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On groups with many subgroups satisfying a transitive normality relation

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Abstract. A group G is said to be a T-*group* if normality in G is a transitive relation. Clearly, as a simple group has the property T, it follows that T is not subgroup closed. A group G is called a $\bar{T}$-*group* if all its subgroups are T-groups. In this note the structure of groups all of whose (proper) subgroups either are nilpotent or satisfy the property $\bar{T}$ will be investigated.

Keywords: T-group, nilpotent group, Fitting subgroup

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In memory of Francesco de Giovanni

1 Introduction

A group G is said to be a T-*group* if normality in G is a transitive relation, i.e., if every subnormal subgroup of G is normal. The structure of finite soluble T-groups has been described by Gaschütz [6], while Robinson [8] investigated infinite soluble groups with the property T. In particular, it turns out that soluble T-groups are metabelian, locally supersoluble and that any finitely generated soluble T-group either is finite or abelian. As every element of a (soluble) T-group G fixes by conjugation every subgroup of a normal abelian subgroup A of G, a natural tool in the investigation of T-groups are the so-called *power automorphisms*, i.e. the automorphisms of a group G which fix every subgroup of G (see, for instance, [8], [3] and [1]). Clearly, a simple group has the pro-

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perty T, so that T is not subgroup closed. A group G is called a $\bar{T}$*-group* if all its subgroups are T-groups. It can be proved that every finite soluble T-group satisfies the property $\bar{T}$, and that the finite $\bar{T}$-groups are soluble.

Let $\mathfrak{P}$ be a class of group. Recall that a group G is said to be a *minimal non-*$\mathfrak{P}$*-group* if G does not belong to the class $\mathfrak{P}$, but all proper subgroups of G are $\mathfrak{P}$-groups. Minimal non-$\mathfrak{P}$-groups have been investigated by several authors for various classes of groups $\mathfrak{P}$. For instance, finite minimal non-nilpotent groups were analysed by Schmidt [11] in 1924, while Robinson [9] has classified finite *minimal non-*T*-groups* showing in particular that they fall into *seven* types. It is easy to prove that a finite minimal non-nilpotent group (minimal non-T-group, respectively) is soluble. Note that the latter result cannot be extended to infinite groups as the consideration of a Tarski group shows. Moreover, it can be proved that a minimal non-T-group is finite if it has no infinite simple sections (see [4, Proposition 1]). It is an open question whether there exists an infinite finitely generated locally graded minimal non-T-group (here a group G is called *locally graded* if each finitely generated non-trivial subgroup of G contains a proper subgroup of finite index). On the other hand, a finitely generated locally graded minimal non-nilpotent group is finite (see [2, Theorem 145]), while the infinite dihedral 2-group and the *Heineken-Mohamed groups* (see [7]) are examples of infinite soluble minimal non-nilpotent groups. More in general, a non-finitely generated minimal non-nilpotent group is a countable locally finite p-group (for some prime p) which either is a Chernikov p-group or a group of Heineken-Mohamed type (see [2, Proposition 144]).

The aim of this short article is to investigate the structure of groups all of whose proper subgroups either are nilpotent or satisfy the property $\bar{T}$. In particular, for an infinite generalized soluble group with the latter property it is proved that only the extreme and unavoidable cases can occur. Moreover, some properties of finite groups G such that $FitG = C_G(G')$ and all their proper subgroups either are nilpotent or T-groups are pointed out.

Most of our notation is standard and can be found in [10].

2 Results

Let G be a T-group. If x is an element of a normal nilpotent subgroup H of G, then $\langle x \rangle$ is normal in G and hence $G' \leq C_G(x)$. It follows that $H \leq C_G(G')$ and thus the Fitting subgroup $FitG$ of G coincides with $C_G(G')$.

In the following we will say that a group G satisfies the *Fitting property* if $FitG = C_G(G')$. Denote by $\mathfrak{X}$ the class of groups satisfying the Fitting property all of whose proper subgroups either are nilpotent or $\bar{T}$-groups. Our first result shows in particular that a finite $\mathfrak{X}$-group G is soluble and hence $G' \leq Z(FitG)$.

Lemma 1. *Let G be a finite group all of whose proper subgroups are either nilpotent or satisfy the property $\bar{T}$. Then G is soluble.*

Proof. As a finite $\bar{T}$-group (a finite nilpotent group, respectively) is supersoluble, then the statement follows from a celebrated Huppert's theorem (see, for instance, [10, 9.4.4]). $\boxed{QED}$

Clearly, $Alt(4)$ is an $\mathfrak{X}$-group, while $SL(2,3)$ is a minimal non-T-group which does not satisfy the Fitting property. On the other hand, the consideration of the direct product $D_8 \times S_3$ shows that there exist $\mathfrak{X}$-groups which neither are minimal non-nilpotent nor minimal non-T-groups.

Lemma 2. *Let G be a finite $\mathfrak{X}$-group. If N is a normal subgroup of G, then the factor group G/N either is nilpotent or an $\mathfrak{X}$-group.*

Proof. Clearly, arguing by induction on the order of G, we may suppose that N is a minimal normal subgroup of G. If N is not contained in the Frattini subgroup $\Phi(G)$ of G, then there exists a maximal subgroup M of G such that $N \not\leq M$, so that $G = MN$. It follows that G/N either is nilpotent or an $\mathfrak{X}$-group.

Now suppose that $N \leq \Phi(G)$. In particular, N is contained in $Fit(G)$ by Frattini's theorem. Let H/N be a normal nilpotent subgroup of G/N. If P is a Sylow p-subgroup of H, for some prime p, then PN is normal in G, and hence Capelli-Frattini argument yields that $G = N_G(P)N$. Therefore P is normal in G, and so H is nilpotent. It follows that

$$Fit(G/N) = Fit(G)/N = C_G(G')/N = C_{G/N}(G'N/N).$$

Thus G/N is an $\mathfrak{X}$-group and the statement is proved. $\boxed{QED}$

We shall say that a group G has a *Sylow tower* if every nontrivial homomorphic image of G contains a nontrivial normal Sylow subgroup. Two classical results by Zappa [12] and Doerk [5] respectively, state that both a finite supersoluble group and a finite minimal non-supersoluble group have a Sylow tower.

Proposition 1. *Let G be a finite $\mathfrak{X}$-group. Then the following hold:*
(1) *G has a Sylow tower and all its normal Sylow subgroups have nilpotent class at most 2.*
(2) *Let δ be the set of all primes $p \in \pi(G)$ such that the Sylow p-subgroups of G have nilpotency class at most 2. Then G has a unique Hall δ-subgroup H and each complement of H in G is nilpotent.*

Proof. As both a finite nilpotent group and a finite $\bar{T}$-group are supersoluble, G either is supersoluble or is minimal non-supersoluble. It follows from the quoted results by Zappa and Doerk that G has a nontrivial normal Sylow p-subgroup P, for some prime p. On the other hand, $G' \leq C_G(FitG))$ and hence $[P', P] = \{1\}$. Now an application of Lemma 2 shows that G has a Sylow tower and the Hall δ-subgroup H is normal.

Let K be a complement of H in G, and assume that Q is a nontrivial Sylow q-subgroup of K, for some prime q. Then Q has nilpotent class at least 3, and hence the normalizer $N_G(Q)$ of Q in G is a proper subgroup of G. As a Sylow subgroup of a finite $\bar{T}$-group is a Dedekind group, it follows that $N_G(Q)$ is nilpotent and so even a Carter subgroup of G. Therefore K is nilpotent since the Carter subgroups of G form a single conjugacy class of G. $\boxed{QED}$

Let G be a finite $\bar{T}$-group. If P is a Sylow p-subgroup of G, for some prime p, then $|P'| \leq 2$ and so $P' \leq Z(G)$ since G is metabelian. Thus $G/Z(G)$ is an A-*group*, i.e. a group all of whose Sylow subgroups are abelian. Our next result generalizes the above remark to a finite $\mathfrak{X}$-group in which all Sylow subgroups have nilpotent class at most 2.

Theorem 1. *Let G be a finite $\mathfrak{X}$-group all of whose Sylow subgroups have nilpotency class at most* 2*. Then the factor group $G/Z(G)$ is an A-group.*

Proof. Obviously we may suppose that G neither is nilpotent nor an A-group. Then G contains a non-abelian Sylow p-subgroup P, for some prime p. Let first $|\pi(G)| \geq 3$. By Lemma 1 there exist some proper Hall subgroups X and Y such that $G = XY$ and $P \leq X \cap Y$. Since each of X and Y either is nilpotent or a $\bar{T}$-group, then P' is a central subgroup of G.

Therefore we may assume that $G = PQ$, where Q is a Sylow q-subgroup, for some prime $q \neq p$. Let N be a minimal normal subgroup of G. Since by Lemma 2 the factor group G/N either is nilpotent or a $\bar{T}$-group, we have that

$$[G, P'] \leq N,$$

arguing by the induction on the order of G. Clearly, G can be considered as a monolithic group and N is its monolith. It follows that $Fit(G) = P$ and N is contained in the socle $Soc(P')$ of P'. Moreover, making $Soc(P')$ into a Q-module by conjugation, we have that $N = Soc(P')$ by Maschke's Theorem. Note that G' is not contained in $\Phi(P)$, since G is not nilpotent and $\Phi(P) \leq \Phi(G)$. Thus, another application of Maschke's Theorem yields that

$$P/\Phi(P) = G'\Phi(P)/\Phi(P) \times X/\Phi(P),$$

where X is a proper G-invariant subgroup of P. It follows that $P = G'X$ and X is not abelian, since $G' \leq Z(P)$ and $P' \neq \{1\}$.

It follows by induction that $X' \leq Z(G)$, and so even $N = Soc(P') \leq Z(G)$. Thus P' is cyclic as N is the monolith of G. Clearly, a non-trivial automorphism induced by Q over P' is fixed point free, so that $Q = C_Q(P')$ since $Soc(P') \leq Z(G)$. As a consequence, $P' \leq Z(G)$ and the statement is proved. $\boxed{QED}$

Now we point out a property of the unique Hall δ-subgroup of a finite $\mathfrak{X}$-group (see Proposition 1).

Theorem 2. *Let G be a finite $\mathfrak{X}$-group. If H is the unique Hall δ-subgroup of G, then $H' \cap Z(H) \leq Z(G)$.*

Proof. Clearly we may suppose that G is not nilpotent, H is a proper subgroup of G and $H' \cap Z(H) \neq \{1\}$. Note that the latter condition ensures that H is not an A-group (see [10, 10.1.7]). Let N be a minimal normal subgroup of G contained in H. Arguing by induction on the order of G, we have that

$$[G, H' \cap Z(H)] \leq N.$$

Therefore we may suppose that N is the unique minimal G-invariant subgroup of H. By Proposition 1 there exists a normal Sylow p-subgroup H_p of G, for some prime p. It follows that $N \leq H_p \leq H$, and so even $H' \leq H_p$ since $G' \leq Z(FitG)$. Moreover, as H is not an A-group, we have that H_p is not abelian. Denote by K a complement of H in G. By Proposition 1 K is nilpotent of class at least 3.

If H is nilpotent, then the uniqueness of N yields that $H = H_p = Fit(G)$. It follows that K is abelian, a contradiction. Thus H is a $\bar{T}$-group and $|H_p'| = 2$. On the other hand, H_p' is normal in G, and hence $N = H_p' \leq Z(G)$. Let $k \in K$ and $x \in H' \cap Z(H)$. There exists an element $a \in N$ such that $x^{-1}x^k = a$, so that

$$x^{k^2} = x^k a^k = x^k a = xa^2 = x.$$

Since $2 \notin \pi(K)$, it follows that $K = C_K(H' \cap Z(H))$ and so even $H' \cap Z(H) \leq Z(G)$, as required. $\boxed{QED}$

Our final results will concern *infinite* groups all of whose proper subgroups either are nilpotent or satisfy the property $\bar{T}$.

Theorem 3. *Let G be a group in which every proper subgroup either is nilpotent or satisfies the property $\bar{T}$. If G is not finitely generated, then G either is nilpotent or a $\bar{T}$-group or a minimal non-nilpotent group.*

Proof. Clearly, the finitely generated subgroups of G either are all nilpotent or all satisfy the property $\bar{T}$. Let H be a proper subgroup of G which is not nilpotent. Then H is a $\bar{T}$-group, and hence all finitely generated subgroups of G have the property $\bar{T}$. It follows that G is a T-group since the property T is *local* (see [8, Corollary 2 to Lemma 2.1.1]). Let K be a proper subgroup of G, and assume that it is nilpotent. If x and y are elements of K, then $\langle x, y\rangle$ is a

Dedekind group, so that $\langle x\rangle^y = \langle x\rangle$. It follows that also K is a Dedekind group. Thus G is a $\bar{T}$-group, and the statement is proved. $\boxed{QED}$

Proposition 2. *Let G be a group with no infinite simple sections. If every proper subgroup of G either is nilpotent or satisfies the property $\bar{T}$, then G is soluble.*

Proof. It is well known that a $\bar{T}$-group without infinite simple sections is soluble. Furthermore, we recalled in the introduction that a non-finitely generated minimal non-nilpotent group is likewise soluble. Therefore, by Theorem 3 we may suppose that G is finitely generated, and hence there exists a proper normal subgroup N of G such that the factor group G/N is finite. Since N either is nilpotent or satisfies the property $\bar{T}$ and G/N is soluble by Lemma 1, we have that G is soluble too. $\boxed{QED}$

Theorem 4. *Let G be a finitely generated group with no infinite simple sections. If every proper subgroup of G either is nilpotent or satisfies the property $\bar{T}$, then G either is nilpotent or finite.*

Proof. By Proposition 2, G is soluble. Clearly, we may assume that G contains a proper subgroup which is not nilpotent. Let M be a maximal subgroup of G containing the commutator subgroup G' of G. As a finitely generated soluble $\bar{T}$-group either is abelian or finite, M satisfies the maximal condition on subgroups. It follows that G is polycyclic. Let N be a maximal non-nilpotent subgroup of G. Then N is finite, since it is a finitely generated $\bar{T}$-group. Moreover, by a Mal'cev's Theorem (see [10, 5.4.16]) also the index $|G : N|$ is finite. Therefore G itself is finite, as required. $\boxed{QED}$

As a consequence of the last result we obtain the following theorem already quoted in the introduction (see [4, Proposition 1]).

Corollary 1. *Let G be a minimal non-T-group with no infinite simple sections. Then G is finite.*

Proof. Assume for a contradiction that G is infinite. Since the property T is local, we have that G is finitely generated, and so even nilpotent by Theorem 4. Let A be a proper normal subgroup of G of finite index. As A is a non-periodic Dedekind group then it is abelian and we may suppose that it is torsion free (with finite rank). Let p and q be some distict primes numbers, and put $m = pq$. If k is a positive integer, then the factor group G/A^{m^k} is finite and its Sylow subgroups are proper, and hence abelian. Therefore G/A^{m^k} is abelian, so that

$$G' \leq \bigcap_{k\in\mathbb{N}} A^{m^k} = \{1\}.$$

Thus G is abelian and this last contradiction proves the statement. $\boxed{QED}$

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