}\sigma_z}(D\upsilon-\gamma)d\sigma_z(\gamma)}_{:=0} \\ & = a(z,\upsilon,D\upsilon). \end{cases}$$

Moreover, as $L^{p'(z)}(\Omega;\mathbb{M}^{m\times n})$ is reflexive, it follows that $(a(z,\upsilon_k,D\upsilon_k))_k$ converges in $L^{p'(z)}(\Omega;\mathbb{M}^{m\times n})$ and its weak $L^{p(z)}$-limit is also $\bar{a}(z) = a(z,\upsilon,D\upsilon)$. This and the Hölder inequality imply

$$\left|\left\langle \mathcal{V}'(\upsilon_k) - \mathcal{V}'(\upsilon), \varphi\right\rangle\right| \le \int_\Omega |a(z,\upsilon_k,D\upsilon_k) - a(z,\upsilon,D\upsilon)|\,|D\varphi|\;\mathrm{d}z$$

and thus,

$$\left\|\mathcal{V}'(\upsilon_k) - \mathcal{V}'(\upsilon)\right\| \le \|a(z,\upsilon_k,D\upsilon_k) - a(z,\upsilon,D\upsilon)\|_{p'(z)} \longrightarrow 0$$

as $k \to \infty$. QED

Lemma 5. *The functional $\mathcal{V}$ is bounded from below, coercive and weakly lower semi-continuous.*

Proof. According to (3.6) and Hölder's inequality, it is obvious that $\mathcal{V}$ is bounded from below. By utilizing (1.5), we can deduce the following inequalities:

$$\begin{cases} \mathcal{V}(\upsilon) & = \displaystyle\int_\Omega \mathcal{A}(z,\upsilon,D\upsilon)\;\mathrm{d}z + \int_\Omega \frac{1}{p(z)}|\upsilon|^{p(z)}\;\mathrm{d}z - \int_\Omega f\upsilon\;\mathrm{d}z \\ & \ge \frac{1}{p(z)}\displaystyle\int_\Omega |D\upsilon|^{p(z)}dx - \|f\|_{p'(z)}\|\upsilon\|_{p(z)}, \\ & \ge \frac{1}{p(z)}\displaystyle\int_\Omega |D\upsilon|^{p(z)}\;\mathrm{d}z - c\|\upsilon\|_{1,p(z)} \longrightarrow +\infty, \end{cases}$$

as $\|\upsilon\|_{1,p(z)} \to \infty$, due to the continuous embedding of $W_0^{1,p(z)}(\Omega;\mathbb{R}^m)$ within $L^{p(z)}(\Omega;\mathbb{R}^m)$. Thus, $\mathcal{V}$ can be considered coercive. Let $(\upsilon_k) \subset W_0^{1,p(z)}(\Omega;\mathbb{R}^m)$ be a sequence that weakly converges to υ in $W_0^{1,p(z)}(\Omega;\mathbb{R}^m)$. Consequently,

$\upsilon_k \to \upsilon$ in $L^{p(z)}(\Omega;\mathbb{R}^m)$ and in measure on Ω (for a subsequence indexed by k), owing to the compact embedding of $W_0^{1,p(z)}(\Omega;\mathbb{R}^m)$ in $L^{p(z)}(\Omega;\mathbb{R}^m)$. Since $\sigma_z = \delta_{D\upsilon(z)}$ for almost every $z \in \Omega$ by Lemma 3, Lemma 2 implies $D\upsilon_k \to D\upsilon$ in measure. Additionally, $(\mathcal{A}(z,\upsilon_k,D\upsilon_k))_k$ is equiintegrable by (3.6). This leads us to conclude from Lemma 4 that

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \mathcal{A}(z,\upsilon,\gamma)\ \mathrm{d}\sigma_z(\gamma)\ \mathrm{d}z \le \liminf_{k\to\infty} \int_\Omega \mathcal{A}(z,\upsilon_k,D\upsilon_k)\ \mathrm{d}z. \tag{3.8}$$

Moreover, under assumption (1.4) and the relation $a(z,\upsilon,\hbar) = \frac{\partial}{\partial\hbar}\mathcal{A}(z,\upsilon,\hbar)$, it follows that $\hbar \mapsto \mathcal{A}(z,\upsilon,\hbar)$ is convex, which implies,

$$\underbrace{\mathcal{A}(z,\upsilon,\gamma)}_{=:Q(\gamma)} \ge \underbrace{\mathcal{A}(z,\upsilon,D\upsilon) + a(z,\upsilon,D\upsilon) : (\gamma - D\upsilon)}_{=:S(\gamma)}, \quad \forall\gamma \in \mathbb{M}^{m\times n}.$$

Given that $\gamma \mapsto Q(\gamma)$ is a C^1-function by Proposition 2, for $\tau \in \mathbb{R}$, we have

$$\frac{Q(\gamma+t\hbar)-Q(\gamma)}{t} \le \frac{S(\gamma+t\hbar)-S(\gamma)}{t} \quad \text{for} \quad t<0$$

and

$$\frac{Q(\gamma+t\hbar)-Q(\gamma)}{t} \ge \frac{S(\gamma+t\hbar)-S(\gamma)}{t} \quad \text{for} \quad t>0.$$

Consequently, $\nabla Q = \nabla S$, i.e.,

$$\mathcal{A}(z,\upsilon,\gamma) = \mathcal{A}(z,\upsilon,D\upsilon) \quad \text{for all } \gamma \in \operatorname{supp}\sigma_z. \tag{3.9}$$

Returning to (3.8), it follows from (3.9) that

$$\begin{aligned}\int_\Omega \int_{\mathbb{M}^{m\times n}} \mathcal{A}(z,\upsilon,\gamma)\ \mathrm{d}\sigma_z(\gamma) &= \int_\Omega \int_{\operatorname{supp}\nu_z} \mathcal{A}(z,\upsilon,D\upsilon)\ \mathrm{d}\sigma_z(\gamma)\ \mathrm{d}z \\ &= \int_\Omega \mathcal{A}(z,\upsilon,D\upsilon)\ \mathrm{d}z \\ &\le \liminf_{k\to\infty} \int_\Omega \mathcal{A}(z,\upsilon_k,D\upsilon_k)\ \mathrm{d}z.\end{aligned}$$

Consequently, we can infer that

$$\mathcal{V}(\upsilon) \le \liminf_{k\to\infty} \mathcal{V}(\upsilon_k).$$

This implies that $\mathcal{V}$ is weakly lower semi-continuous, thus completing the proof. QED

As $\mathcal{V}$ is proper, weakly semi-continuous and coercive, it follows that $\mathcal{V}$ has a minimizer which is in fact a weak energy solution of $(\mathcal{P})$. Consequently, the main result's proof is comprehensive.

3.2 Uniqueness

Assume υ_1 and υ_2 are two weak solutions of the problem $(\mathcal{P})$. According to Definition 1 with υ_1 as a weak solution, we choose $\phi = \upsilon_1 - \upsilon_2$ to get

$$\int_\Omega a(z, \upsilon_1, D\upsilon_1) : D(\upsilon_1 - \upsilon_2)\, dz + \int_\Omega |\upsilon_1|^{p(z)-2}\upsilon_1(\upsilon_1 - \upsilon_2)\, dz = \int_\Omega f(z)(\upsilon_1 - \upsilon_2)\, dz. \quad (3.10)$$

Likewise, with υ_2 as a weak solution, we take $\phi = \upsilon_2 - \upsilon_1$, which yields

$$\int_\Omega a(z, \upsilon_2, D\upsilon_2) : D(\upsilon_2 - \upsilon_1)\, dz + \int_\Omega |\upsilon_2|^{p(z)-2}\upsilon_2(\upsilon_2 - \upsilon_1)\, dz = \int_\Omega f(z)(\upsilon_2 - \upsilon_1)\, dz. \quad (3.11)$$

By summing equations (3.10) and (3.11), we arrive at

$$\int_\Omega \Big(a(z, \upsilon_1, D\upsilon_1) - a(z, \upsilon_2, D\upsilon_2)\Big) : D(\upsilon_1 - \upsilon_2)\, dz + \int_\Omega \Big(|\upsilon_1|^{p(z)-2}\upsilon_1 - |\upsilon_2|^{p(z)-2}\upsilon_2\Big)(\upsilon_1 - \upsilon_2)\, dz = 0. \quad (3.12)$$

Using relation (1.4), we can deduce from equation (3.12) that

$$\int_\Omega \Big(|\upsilon_1(z)|^{p(z)-2}\upsilon_1(z) - |\upsilon_2(z)|^{p(z)-2}\upsilon_2(z)\Big)(\upsilon_1(z) - \upsilon_2(z))\, dz = 0. \quad (3.13)$$

Since $p^- > 1$, the following relation is true for any $\xi, \eta \in \mathbb{R}^m, \xi \neq \eta$ (See p. 4 in [20])

$$\Big(|\xi|^{p(z)-2}\xi - |\eta|^{p(z)-2}\eta\Big)(\xi - \eta) > 0. \quad (3.14)$$

Consequently, from equation (3.13), we have

$$\Big(|\upsilon_1(z)|^{p(z)-2}\upsilon_1(z) - |\upsilon_2(z)|^{p(z)-2}\upsilon_2(z)\Big)(\upsilon_1(z) - \upsilon_2(z))\, dz = 0, \text{ a.e. } z \in \Omega. \quad (3.15)$$

Employing relation (3.14), we conclude that

$$\upsilon_1(z) = \upsilon_2(z) \text{ a.e. } z \in \Omega. \tag{3.16}$$

Thus, this demonstrates the uniqueness of the solution.

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