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On ideal and subalgebra coefficients in a class of k-algebras

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Abstract. Let k be a commutative field with prime field k_0 and A a k-algebra. Moreover, assume that there is a k-vector space basis Ω of A that satisfies the following condition: for all $\omega_1, \omega_2 \in \Omega$, the product $\omega_1\omega_2$ is contained in the k_0-vector space spanned by Ω.

It is proven that the concept of *minimal field of definition* from polynomial rings and semigroup algebras can be generalized to the above class of (not necessarily associative) k-algebras. Namely, let U be a one-sided ideal in A or a k-subalgebra of A. Then there exists a smallest $k' \leq k$ with U—as one-sided ideal resp. as k-algebra—being generated by elements in the k'-vector space spanned by Ω.

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1 Introduction

For a polynomial ring $R := k[X_1, \ldots, X_n]$ over a commutative field k, it is well-known that to each ideal $I \subseteq R$ a unique so-called *minimal field of definition* can be associated (e. g. [9, Chapter I, §7, Lemma 2], [3, Chapter III, Theorem 7]). In other words, there is a smallest subfield $k' \leq k$ such that both I has a basis B with the coefficients of all polynomials in B being contained in k', and k' is a subfield of each field spanned over the prime field by the coefficients of a basis of I.

Robbiano and Sweedler [5] (cf. also [4]) show that generators of this minimal field of definition can be derived from a reduced Gröbner basis of the ideal in question. Using SAGBI—a subalgebra analogue of reduced Gröbner bases of ideals—they are able to carry over the notion of minimal field of definition from ideals to k-subalgebras of R [5, Theorem 3.9]. In [7, Theorem 19] and [8] it is shown, that the concept of minimal field of definition is in fact well-defined for one-sided ideals and k-subalgebras of arbitrary semigroup algebras.

This contribution shows that the concept of minimal field of definition naturally carries over to more general k-algebras than only semigroup algebras. For showing this, we define the minimal field of definition with respect to a fixed vector space basis Ω of the k-algebra in question.—For Ω being the canonical basis S of a semigroup algebra $k[S]$, this yields the established definitions, but with this basis-dependent approach, the existence of a minimal field of definition can be established for one-sided ideals and k-subalgebras of a significantly larger class of—not necessarily associative—k-algebras.

2 The minimal field of definition

Let k be a commutative field with prime field k_0, A a (not necessarily associative) k-algebra, and Ω a k-vector space basis of A. For subfields $k' \leq k$, we write $\langle\Omega\rangle_{k'-\mathrm{vec}}$ for the k'-vector space spanned by Ω, and we use the following terminology:

1 Definition. Let I be a left or right ideal in A. Then we call a subfield $k' \leq k$ *minimal field of definition* of I *with respect to* Ω if both

(i) there exists a subset $G' \subseteq \langle\Omega\rangle_{k'-\mathrm{vec}}$ such that I as left resp. right ideal is generated by G', and

(ii) for each $G \subseteq A$ generating I as a left resp. right ideal, the field k' is contained in the extension field of k_0 generated by the k-coefficients of the elements in G w.r.t. the basis Ω.

Analogously, for a k-subalgebra B of A, a subfield $k' \leq k$ is called *minimal field of definition* of A *with respect to* Ω if both

(i) there exists a subset $G' \subseteq \langle\Omega\rangle_{k'-\mathrm{vec}}$ such that B as k-algebra is generated by G', and

(ii) for each $G \subseteq A$ generating B as a k-algebra, the field k' is contained in the extension field of k_0 generated by the k-coefficients of the elements in G w.r.t. the basis Ω.

Note that for the special case of $A = k[S]$ being a semigroup algebra over k and $\Omega = S$, the above definition coincides with [8, Definition 1]; in particular for $S = \mathbb{N}_0^n$ we obtain the established definitions for polynomial rings. Also it is obvious, that the minimal field of definition of a one-sided ideal resp. of a k-subalgebra is either unique or does not exist at all. Carrying over the arguments from the proof of [8, Theorem 1], one can verify that the latter case cannot occur, if the vector space Ω is "suitably closed" under multiplication:

2 Theorem. *Let B be a one-sided ideal in A or a k-subalgebra of A. If for all $\omega_1, \omega_2 \in \Omega$ the inclusion $\omega_1\omega_2 \in \langle\Omega\rangle_{k_0-\mathrm{vec}}$ holds, then there exists a minimal field of definition $k' \leq k$ of B with respect to Ω.*

Proof. First, we choose $\Sigma \subseteq \Omega$ such that $\{\sigma + B : \sigma \in \Sigma\}$ is a basis of the k-vector space A/B. Next, denote by $\overline{\Sigma} := \Omega \setminus \Sigma$ the complement of Σ in Ω. Then, by construction of Σ, for each $\overline{\sigma} \in \overline{\Sigma}$ there are uniquely determined coefficients $\alpha_{\overline{\sigma}\sigma} \in k$ with

$$\overline{\sigma} - \sum_{\sigma\in\Sigma} \alpha_{\overline{\sigma}\sigma} \cdot \sigma \in B \tag{1}$$

and $\alpha_{\overline{\sigma}\sigma} = 0$ for all but finitely many $\sigma \in \Sigma$. Next, we define a field k' as

$$k' := k_0(\{\alpha_{\overline{\sigma}\sigma} : \overline{\sigma} \in \overline{\Sigma}, \sigma \in \Sigma\}). \tag{2}$$

We want to prove that k' is is the minimal field of definition of B: for this, we first have to show that B—as one-sided ideal resp. as k-algebra— can be generated by a subset $G_{k'} \subseteq \langle\Omega\rangle_{k'-\mathrm{vec}}$. For showing the latter, it is sufficient to prove that each element of B is already contained in the k-vector space spanned by all elements of the form (1). So choose

$$f = \sum_{\overline{\sigma}\in\overline{\Sigma}} \beta_{\overline{\sigma}} \cdot \overline{\sigma} + \sum_{\sigma\in\Sigma} \beta_\sigma \cdot \sigma \in B$$

arbitrary (in particular only finitely many coefficients $\beta_{\overline{\sigma}}, \beta_\sigma \in k$ are non-zero). By adjusting the coefficients in the second sum accordingly, we can write f in the form

$$f = \sum_{\overline{\sigma}\in\overline{\Sigma}} \beta_{\overline{\sigma}} \cdot \left(\overline{\sigma} - \sum_{\sigma\in\Sigma} \alpha_{\overline{\sigma}\sigma} \cdot \sigma\right) + \sum_{\sigma\in\Sigma} \gamma_\sigma \cdot \sigma$$

with $\alpha_{\overline{\sigma}\sigma}$ as in (1), $\gamma_\sigma \in k$, and all summations being finite. Now by construction, both f and the $\overline{\sigma} - \sum_{\sigma\in\Sigma} \alpha_{\overline{\sigma}\sigma} \cdot \sigma$ are contained in B. So from B being a k-vector space we conclude that $\sum_{\sigma\in\Sigma} \gamma_\sigma \cdot \sigma \in B$. Because of $\{\sigma + B : \sigma \in \Sigma\}$ being a k-vector space basis of A/B, we thus have $\gamma_\sigma = 0$ for all $\sigma \in \Sigma$, and

$$G_{k'} := \left\{\overline{\sigma} - \sum_{\sigma\in\Sigma} \alpha_{\overline{\sigma}\sigma} \cdot \sigma \;\middle|\; \overline{\sigma} \in \overline{\Sigma}\right\} \subseteq \langle\Omega\rangle_{k'-\mathrm{vec}}$$

generates B as k-vector space as desired.

Now let G be an arbitrary generating set of B as one-sided ideal resp. as k-algebra, and denote by Θ the set of all coefficients that occur when expressing each element of G as k-linear combination in Ω. Thus, we are left to verify the inclusion $k' \subseteq k_0(\Theta)$ where

$$k_0(\Theta) = k_0\Big(\{\alpha \mid \exists g \in G : g = \sum_{\omega\in\Omega} \lambda_\omega \cdot \omega \ \ , \text{ and } \alpha = \lambda_{\omega_0} \text{ for some } \omega_0 \in \Omega\}\Big).$$

To do so, we prove that for each $\overline{\sigma} \in \overline{\Sigma}$ the coefficients $\{\alpha_{\overline{\sigma}\sigma} : \sigma \in \Sigma\}$ from expression (1) are contained in $k_0(\Theta)$: fix $\overline{\sigma} \in \overline{\Sigma}$ arbitrary, then $\overline{\sigma} - \sum_{b\in\Sigma} \alpha_{\overline{\sigma}\sigma} \cdot \sigma \in B$. If B is a left or right ideal in A, then there is a representation

$$\overline{\sigma} - \sum_{\sigma\in\Sigma} \alpha_{\overline{\sigma}\sigma} \cdot \sigma = \begin{cases} \sum_{g\in G} a_g \cdot g \text{ with } a_g \in A \cup k, & \text{if } B \text{ is a left ideal.} \\ \sum_{g\in G} g \cdot a_g \text{ with } a_g \in A \cup k, & \text{if } B \text{ is a right ideal.} \end{cases} \tag{3}$$

For the case where G generates B as k-subalgebra of A, denote by $\mathrm{Mag}(X)$ the free non-associative magma generated by the indeterminates $X := \{X_g : g \in G\}$. By $k\{X\}$ we denote the free magma algebra consisting of all finite k-linear combinations of elements in $\mathrm{Mag}(X)$ (cf. [1]). Then each element of B can be obtained from a suitable $q \in k\{X\}$ through the "evaluation" $X_g \mapsto g$ ($g \in G$). Denoting this "evaluation" by $q(G)$, we can say that for G generating B as k-subalgebra there is a representation

$$\overline{\sigma} - \sum_{\sigma\in\Sigma} \alpha_{\overline{\sigma}\sigma} \cdot \sigma = q(G) \text{ with } q \in k\{X\}. \tag{4}$$

To complete the proof, it is sufficient to verify that

$$a_g \in \langle\Omega\rangle_{k_0(\Theta)-\mathrm{vec}} \cup k_0(\Theta) \text{ resp. } q \in k_0(\Theta)\{X\} \tag{5}$$

holds, as then the required inclusion $\{\alpha_{\overline{\sigma}\sigma} : \sigma \in \Sigma\} \subseteq k_0(\Theta)$ follows from equation (3) resp. (4) and the assumption that the product of any two basis elements $\omega_1, \omega_2 \in \Omega$ is contained in $\langle\Omega\rangle_{k_0-\mathrm{vec}}$.

To prove relation (5), we introduce new indeterminates for the non-zero coefficients of the a_g resp. of q:

- For B an ideal we either have $a_g \in A$ or both $a_g \notin A$ and $a_g \in k$.

  For $a_g = \sum_{\omega\in\Omega} \gamma_{g\omega} \cdot \omega \in A$ we set

  $$A_g := \sum_{\omega\in\Omega} Z_{g\omega} \cdot \omega$$

  with new indeterminates Z_η and $Z_\eta = 0$ if the corresponding coefficient γ_η is zero. In the same manner we replace each $a_g \in k$, $a_g \notin A$, by a new indeterminate $A_g := Z_g$.

- For B a k-subalgebra and $q = \sum_{t\in \mathrm{Mag}(X)} \gamma_t \cdot t$ we set

$$Q := \sum_{t\in \mathrm{Mag}(X)} Z_t \cdot t$$

with new indeterminates Z_η and $Z_\eta = 0$ if the corresponding coefficient γ_η is zero.

Now replacing in the sum $\sum_{g\in G} A_g \cdot g$ (resp. $\sum_{g\in G} g \cdot A_g$) resp. in $Q(G)$ the elements $g \in G$ by their corresponding expressions $g = \sum_{\omega\in\Omega} \lambda_\omega \cdot \omega$, $\lambda_\omega \in k_0(\Theta)$, we get an equation of the form

$$\left.\begin{array}{l} \sum_{g\in G} A_g \cdot g \\ \text{resp. } \sum_{g\in G} g \cdot A_g \\ \text{resp. } Q(G) \end{array}\right\} = \sum_{\overline{\sigma}\in\overline{\Sigma}} p_{\overline{\sigma}} \cdot \overline{\sigma} + \sum_{\sigma\in\Sigma} p_\sigma \cdot \sigma$$

where the p_μ are polynomials in the indeterminates Z_η of total degree ≤ 1 and where the coefficients of each p_μ are contained in $k_0(\Theta)$.—As we will only be interested in the values of the polynomials p_μ when specializing the indeterminates Z_η to values in k, we can assume that multiplication with the p_μ is associative and that the p_μ commute with the elements in $S = \Sigma \cup \overline{\Sigma}$.

Owing to equation (3) we know that the linear system of equations

$$\begin{aligned} p_{\overline{\sigma}} &= 1 \\ p_{\overline{\sigma}'} &= 0 \quad \text{for } \overline{\sigma}' \in \overline{\Sigma} \setminus \{\overline{\sigma}'\} \end{aligned} \tag{6}$$

has a solution over k. Thus, as all coefficients of these linear equations lie in $k_0(\Theta)$ we conclude that there is also a solution over $k_0(\Theta)$. If we substitute this solution for the indeterminates Z_η into the A_g resp. into Q we obtain an equation

$$\left.\begin{array}{l} \sum_{g\in G} a'_g \cdot g \\ \text{resp. } \sum_{g\in G} g \cdot a'_g \\ \text{resp. } q'(G) \end{array}\right\} = \overline{\sigma} - \sum_{\sigma\in\Sigma} \epsilon_{\overline{\sigma}\sigma} \cdot \sigma \tag{7}$$

where all $\epsilon_{\overline{\sigma}\sigma}$ are contained in $k_0(\Theta)$ and $a'_g \in \langle\Omega\rangle_{k_0(\Theta)-\mathrm{vec}} \cup k_0(\Theta)$ resp. $q' \in k_0(\Theta)\{X\}$. As $\{\sigma + B : \sigma \in \Sigma\}$ is a k-vector space basis of A/B, and as the element on the left-hand side of equation (7) is contained in B, the $\epsilon_{\overline{\sigma}\sigma}$ and the $\alpha_{\overline{\sigma}\sigma}$ must coincide, and the proof is complete. QED

A natural example of a k-algebra that allows for a vector space basis Ω as required in Theorem 2 is the Weyl algebra:

3 Example. Let k be of characteristic 0 and A the Weyl algebra of dimension n, i. e. the free associative k-algebra $k\langle X_1, \ldots, X_n, \partial_1, \ldots, \partial_n\rangle$ modulo the relations

$$X_iX_j = X_jX_i,\ \partial_i\partial_j = \partial_j\partial_i,\ \partial_iX_j = X_j\partial_i \text{ for } i \neq j, \text{ and } \partial_iX_i = X_i\partial_i + 1.$$

Then a vector space basis of A is given by

$$\Omega = \{X_1^{\alpha_1} \ldots X_n^{\alpha_n}\partial_1^{\beta_1} \ldots \partial_n^{\beta_n} : \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbb{N}_0\}$$

(see, e. g., [6, Section 1.1]). Further on, using the *Leibnitz formula* (e. g., [6, Theorem 1.1.1]) one easily verifies that for $\omega_1, \omega_2 \in \Omega$ arbitrary the inclusion $\omega_1\omega_2 \in \langle\Omega\rangle_{\mathbb{Q}-\text{vec}}$ holds.

It comes to no surprise that the condition in Theorem 2 is also fulfilled by several families of k-algebras arising as vector spaces of linear transformations:

4 Example. Let V be an n-dimensional k-vector space and consider the corresponding general linear algebra $\mathfrak{gl}(n, V)$. Namely, $\mathfrak{gl}(n, V)$ is the Lie algebra defined by the set of linear transformations of V with the bracket operation $[x, y] = xy - yx$. Note that, in particular, $\mathfrak{gl}(n, V)$ is in general non-associative.

Fixing a base for V, the algebra $\mathfrak{gl}(n, V)$ may be identified with the set of all $n \times n$ matrices over k. Then the set Ω of elementary matrices

$$\Omega := \{e_{ij} |\ 1 \leq i, j \leq n\}$$

(where $e_{ij} = (\delta_{ij})_{1 \leq i,j \leq n}$ is the matrix with 1 at the entry (i, j) and 0 elsewhere) is a vector space basis for $\mathfrak{gl}(n, V)$ satisfying the condition of Theorem 2. Just observe that for $1 \leq i, j, k, l \leq n$ we have

$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj}.$$

In a similar way it is easy to check that several other classical algebras—e. g. the special linear or the symplectic algebra—are valid examples (see [2, Chapter 1] for definitions and standard bases).

3 Conclusion

The above discussion shows that the notion of minimal field of definition is well-defined for one-sided ideals and subalgebras in k-algebras where a "suitable" k-vector space basis exists. As evidenced by the examples, the requirement for being "suitable" is relatively natural and fulfilled by several important families of k-algebras. Unfortunately, the given proof does not yield an analogous result for two-sided ideals, and it remains to be clarified to what extent the concept of a minimal field of definition is meaningful for two-sided ideals in non-commutative k-algebras.

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