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Ideals as Generalized Prime Ideal Factorization of Submodules

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Abstract. For a submodule N of an R-module M, a unique product of prime ideals in R is assigned, which is called the generalized prime ideal factorization of N in M, and denoted as $\mathcal{P}_M(N)$. But for a product of prime ideals $\mathfrak{p}_1 \cdots \mathfrak{p}_n$ in R and an R-module M, there may not exist a submodule N in M with $\mathcal{P}_M(N) = \mathfrak{p}_1 \cdots \mathfrak{p}_n$. In this article, for an arbitrary product of prime ideals $\mathfrak{p}_1 \cdots \mathfrak{p}_n$ and a module M, we find conditions for the existence of submodules in M having $\mathfrak{p}_1 \cdots \mathfrak{p}_n$ as their generalized prime ideal factorization.

Keywords: prime submodule, prime filtration, Noetherian ring, prime ideal factorization, regular prime extension filtration

MSC 2020 classification: primary 13A05, secondary 13A15, 13E05, 13E15

1 Introduction

Throughout this article, R denotes a commutative Noetherian ring with identity and M will be a finitely generated unitary R-module. The reference for standard terminology and notations will be [3] and [4].

A proper submodule N of an R-module M is called a *prime submodule* of M if for any $a \in R$ and $x \in M$, $ax \in N$ implies $a \in (N : M)$ or $x \in N$. If N is a prime submodule of M, then $(N : M) = \mathfrak{p}$, a prime ideal in R, and we say N is a $\mathfrak{p}$*-prime submodule* of M. Let N and K be submodules of M. Then K is called a $\mathfrak{p}$*-prime extension of* N in M if N is a $\mathfrak{p}$-prime submodule of K, and it is denoted as $N \overset{\mathfrak{p}}{\subset} K$. In this case, $\mathrm{Ass}(K/N) = \{\mathfrak{p}\}$.

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Let N be a proper submodule of an R-module M. Then we have $\mathfrak{p} \in \mathrm{Ass}(M/N)$ if and only if there exists a $\mathfrak{p}$-prime extension of N in M [1, Lemma 3]. A $\mathfrak{p}$-prime extension K of N is said to be *maximal* if K is maximal among the submodules of M which are $\mathfrak{p}$-prime extensions of N in M. Since M is Noetherian, maximal $\mathfrak{p}$-prime extensions exist. A filtration of submodules $\mathcal{F} : N = M_0 \overset{\mathfrak{p}_1}{\subset} M_1 \subset \cdots \subset M_{n-1} \overset{\mathfrak{p}_n}{\subset} M_n = M$ is called a *maximal prime extension (MPE) filtration* of M over N, if $M_{i-1} \overset{\mathfrak{p}_i}{\subset} M_i$ is a maximal $\mathfrak{p}_i$-prime extension in M for $1 \leq i \leq n$. It is proved that $\mathrm{Ass}(M/M_{i-1}) = \{\mathfrak{p}_i, \ldots, \mathfrak{p}_n\}$ for each $1 \leq i \leq n$ [1, Proposition 14]. Hence, the set of prime ideals which occur in any MPE filtration of M over N is exactly equal to $\mathrm{Ass}(M/N)$.

A maximal $\mathfrak{p}$-prime extension K of N is said to be *regular* if $\mathfrak{p}$ is a maximal element in $\mathrm{Ass}(M/N)$, and the filtration $\mathcal{F} : N = M_0 \overset{\mathfrak{p}_1}{\subset} M_1 \subset \cdots \subset M_{n-1} \overset{\mathfrak{p}_n}{\subset} M_n = M$ is called a *regular prime extension (RPE) filtration* of M over N if $M_{i-1} \overset{\mathfrak{p}_i}{\subset} M_i$ is a regular $\mathfrak{p}_i$-prime extension in M for $1 \leq i \leq n$. In this case, for each $i < j$, $M_i \overset{\mathfrak{p}_{i+1}}{\subset} M_{i+1} \cdots \subset M_{j-1} \overset{\mathfrak{p}_j}{\subset} M_j$ is also an RPE filtration of M_j over M_i. Since RPE filtrations are MPE filtrations, $\mathrm{Ass}(M_j/M_i) = \{\mathfrak{p}_{i+1}, \ldots, \mathfrak{p}_j\}$ for $1 \leq i < j \leq n$. In particular, $\mathrm{Ass}(M/N) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$.

The following lemma gives the condition for interchanging the occurrences of prime ideals in an RPE filtration.

Lemma 1.1. [1, Lemma 20] Let N be a proper submodule of M and $N = M_0 \subset \cdots \subset M_{i-1} \overset{\mathfrak{p}_i}{\subset} M_i \overset{\mathfrak{p}_{i+1}}{\subset} M_{i+1} \subset \cdots \subset M_n = M$ be an RPE filtration of M over N. If $\mathfrak{p}_{i+1} \not\subseteq \mathfrak{p}_i$ for some i, then there exists a submodule K_i of M such that $N = M_0 \subset \cdots \subset M_{i-1} \overset{\mathfrak{p}_{i+1}}{\subset} K_i \overset{\mathfrak{p}_i}{\subset} M_{i+1} \subset \cdots \subset M_n = M$ is an RPE filtration of M over N.

RPE filtrations satisfy the following uniqueness property.

Lemma 1.2. [1, Theorem 22] For a proper submodule N of M, the number of times a prime ideal $\mathfrak{p}$ occurs in any RPE filtration of M over N is unique, and hence, any two RPE filtrations of M over N have the same length.

The submodules which occur in an RPE filtration are characterized as follows.

Lemma 1.3. [2, Lemma 3.1]. Let N be a proper submodule of an R-module M. If $N = M_0 \overset{\mathfrak{p}_1}{\subset} M_1 \subset \cdots \subset M_{n-1} \overset{\mathfrak{p}_n}{\subset} M_n = M$ is an RPE filtration of M over N, then $M_i = \{x \in M \mid \mathfrak{p}_1 \cdots \mathfrak{p}_i x \subseteq N\}$ for $1 \leq i \leq n$.

Hence, the product of prime ideals that occur in any two RPE filtrations of M over N is the same. This product is called the *generalized prime ideal factorization* of N in M and denoted as $\mathcal{P}_M(N)$ in [5], and sufficient conditions for $\mathcal{P}_M(\mathfrak{p}_1 \cdots \mathfrak{p}_n M) = \mathfrak{p}_1 \cdots \mathfrak{p}_n$ were found, where $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are prime ideals in

R [5, Theorem 2.14].

There may be products of prime ideals that are not the generalized prime ideal factorization of any submodule of a given module.

Example 1.4. Let $R = \frac{k[x,y,z]}{(xy-z^2, x^2-yz)}$ and $\overline{x}, \overline{y}, \overline{z}$ denote the images of x, y, z respectively in R. Let $\mathfrak{p}$ be the prime ideal $(\overline{x}, \overline{z})$. Then $(\mathfrak{p}^2 : \mathfrak{p}) = (\overline{x}, \overline{y}, \overline{z})$. Suppose there exists an ideal $\mathfrak{a}$ in R with $\mathcal{P}_R(\mathfrak{a}) = \mathfrak{p}^2$. Then there exists an RPE filtration $\mathfrak{a} \overset{\mathfrak{p}}{\subset} \mathfrak{a}_1 \overset{\mathfrak{p}}{\subset} R$ and therefore, $\mathrm{Ass}(R/\mathfrak{a}) = \{\mathfrak{p}\}$. By Lemma 1.3, $\mathfrak{a}_1 = (\mathfrak{a} : \mathfrak{p})$, and since $\mathfrak{p}^2 \subseteq \mathfrak{a}$, we have $(\mathfrak{p}^2 : \mathfrak{p}) \subseteq (\mathfrak{a} : \mathfrak{p})$. Since $(\overline{x}, \overline{y}, \overline{z}) = (\mathfrak{p}^2 : \mathfrak{p}) \subseteq (\mathfrak{a} : \mathfrak{p}) = \mathfrak{a}_1 \subsetneq R$ and $(\overline{x}, \overline{y}, \overline{z})$ is a maximal ideal, $(\overline{x}, \overline{y}, \overline{z}) = (\mathfrak{a} : \mathfrak{p})$. Then $(\overline{x}, \overline{y}, \overline{z}) = (\mathfrak{a} : p)$ for every $p \in \mathfrak{p} \setminus \mathfrak{a}$. This would imply that $(\overline{x}, \overline{y}, \overline{z}) \in \mathrm{Ass}(R/\mathfrak{a})$, a contradiction. Therefore, an ideal $\mathfrak{a}$ in R cannot have $\mathcal{P}_R(\mathfrak{a}) = \mathfrak{p}^2$.

In this article, for a product of prime ideals $\mathfrak{p}_1 \cdots \mathfrak{p}_n$ ($\mathfrak{p}_i$'s not necessarily distinct), we find conditions for the existence of submodules N of M with $\mathcal{P}_M(N) = \mathfrak{p}_1 \cdots \mathfrak{p}_n$. We also give a necessary and sufficient condition for $\mathcal{P}_M(\mathfrak{p}_1 \cdots \mathfrak{p}_n M) = \mathfrak{p}_1 \cdots \mathfrak{p}_n$.

2 Ideals as Generalized Prime Ideal Factorization of Submodules

Lemma 2.1. Let N be a submodule of M and $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be some minimal prime ideals in $\mathrm{Ass}(M/N)$. Then there exists a submodule K of M containing N such that $\mathcal{P}_M(K) = \mathfrak{p}_1 \cdots \mathfrak{p}_r$.

Proof. Let $N = M_0 \overset{\mathfrak{q}_1}{\subset} M_1 \subset \cdots \subset M_{n-1} \overset{\mathfrak{q}_n}{\subset} M_n = M$ be an RPE filtration of M over N. Since $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_n\} = \mathrm{Ass}(M/N)$, for each $1 \leq i \leq r$, $\mathfrak{p}_i = \mathfrak{q}_j$ for some j. Since $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are minimal, we can reorder $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ such that $\mathfrak{q}_j \not\subset \mathfrak{q}_k$ for $1 \leq j < k \leq n$ and $\mathfrak{q}_{n-r+i} = \mathfrak{p}_i$ for $1 \leq i \leq r$. So using Lemma 1.1 sufficient times we can have an RPE filtration

$$N = K_0 \subset K_1 \subset \cdots \subset K_{n-r} \overset{\mathfrak{p}_1}{\subset} K_{n-r+1} \overset{\mathfrak{p}_2}{\subset} \cdots \subset K_{n-1} \overset{\mathfrak{p}_r}{\subset} K_n = M$$

of M over N. Then

$$K_{n-r} \overset{\mathfrak{p}_1}{\subset} K_{n-r+1} \overset{\mathfrak{p}_2}{\subset} \cdots \subset K_{n-1} \overset{\mathfrak{p}_r}{\subset} K_n = M$$

is an RPE filtration. So if $K = K_{n-r}$, then K is a submodule of M containing N with $\mathcal{P}_M(K) = \mathfrak{p}_1 \cdots \mathfrak{p}_r$. $\boxed{QED}$

Now we show that for a prime ideal $\mathfrak{p}$ in R, $\mathfrak{p} \in \mathrm{Supp}(M)$ is a necessary and sufficient condition for the existence of a submodule N in M with $\mathcal{P}_M(N) = \mathfrak{p}$. More generally, we have the following result.

Proposition 2.2. Let M be an R-module and $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be prime ideals in R such that $\mathfrak{p}_i \not\subseteq \mathfrak{p}_j$ for every $i, j \in \{1, \ldots, n\}$ with $i \neq j$. Then the following are equivalent:

(1) $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} \subseteq \mathrm{Supp}(M)$;

(2) $\mathfrak{p}_i \in \mathrm{Supp}(M/\mathfrak{p}_1 \cdots \mathfrak{p}_n M)$ for every $1 \leq i \leq n$;

(3) $\mathfrak{p}_i$ is minimal in $\mathrm{Ass}(M/\mathfrak{p}_1 \cdots \mathfrak{p}_n M)$ for every $1 \leq i \leq n$;

(4) There exists a submodule N in M with $\mathcal{P}_M(N) = \mathfrak{p}_1 \cdots \mathfrak{p}_n$.

Proof. (i) $\Rightarrow$ (ii): Suppose $\mathfrak{p}_i \notin \mathrm{Supp}(M/\mathfrak{p}_1 \cdots \mathfrak{p}_n M)$ for some i. Then we have $(M/\mathfrak{p}_1 \cdots \mathfrak{p}_n M)_{\mathfrak{p}_i} = 0$. So we get $M_{\mathfrak{p}_i} = (\mathfrak{p}_1 \cdots \mathfrak{p}_n) M_{\mathfrak{p}_i}$. Since $(\mathfrak{p}_1 \cdots \mathfrak{p}_n)_{\mathfrak{p}_i} \subseteq \mathfrak{p}_i R_{\mathfrak{p}_i}$, by Nakayama's lemma $M_{\mathfrak{p}_i} = 0$. Therefore $\mathfrak{p}_i \notin \mathrm{Supp}(M)$.

(ii) $\Rightarrow$ (iii): If $\mathfrak{q} \in \mathrm{Supp}(M/\mathfrak{p}_1 \cdots \mathfrak{p}_n M)$, then $\mathfrak{p}_1 \cdots \mathfrak{p}_n \subseteq \mathfrak{q}$, and therefore $\mathfrak{q}$ contains some $\mathfrak{p}_i$. So the set of minimal elements of $\mathrm{Supp}(M/\mathfrak{p}_1 \cdots \mathfrak{p}_n M)$ is contained in $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. Since $\mathfrak{p}_i \not\subseteq \mathfrak{p}_j$ for all $i \neq j$, $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are minimal elements in $\mathrm{Supp}(M/\mathfrak{p}_1 \cdots \mathfrak{p}_n M)$. Therefore $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are minimal in $\mathrm{Ass}(M/\mathfrak{p}_1 \cdots \mathfrak{p}_n M)$.

(iii) $\Rightarrow$ (iv): Since $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are minimal in $\mathrm{Ass}(M/\mathfrak{p}_1 \cdots \mathfrak{p}_n M)$, by Lemma 2.1, there exists a submodule N of M with $\mathcal{P}_M(N) = \mathfrak{p}_1 \cdots \mathfrak{p}_n$.

(iv) $\Rightarrow$ (i): Since $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are the prime ideals which occur in an RPE filtration of M over N, $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} = \mathrm{Ass}(M/N) \subseteq \mathrm{Supp}(M)$. $\boxed{QED}$

Corollary 2.3. Let $\mathfrak{p}$ be a prime ideal in R. Then $\mathfrak{p} \in \mathrm{Supp}(M)$ if and only if there exists a submodule N in M with $\mathcal{P}_M(N) = \mathfrak{p}$.

In Proposition 2.2, the prime ideals are distinct. Now we find conditions for the product of prime ideals that need not be distinct to be a generalized prime ideal factorization of some submodule.

Proposition 2.4. Let $\mathfrak{p}$ be a prime ideal in R and r be a positive integer. If $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{p}^{r-1}M/\mathfrak{p}^r M)$, then there exists a submodule N in M such that $\mathcal{P}_M(N) = \mathfrak{p}^r$.

Proof. Let $N = \{x \in M \mid (\mathfrak{p}^r M : x) \not\subseteq \mathfrak{p}\}$. Let $x_1, x_2 \in N$ and $u \in R$. Then there exists $a_1 \in (\mathfrak{p}^r M : x_1) \setminus \mathfrak{p}$ and $a_2 \in (\mathfrak{p}^r M : x_2) \setminus \mathfrak{p}$. Then $a_1 a_2 \in (\mathfrak{p}^r M : ux_1 + x_2) \setminus \mathfrak{p}$, which implies that $ux_1 + x_2 \in N$. Hence N is a submodule of M. Since $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{p}^{r-1}M/\mathfrak{p}^r M)$, there exists $x \in \mathfrak{p}^{r-1}M$ such that $\mathfrak{p} = (\mathfrak{p}^r M : x)$. This implies $x \notin N$. Therefore N is a proper submodule of M. Also, $N \supseteq \mathfrak{p}^r M$.

We claim that $\mathrm{Ass}(M/N) = \{\mathfrak{p}\}$. Let $\mathfrak{q} \in \mathrm{Ass}(M/N)$. Then $\mathfrak{p}^r \subseteq \mathfrak{q}$ since $\mathfrak{p}^r M \subseteq N$. Therefore $\mathfrak{p} \subseteq \mathfrak{q}$. Now $\mathfrak{q} = (N : z)$ for some $z \in M$, $z \notin N$, that is, $(\mathfrak{p}^r M : z) \subseteq \mathfrak{p}$. Let $a \in \mathfrak{q}$. Then $az \in N$, which gives $(\mathfrak{p}^r M : az) \not\subseteq \mathfrak{p}$. Let $b \in R \setminus \mathfrak{p}$ such that $baz \in \mathfrak{p}^r M$, i.e., $ba \in (\mathfrak{p}^r M : z) \subseteq \mathfrak{p}$. This implies $a \in \mathfrak{p}$. Therefore $\mathfrak{q} \subseteq \mathfrak{p}$. Hence $\mathrm{Ass}(M/N) = \{\mathfrak{p}\}$.

If $N = M_0 \overset{\mathfrak{p}_1}{\subset} M_1 \subset \cdots \subset M_{k-1} \overset{\mathfrak{p}_k}{\subset} M_k = M$ is an RPE filtration of M over N, then $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\} = \mathrm{Ass}(M/N) = \{\mathfrak{p}\}$. So $\mathcal{P}_M(N) = \mathfrak{p}^k$. Suppose $k < r$. Then $\mathfrak{p}^{r-1} \subseteq \mathfrak{p}^k$, which implies $\mathfrak{p}^{r-1}M \subseteq \mathfrak{p}^k M \subseteq N$. So, for every $x \in \mathfrak{p}^{r-1}M$, $(\mathfrak{p}^r M : x) \not\subseteq \mathfrak{p}$. But $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{p}^{r-1}M/\mathfrak{p}^r M)$ implies $\mathfrak{p} = (\mathfrak{p}^r M : x)$ for some $x \in \mathfrak{p}^{r-1}M$, a contradiction. Therefore, $k \geq r$, and this implies $M_r \subseteq M_k = M$. By Lemma 1.3, $M_r = \{x \in M \mid \mathfrak{p}^r x \subseteq N\}$. For any $x \in M$, $\mathfrak{p}^r x \subseteq \mathfrak{p}^r M \subseteq N$. Therefore $M_r = M$. So, $N \overset{\mathfrak{p}}{\subset} M_1 \overset{\mathfrak{p}}{\subset} \cdots \overset{\mathfrak{p}}{\subset} M_r = M$ is an RPE filtration of M over N, and hence $\mathcal{P}_M(N) = \mathfrak{p}^r$. $\boxed{QED}$

In Example 1.4, $\mathfrak{p} \notin \mathrm{Ass}(\mathfrak{p}/\mathfrak{p}^2) = \{(\overline{x}, \overline{y}, \overline{z})\}$. So $\mathfrak{p}$ need not be an element of $\mathrm{Ass}(\mathfrak{p}^{r-1}M/\mathfrak{p}^r M)$ even if $\mathfrak{p}^r M \subsetneq \mathfrak{p}^{r-1}M$.

Theorem 2.5. Let M be an R-module, $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be distinct prime ideals in R ordered as $\mathfrak{p}_i \not\subset \mathfrak{p}_j$ for $i < j$, and $r_1, \ldots, r_n$ be positive integers. If $\mathfrak{p}_i \in \mathrm{Supp}({\mathfrak{p}_i}^{r_i - 1}{\mathfrak{p}_{i+1}}^{r_{i+1}} \cdots {\mathfrak{p}_n}^{r_n} M)$ for $i = 1, \ldots, n$, then there exists a submodule N in M such that $\mathcal{P}_M(N) = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_n}^{r_n}$.

Proof. We prove by induction on n. If $n = 1$, $\mathfrak{p}_1 \in \mathrm{Supp}({\mathfrak{p}_1}^{r_1 - 1}M)$ and by Proposition 2.2, $\mathfrak{p}_1 \in \mathrm{Ass}({\mathfrak{p}_1}^{r_1-1}M/{\mathfrak{p}_1}^{r_1}M)$. Then by Proposition 2.4, there exists a submodule N in M with $\mathcal{P}_M(N) = {\mathfrak{p}_1}^{r_1}$. Now let $n > 1$, and assume that the result is true for $n - 1$ prime ideals. Then there exists a submodule L in M with $\mathcal{P}_M(L) = {\mathfrak{p}_2}^{r_2} \cdots {\mathfrak{p}_n}^{r_n}$. That is, we have an RPE filtration

$$L \overset{\mathfrak{p}_2}{\subset} L_1^{(2)} \overset{\mathfrak{p}_2}{\subset} \cdots \overset{\mathfrak{p}_2}{\subset} L_{r_2}^{(2)} \overset{\mathfrak{p}_3}{\subset} L_1^{(3)} \overset{\mathfrak{p}_3}{\subset} \cdots \subset L_{r_n - 1}^{(n)} \overset{\mathfrak{p}_n}{\subset} L_{r_n}^{(n)} = M. \tag{2.1}$$

Then ${\mathfrak{p}_2}^{r_2} \cdots {\mathfrak{p}_n}^{r_n} M \subseteq L$.

So, we have $\mathrm{Ann}({\mathfrak{p}_1}^{r_1-1}L) \subseteq \mathrm{Ann}({\mathfrak{p}_1}^{r_1-1}{\mathfrak{p}_2}^{r_2} \cdots {\mathfrak{p}_n}^{r_n}M) \subseteq \mathfrak{p}_1$ since $\mathfrak{p}_1 \in \mathrm{Supp}({\mathfrak{p}_1}^{r_1-1}{\mathfrak{p}_2}^{r_2} \cdots {\mathfrak{p}_n}^{r_n}M)$. That is, $\mathfrak{p}_1 \in \mathrm{Supp}({\mathfrak{p}_1}^{r_1-1}L)$, and by Proposition 2.2, $\mathfrak{p}_1 \in \mathrm{Ass}({\mathfrak{p}_1}^{r_1-1}L/{\mathfrak{p}_1}^{r_1}L)$. Then by Proposition 2.4, there exists a submodule N in L such that $\mathcal{P}_L(N) = {\mathfrak{p}_1}^{r_1}$. That is, we have the RPE filtration

$$N \overset{\mathfrak{p}_1}{\subset} L_1^{(1)} \overset{\mathfrak{p}_1}{\subset} L_2^{(1)} \subset \cdots \overset{\mathfrak{p}_1}{\subset} L_{r_1}^{(1)} = L. \tag{2.2}$$

Next, we show that

$$\begin{aligned} N = L_0^{(1)} \overset{\mathfrak{p}_1}{\subset} L_1^{(1)} \overset{\mathfrak{p}_1}{\subset} L_2^{(1)} \subset \cdots \overset{\mathfrak{p}_1}{\subset} L_{r_1}^{(1)} = L \overset{\mathfrak{p}_2}{\subset} L_1^{(2)} \overset{\mathfrak{p}_2}{\subset} \cdots \\ \overset{\mathfrak{p}_2}{\subset} L_{r_2}^{(2)} \overset{\mathfrak{p}_3}{\subset} L_1^{(3)} \subset \cdots \overset{\mathfrak{p}_{n-1}}{\subset} L_{r_{n-1}}^{(n-1)} \overset{\mathfrak{p}_n}{\subset} L_1^{(n)} \overset{\mathfrak{p}_n}{\subset} \cdots \overset{\mathfrak{p}_n}{\subset} L_{r_n}^{(n)} = M \end{aligned} \tag{2.3}$$

is an RPE filtration of M over N, which would imply that $\mathcal{P}_M(N) = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_n}^{r_n}$. Since the filtration (2.1) is already an RPE filtration, it is enough to show that $L_{j-1}^{(1)} \overset{\mathfrak{p}_1}{\subset} L_j^{(1)}$ is a regular prime extension in M for $1 \leq j \leq r_1$.

From (2.2) we have that $L^{(1)}_{j-1} \subset L^{(1)}_j$ is a $\mathfrak{p}_1$-prime extension for every $1 \leq j \leq r_1$. Suppose $L^{(1)}_{j-1} \overset{\mathfrak{p}_1}{\subset} L^{(1)}_j$ is not a maximal $\mathfrak{p}_1$-prime extension in M for some j. Then there exists a submodule $K \supset L^{(1)}_j$ such that $L^{(1)}_{j-1} \overset{\mathfrak{p}_1}{\subset} K$ is a $\mathfrak{p}_1$-prime extension in M. Since $L^{(1)}_{j-1} \overset{\mathfrak{p}_1}{\subset} L^{(1)}_j$ is a maximal $\mathfrak{p}_1$-prime extension in L, $K \nsubseteq L$. Let $x \in K \setminus L$. For $2 \leq i \leq n$, since $\mathfrak{p}_1 \nsubseteq \mathfrak{p}_i$, there exists $p_i \in \mathfrak{p}_1 \setminus \mathfrak{p}_i$. Then $p_i x \in L^{(1)}_{j-1}$. Since $L^{(1)}_{j-1} \subset L$, from (2.1) we get that $p_i x \in L^{(i)}_k$ for every $2 \leq i \leq n$, $1 \leq k \leq r_i$.

Since $p_n x \in L^{(n)}_{r_n-1}$, $L^{(n)}_{r_n-1} \subset M$ is a $\mathfrak{p}_n$-prime extension, and $p_n \notin \mathfrak{p}_n$, we have $x \in L^{(n)}_{r_n-1}$. Then $p_n x \in L^{(n)}_{r_n-2}$ and $L^{(n)}_{r_n-2} \overset{\mathfrak{p}_n}{\subset} L^{(n)}_{r_n-1}$ is a prime extension implies $x \in L^{(n)}_{r_n-2}$. Repeating this argument $r_n - 3$ times, we get $x \in L^{(n-1)}_{r_{n-1}}$.

Replacing M by $L^{(n-1)}_{r_{n-1}}$ and p_n by p_{n-1} in the previous paragraph, we get $x \in L^{(n-2)}_{r_{n-2}}$. Continuing this process, finally we get $x \in L^{(1)}_{r_1} = L$, a contradiction. Therefore, $L^{(1)}_{j-1} \overset{\mathfrak{p}_1}{\subset} L^{(1)}_j$ is a maximal prime extension in M for every $1 \leq j \leq r_1$, and hence (2.3) is an MPE filtration of M over N.

So, for $1 \leq j \leq r_1$, we get $\mathrm{Ass}(M/L^{(1)}_{j-1}) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ and since $\mathfrak{p}_1 \not\subset \mathfrak{p}_i$ for every $i > 1$, $\mathfrak{p}_1$ is maximal in $\mathrm{Ass}(M/L^{(1)}_{j-1})$. Therefore (2.3) is an RPE filtration of M over N. Hence $\mathcal{P}_M(N) = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_n}^{r_n}$. $\boxed{QED}$

The converse of Theorem 2.5 does not hold. For example, if $\mathfrak{p}_2 \subsetneq \mathfrak{p}_1$ are prime ideals in a ring R and M is the R-module $\frac{R}{\mathfrak{p}_2} \oplus \frac{R}{\mathfrak{p}_2}$, then for its submodule $N = \frac{\mathfrak{p}_1}{\mathfrak{p}_2} \oplus 0$, we have the RPE filtration

$$N = \frac{\mathfrak{p}_1}{\mathfrak{p}_2} \oplus 0 \quad \overset{\mathfrak{p}_1}{\subset} \quad \frac{R}{\mathfrak{p}_2} \oplus 0 \quad \overset{\mathfrak{p}_2}{\subset} \quad \frac{R}{\mathfrak{p}_2} \oplus \frac{R}{\mathfrak{p}_2} = M$$

of M over N. So we have $\mathcal{P}_M(N) = \mathfrak{p}_1\mathfrak{p}_2$. But $\mathfrak{p}_2 M = 0$. Therefore $\mathfrak{p}_1 \notin \mathrm{Supp}(\mathfrak{p}_2 M)$.

Next, we show that if we assume further that $\mathfrak{p}_i \not\subset \mathfrak{p}_j$ for $i \neq j$, the converse of Theorem 2.5 holds. We need the following lemma.

Lemma 2.6. [2, Lemma 2.8] If $N \overset{\mathfrak{p}}{\subset} K$ is a regular $\mathfrak{p}$-prime extension in M, then for any submodule L of M, $N \cap L \overset{\mathfrak{p}}{\subset} K \cap L$ is a regular $\mathfrak{p}$-prime extension in L when $N \cap L \neq K \cap L$.

Theorem 2.7. Let N be a submodule of M with $\mathcal{P}_M(N) = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_n}^{r_n}$, where $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are distinct prime ideals in R and $r_1, \ldots, r_n$ are positive integers. If all the prime ideals in $\mathrm{Ass}(M/N)$ are minimal, then we have $\mathfrak{p}_i \in \mathrm{Supp}({\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_{i-1}}^{r_{i-1}} {\mathfrak{p}_i}^{r_i-1} {\mathfrak{p}_{i+1}}^{r_{i+1}} \cdots {\mathfrak{p}_n}^{r_n} M)$ for $i = 1, \ldots, n$.

Proof. Since $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are minimal, for every i we can reorder $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ such that $\mathfrak{p}_1 = \mathfrak{p}_i$ and by Lemma 1.1, we have an RPE filtration

$$N \overset{\mathfrak{p}_1}{\subset} L_1^{(1)} \overset{\mathfrak{p}_1}{\subset} L_2^{(1)} \subset \cdots \subset L_{r_1-1}^{(1)} \overset{\mathfrak{p}_1}{\subset} L_{r_1}^{(1)} \overset{\mathfrak{p}_2}{\subset} L_1^{(2)} \subset \cdots \overset{\mathfrak{p}_n}{\subset} L_{r_n}^{(n)} = M$$

of M over N. So it is enough to show that $\mathfrak{p}_1 \in \operatorname{Supp}({\mathfrak{p}_1}^{r_1-1}{\mathfrak{p}_2}^{r_2} \cdots {\mathfrak{p}_n}^{r_n} M)$. Clearly ${\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_n}^{r_n} M \subseteq N$ and ${\mathfrak{p}_1}^{r_1-1}{\mathfrak{p}_2}^{r_2} \cdots {\mathfrak{p}_n}^{r_n} M \subseteq L_1^{(1)}$.

We claim that ${\mathfrak{p}_1}^{r_1-1}{\mathfrak{p}_2}^{r_2} \cdots {\mathfrak{p}_n}^{r_n} M \not\subseteq N$. Let $x \in L_{r_1}^{(1)} \setminus L_{r_1-1}^{(1)}$. Then ${\mathfrak{p}_1}^{r_1-1} x \subseteq L_1^{(1)}$ and ${\mathfrak{p}_1}^{r_1-1} x \not\subseteq N$. So there exists $b \in {\mathfrak{p}_1}^{r_1-1}$ such that $bx \in L_1^{(1)} \setminus N$. Choose $a_j \in \mathfrak{p}_j \setminus \mathfrak{p}_1$ for every $2 \leq j \leq n$ and let $a = \prod_{2 \leq j \leq n} a_j^{r_j}$. Then $bax \in {\mathfrak{p}_1}^{r_1-1}{\mathfrak{p}_2}^{r_2} \cdots {\mathfrak{p}_n}^{r_n} M$. Suppose $bax \in N$. Then, since $bx \in L_1^{(1)} \setminus N$ and $N \overset{\mathfrak{p}_1}{\subset} L_1^{(1)}$ is a $\mathfrak{p}_1$-prime extension, we get $a \in \mathfrak{p}_1$, a contradiction. So $bax \notin N$. Therefore ${\mathfrak{p}_1}^{r_1-1}{\mathfrak{p}_2}^{r_2} \cdots {\mathfrak{p}_n}^{r_n} M \not\subseteq N$. So we have

$$\begin{aligned} N \cap ({\mathfrak{p}_1}^{r_1-1}{\mathfrak{p}_2}^{r_2} \cdots {\mathfrak{p}_n}^{r_n} M) &\subsetneq {\mathfrak{p}_1}^{r_1-1}{\mathfrak{p}_2}^{r_2} \cdots {\mathfrak{p}_n}^{r_n} M \\ &= L_1^{(1)} \cap ({\mathfrak{p}_1}^{r_1-1}{\mathfrak{p}_2}^{r_2} \cdots {\mathfrak{p}_n}^{r_n} M). \end{aligned}$$

Then by Lemma 2.6,

$$N \cap ({\mathfrak{p}_1}^{r_1-1}{\mathfrak{p}_2}^{r_2} \cdots {\mathfrak{p}_n}^{r_n} M) \quad \overset{\mathfrak{p}_1}{\subset} \quad {\mathfrak{p}_1}^{r_1-1}{\mathfrak{p}_2}^{r_2} \cdots {\mathfrak{p}_n}^{r_n} M$$

is a regular $\mathfrak{p}_1$-prime extension in ${\mathfrak{p}_1}^{r_1-1}{\mathfrak{p}_2}^{r_2} \cdots {\mathfrak{p}_n}^{r_n} M$. Then by Corollary 2.3, $\mathfrak{p}_1 \in \operatorname{Supp}({\mathfrak{p}_1}^{r_1-1}{\mathfrak{p}_2}^{r_2} \cdots {\mathfrak{p}_n}^{r_n} M)$. $\boxed{QED}$

From Theorems 2.5 and 2.7, we get the following corollary.

Corollary 2.8. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be distinct prime ideals in R with $\mathfrak{p}_i \not\subset \mathfrak{p}_j$ for $i \neq j$, and $r_1, \ldots, r_n$ be positive integers. Then ${\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_n}^{r_n}$ is the generalized prime ideal factorization of some submodule of M if and only if $\mathfrak{p}_i \in \operatorname{Supp}({\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_{i-1}}^{r_{i-1}} {\mathfrak{p}_i}^{r_i-1} {\mathfrak{p}_{i+1}}^{r_{i+1}} \cdots {\mathfrak{p}_n}^{r_n} M)$ for every $1 \leq i \leq n$.

In [5] we have found conditions for $\mathcal{P}_M(\mathfrak{p}_1 \cdots \mathfrak{p}_n M) = \mathfrak{p}_1 \cdots \mathfrak{p}_n$ [5, Theorem 2.14] and showed that this need not always be true [5, Example 2.5]. Now for an R-module M and a product of prime ideals $\mathfrak{a} = \mathfrak{p}_1 \cdots \mathfrak{p}_n$ ($\mathfrak{p}_i$'s not necessarily distinct), we give a necessary and sufficient condition for $\mathcal{P}_M(\mathfrak{a}M) = \mathfrak{a}$.

Theorem 2.9. Let M be an R-module and $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be prime ideals in R, not necessarily distinct, with $\mathfrak{p}_i$ maximal among $\{\mathfrak{p}_i, \ldots, \mathfrak{p}_n\}$ for $1 \leq i \leq n$. Let $\mathfrak{a} = \mathfrak{p}_1 \cdots \mathfrak{p}_n$, $\mathfrak{a}_0 = R$, and $\mathfrak{a}_i = \mathfrak{p}_1 \cdots \mathfrak{p}_i$ for $i = 1, \ldots, n-1$. Then $\mathcal{P}_M(\mathfrak{a}M) = \mathfrak{a}$ if and only if $\operatorname{Ass}\big(\frac{(\mathfrak{a}M:\mathfrak{a}_i)}{(\mathfrak{a}M:\mathfrak{a}_{i-1})}\big) = \{\mathfrak{p}_i\}$ for every $1 \leq i \leq n$.

Proof. If $\operatorname{Ass}\big(\frac{(\mathfrak{a}M:\mathfrak{a}_i)}{(\mathfrak{a}M:\mathfrak{a}_{i-1})}\big) = \{\mathfrak{p}_i\}$ for every $1 \leq i \leq n$, we show that

$$\mathfrak{a}M \overset{\mathfrak{p}_1}{\subset} (\mathfrak{a}M : \mathfrak{a}_1) \overset{\mathfrak{p}_2}{\subset} (\mathfrak{a}M : \mathfrak{a}_2) \subset \cdots \subset (\mathfrak{a}M : \mathfrak{a}_{n-1}) \overset{\mathfrak{p}_n}{\subset} (\mathfrak{a}M : \mathfrak{a}) = M \qquad (2.4)$$

is an RPE filtration.

$\text{Ass}\big(\frac{(\mathfrak{a}M:\mathfrak{a}_i)}{(\mathfrak{a}M:\mathfrak{a}_{i-1})}\big) = \{\mathfrak{p}_i\}$ implies that there exists a regular $\mathfrak{p}_i$-prime extension K of $(\mathfrak{a}M : \mathfrak{a}_{i-1})$ in $(\mathfrak{a}M : \mathfrak{a}_i)$. Then $K = \{x \in (\mathfrak{a}M : \mathfrak{a}_i) \mid \mathfrak{p}_i x \subseteq (\mathfrak{a}M : \mathfrak{a}_{i-1})\}$ by Lemma 1.3. For every $x \in (\mathfrak{a}M : \mathfrak{a}_i)$, $\mathfrak{a}_{i-1}\mathfrak{p}_i x = \mathfrak{a}_i x \subseteq \mathfrak{a}M$, that is, $\mathfrak{p}_i x \subseteq (\mathfrak{a}M : \mathfrak{a}_{i-1})$. Therefore, $K = (\mathfrak{a}M : \mathfrak{a}_i)$, and hence $(\mathfrak{a}M : \mathfrak{a}_i)$ is the unique regular $\mathfrak{p}_i$-prime extension of $(\mathfrak{a}M : \mathfrak{a}_{i-1})$ in $(\mathfrak{a}M : \mathfrak{a}_i)$. Suppose it is not maximal in M. Then there exists $x \in M \setminus (\mathfrak{a}M : \mathfrak{a}_i)$ such that $\mathfrak{p}_i x \subseteq (\mathfrak{a}M : \mathfrak{a}_{i-1})$, i.e., $x \in (\mathfrak{a}M : \mathfrak{a}_{i-1}\mathfrak{p}_i) = (\mathfrak{a}M : \mathfrak{a}_i)$, a contradiction. So $(\mathfrak{a}M : \mathfrak{a}_i)$ is a maximal $\mathfrak{p}_i$-prime extension of $(\mathfrak{a}M : \mathfrak{a}_{i-1})$ in M for every i. Therefore (2.4) is an MPE filtration of M over $\mathfrak{a}M$. This implies that $\text{Ass}\big(\frac{M}{(\mathfrak{a}M:\mathfrak{a}_{i-1})}\big) = \{\mathfrak{p}_i, \dots, \mathfrak{p}_n\}$ for every $1 \leq i \leq n$. Since $\mathfrak{p}_i$ is maximal among $\{\mathfrak{p}_i, \dots, \mathfrak{p}_n\}$, $\mathfrak{p}_i$ is maximal in $\text{Ass}\big(\frac{M}{(\mathfrak{a}M:\mathfrak{a}_{i-1})}\big)$. Therefore (2.4) is an RPE filtration. Hence $\mathcal{P}_M(\mathfrak{a}M) = \mathfrak{p}_1 \cdots \mathfrak{p}_n = \mathfrak{a}$.

Conversely, suppose that $\mathcal{P}_M(\mathfrak{a}M) = \mathfrak{a} = \mathfrak{p}_1 \cdots \mathfrak{p}_n$. Since $\mathfrak{p}_i$ is maximal among $\{\mathfrak{p}_i, \dots, \mathfrak{p}_n\}$ for every $1 \leq i \leq n$, we can construct an RPE filtration

$$\mathfrak{a}M = M_0 \overset{\mathfrak{p}_1}{\subset} M_1 \overset{\mathfrak{p}_2}{\subset} M_2 \subset \cdots M_{n-1} \overset{\mathfrak{p}_n}{\subset} M_n = M$$

of M over $\mathfrak{a}M$. Then by Lemma 1.3, $M_i = \{x \in M \mid \mathfrak{p}_1 \cdots \mathfrak{p}_i x \subseteq \mathfrak{a}M\}$, i.e., $M_i = (\mathfrak{a}M : \mathfrak{a}_i)$ for every $1 \leq i \leq n$. Then clearly $\text{Ass}\big(\frac{(\mathfrak{a}M:\mathfrak{a}_i)}{(\mathfrak{a}M:\mathfrak{a}_{i-1})}\big) = \text{Ass}\big(\frac{M_i}{M_{i-1}}\big) = \{\mathfrak{p}_i\}$ for every $1 \leq i \leq n$. $\boxed{QED}$

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