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Fano configurations in translation planes of large dimension

Norman L. Johnson
Department of Mathematics, University of Iowa,
Iowa City, Iowa 52242, USA
njohnson@math.uiowa.edu

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Abstract. It is shown that there are a large variety of André and generalized André planes of large dimension that admit Fano configurations. Furthermore, infinite subregular planes of various types admit Fano configurations.

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1 Introduction

Recently J. C. Fisher and the author (Fisher and Johnson [1]) considered the question of the existence of Fano configurations in finite subregular translation planes. Perhaps the idea for this study arose in 1954, when Hanna Neumann [6] showed, using coordinates, that in any 'projective' Hall plane of odd order, there is always a projective subplane of order 2—a Fano plane. It is well known that a Hall plane of order q^2 may be characterized as a translation plane obtained from a Desarguesian affine plane of order q^2 by the replacement of a regulus net. More generally, a 'subregular plane' is a translation plane constructed from a Desarguesian affine plane by the replacement of a set of disjoint regulus nets. In the previous article, arbitrary *subregular* planes of odd order that may be constructed from a Desarguesian affine plane by the replacement of a set of t disjoint regulus nets are considered and in a wide variety of cases, such translation planes admit Fano configurations. If one considers the maximal possible cardinality set of $q-1$ disjoint regulus nets then a multiple derivation of all of these nets leads to another Desarguesian plane. Hence, it is not true that any subregular plane admits Fano configurations. However, we show that if t is less than or equal to roughly a quarter of the possible $(q-1)$ then such planes always admit Fano configurations.

The principal result of Fisher and Johnson is as follows.

1 Theorem (Fisher and Johnson [1])**.** *Let Σ be a Desarguesian affine plane of odd order q^2 and let Λ be a set of $k+1$ disjoint reguli. Let π denote the* *translation plane obtained by a multiple derivation replacement of the reguli. If* $k+1 < \frac{(q+1)}{4}$ *then the projective extension of* π *contains a Fano configuration.*

For André planes the above result can be strengthened to conclude that if the number of disjoint reguli is $< 3\frac{(q-1)}{8}$, then again the plane admits a Fano configuration.

But subregular planes represent only a small part of the known translation planes. In particular, there are a variety of translation planes of large dimension and, of course, there are many infinite translation planes for which the question of the existence of Fano configurations has never been asked. So, the following problem arises:

2 Problem. What classes of large-dimension translation planes of odd order or of infinite order and characteristic not 2 admit Fano configurations?

So, in this article, large-dimension planes with spreads in $\mathrm{PG}(2t-1,q)$, for $t>2$, or infinite planes of characteristic not 2 are considered with regard to the existence, if any, of Fano configurations. A very general construction is given and then applied to André planes of odd order q^n, n even, and kernel containing $\mathrm{GF}(q)$ and various other types of generalized André planes.

Actually, there are a tremendous variety of generalized André planes, and in the background section, we describe a large class, which we call 'mixed generalized André planes' of order q^n and kernel $\mathrm{GF}(q)$. Our main result shows that if n is even and the number of 'big' nets is roughly $(q-1)/4$, we always obtain Fano configurations (see the background section for the definitions).

3 Theorem. *Let* π *be any mixed generalized André plane of odd order* q^n *and kernel containing* $\mathrm{GF}(q)$.

(1) Assume that n *is even and that the number of big nets is*

$$\leq \left[\frac{(q-1)}{4} \Big/ \left(\frac{q-1}{2}, n/2\right)\right].$$

Then the projective extension of π *admits a Fano configuration.*

(2) Also, if $(n/2,(q-1)/2)=1$ *then*

(a) if $q \equiv -1 \bmod 4$ *assume that the number of big nets is*

$$\leq (q-1)/4$$

and

(b) if $q \equiv 1 \bmod 4$ *assume that the number of big nets is*

$$\leq [(q+1)/4].$$

Then the projective extension of π admits a Fano configuration.

The methods are also valid for arbitrary characteristic. For example:

4 Theorem. *Derivation of any Pappian spread coordinatized by a field $K = F[\theta]$ such that $\theta^2 = \gamma$ is a nonsquare in F produces a Hall plane whose projective extension contains a Fano subplane.*

2 Background

Initially, the ambient space is the Desarguesian affine plane of order q^n where n is even, although the procedure works just as well in affine planes that admit a suitable Desarguesian net of degree $(q^n - 1)/(q - 1) + 4$ and order q^n.

The first theorem sets up the conditions for the existence of Fano configurations in arbitrary André translation planes of order q^n. Actually, the results involve translation planes which are not always André, but which are made up of André replacement nets of varying sizes. All of the translation planes will have square order. In order that the reader fully appreciate the tremendous variety of the translation planes that are considered, some specific background information on André planes and generalized André planes will be included. This background is taken from the author's work Johnson [5].

5 Definition. A 'q^e-fan' in a $2ds$-dimensional vector space over K isomorphic to $\mathrm{GF}(q)$ is a set of $(q^w - 1)/(q^e - 1)$ mutually disjoint K-subspaces of dimension ds that are in an orbit under a field group F_w^* of order $(q^d - 1)$, where F_w contains K and such that F_w^* is fixed-point-free, and where $w = d$ or $2d$ and in the latter case s is odd.

Fans arise in translation planes as follows:

6 Corollary. *Let π be a translation plane of order q^{ds} and kernel containing K isomorphic to $\mathrm{GF}(q)$ that admits a fixed-point-free field group F_w^* of order $(q^w - 1)$ containing K^*, where $w = d$ or $2d$ and s is odd if $w = 2d$.*

Then, for any component orbit Γ, there is a divisor e_Γ of w such the orbit length of Γ is $(q^w - 1)/(q^{e_L} - 1)$ so that Γ is a q^{e_Γ}-fan.

2.1 Multiple André replacement

Let Σ be a Desarguesian plane of order q^{ds} where q is a prime power. Let F_{ds} denote the field isomorphic to $\mathrm{GF}(q^{ds})$ coordinatizing Σ.

7 Definition. A 'generalized André plane' is a translation plane with spread

$$x = 0,\ y = 0,\ y = x^{q^{\lambda(m)}} m;\ m \in F_{ds}\,,$$

where λ is a function from F_{ds}^* to N, the set of natural numbers.

8 Definition. Let the 'q-André net' A_α be defined as follows:

$$A_\alpha = \{\, y = xm;\ m^{(q^{ds}-1)/(q-1)} = \alpha \,\}.$$

An 'André replacement' is defined as follows:

Choose any divisor e of d and consider the 'André replacement net' $A_\alpha^{q^{ef}}$ defined as follows:

$$A_\alpha^{q^{ef}} = \{\, y = x^{q^{ef}}m;\ m^{(q^{ds}-1)/(q-1)} = \alpha,\ (ef, d) = e \,\}.$$

Then if Σ is the associated Desarguesian plane with spread

$$A_\alpha \cup M$$

then there is a constructed translation plane with spread

$$A_\alpha^{q^{ef}} \cup M.$$

This translation plane is called an 'André plane'.

More generally, it is possible to replace ef above by any integer u between 0 and $ds-1$ and also obtain an André replacement. The main point is that the exponent of q is fixed for all components of the associated q-André net.

9 Definition. The collineation subgroup of the associated Desarguesian plane Σ that is in the linear translation complement and acts like the $\mathrm{GF}(q^{ds})$-scalar group is called the '$\mathrm{GF}(q^{ds})$-kernel group'.

Then the $\mathrm{GF}(q^{ds})$-kernel group acts on the André net and if we replace by André replacement as above, this group acts on the constructed André plane.

Choose any divisor e of d and consider the André replacement net $A_\alpha^{q^{ef}}$. Take the subfield F_d of F_{ds} isomorphic to $\mathrm{GF}(q^d)$. This group F_d^* acts on $A_\alpha^{q^{ef}}$ with orbits of length $(q^d-1)/(q^e-1)$ and hence there are exactly $k_e = \frac{(q^{ds}-1)}{(q-1)}\frac{(q^e-1)}{(q^d-1)}$ q^e-fans. Note that this process can be done for any divisor e_β for any André net A_β. Hence, we obtain a variety of q^{e_β}-fans.

10 Definition. The following constructed translation planes are obtained by what is called **'multiple André replacement'**.

11 Theorem. *Let Σ be a Desarguesian affine plane of order q^{ds}. For each of the $q-1$, André nets A_α, choose a divisor e_α of d (these divisors can possibly be equal and/or possibly equal to 1 or d). For each q-André net A_α, there is a corresponding set of k_{e_α} q^{e_α}-fans. Form the corresponding André plane $\Sigma_{(e_\alpha f_\alpha \forall \alpha \in \mathrm{GF}(q))}$ obtained with spread:*

$$y = x^{q^{e_\alpha f_\alpha}}m \text{ for } m^{(q^{ds}-1)/(q-1)} = \alpha,\ x = 0,\ y = 0;\ m \in \mathrm{GF}(q^{ds}),\ (e_\alpha f_\alpha, d) = e_\alpha.$$

Then the spread $\Sigma_{(e_\alpha f_\alpha \forall \alpha \in \mathrm{GF}(q))}$ is a union of $\sum_{\alpha=1}^{q-1}\left(k_{e_\alpha} = \frac{(q^{ds}-1)}{(q-1)}\frac{(q^{e_\alpha}-1)}{(q^d-1)}\right)$ q^{e_α}-fans, together with two q^d-fans $x = 0$ and $y = 0$.

2.2 Multiple q^e-André replacement

Actually, a refinement of the above will produce a more general variety of partition; however, the associated translation planes are not necessarily André planes, but are certainly generalized André planes. For example, we may partition any q-André net of cardinality $(q^{ds}-1)/(q-1)$ into $(q^e-1)/(q-1)$ q^e-André nets of cardinality $(q^{ds}-1)/(q^e-1)$. For the q^e-André nets, the basic replacement components must be the form $y = x^{q^{ef}}m$; however, we may choose the $f's$ independent of each other for the $(q^e-1)/(q-1)$ q^e-André nets. We then may choose another divisor e_1 of d/e to produce a set of $\frac{(q^{ds}-1)}{(q^e-1)}\frac{(q^{ee_1}-1)}{(q^d-1)}$ q^{ee_1}-fans from each of the $(q^e-1)/(q-1)$ q^e-André nets. Furthermore, the partitioning into relative sized André nets can be continued.

2.3 Non-André hyper-reguli; The generalized André planes of Johnson

12 Definition. A 'hyper-regulus' in a vector space V_{2n} of dimension $2n$ over GF(q) is a set of $(q^n-1)/(q-1)$ mutually disjoint n-dimensional subspaces over GF(q), which is covered by another set of $(q^n-1)/(q-1)$ n-dimensional subspaces over GF(q).

When $n = 2$, a hyper-regulus is simply a regulus. It is well known that replacement of a regulus that sits as a partial spread in an affine Desarguesian plane of order q^2 by its opposite regulus produces a Hall plane. If there is a set of mutually line-disjoint reguli that lie in an affine Desarguesian plane of order q^2 then the translation plane obtained by replacement of each regulus in the set by its opposite regulus is called a 'subregular' plane.

2.4 The existence of hyper-reguli of non-André type

13 Theorem. *Let q be any prime power and let n be any composite integer. Then, there exists a hyper-regulus which is not an André hyper-regulus of order q^n and degree $(q^n-1)/(q-1)$. Consider*

$$\begin{aligned} y &= x^q m;\ m^{(q^n-1)/(q^d-1)} = 1, \\ y &= x^{q^{n-(d-1)}} m;\ m^{(q^n-1)/(q-1)} = 1 \text{ but } m^{(q^n-1)/(q^d-1)} \neq 1. \end{aligned}$$

14 Corollary. *The translation plane obtained from a Desarguesian plane by the replacement of an André net by a non-André hyper-regulus is not an André plane.*

15 Definition. The replacements that use subspaces of the form $y = x^{q^{\rho(i)}}m$, as above, are called 'generalized André replacements'.

We now may construct a tremendous variety of 'new' translation planes by generalizations of the previous constructions.

16 Theorem. *Let q be any prime power and let n be any composite integer and let d denote the smallest prime properly dividing n.*

(1) If $(n/d,(q^d-1)/(q-1)) \neq 1$, then there exists a non-André hyper-regulus of order q^n and degree $(q^n-1)/(q-1)$ that admits F_d^*, as a fixed-point-free group with orbits of length $(q^d-1)/(q-1)$.

(2) The translation plane of order q^n obtained by replacement of any such non-André hyper-regulus has a spread that is a union of q-fans or q^d-fans.

2.5 Replacements admitting F_d^*; Fans

We now turn to consider that it is possible to find new generalized André planes of order q^{ds} that admit F_d^* as a fixed-point-free collineation group not all of whose orbits of components are trivial. In this section, there is no restriction on d.

17 Theorem. *Let Σ be a Desarguesian affine plane of order q^{ds} defined by the field F isomorphic to $\mathrm{GF}(q^{ds})$ and let F_d denote the subfield isomorphic to $\mathrm{GF}(q^{ds})^*$. Let F^* and F_d^*, respectively, denote the associated multiplicative groups. Let e be any divisor of d.*

(1) Let $\mathcal{A}_\alpha$ denote the q^e-André net

$$\{\, y = xm;\ m^{(q^{ds}-1)/(q^e-1)} = \alpha \,\},\ \text{and let } \alpha = 1.$$

Next, we consider cosets of $F_d^{*s(q^e-1)}$ in $F_d^{*(q^e-1)}$.

Let

$$\left\{ \alpha_i; i = 1, \ldots, \left(s, \frac{(q^d-1)}{(q^e-1)}\right) \right\}$$

be a set of coset representatives for $F_d^{*s(q^e-1)}$.

Let

$$\mathcal{A}_1^{q^{e\lambda(i)}} : \left\{ y = x^{q^{e\lambda(i)}} m;\ m^{(q^{ds}-1)/(q^d-1)} \in \alpha_i F_d^{*s(q^e-1)} \right\},$$
$$\text{where } (d/e, \lambda(i)) = 1,\ i = 1, \ldots, \left(s, \frac{(q^d-1)}{(q^e-1)}\right).$$

Then, the kernel homology subgroup of Σ corresponding to F_d^* leaves $\mathcal{A}_1^{q^{e\lambda(i)}}$ invariant and has $\frac{(q^{ds}-1)}{(q^d-1)} \Big/ \left(s, \frac{(q^d-1)}{(q^e-1)}\right)$ orbits of length $(q^d-1)/(q^e-1)$.

(2) $\left(s, \frac{(q^d-1)}{(q^e-1)}\right) = 1$ *if and only if* $\left|\mathcal{A}_1^{q^{e\lambda(1)}}\right| = \frac{(q^{ds}-1)}{(q^e-1)}$ *if and only if (for arbitrary* $\lambda(i)$*) we have a standard* q^e*-André replacement.*

(3) *If* $(\lambda(i) - \lambda(j), d/e) = d/e$*, for all* $i, j = 1, 2, \ldots, \left(s, \frac{(q^d-1)}{(q^e-1)}\right)$*, then*

$$\bigcup_{i=1}^{(q^{ds}-1)/(q^d-1)} \mathcal{A}_1^{q^{e\lambda(i)}}$$

forms a generalized André replacement, admitting F_d^* *as a fixed-point-free collineation group. Hence, we have a generalized André replacement that is a union of* q^e*-fans.*

If, in addition, $(\lambda(i), ds/e) = 1$ *for all* $i = 1, \ldots, \left(s, \frac{(q^d-1)}{(q^e-1)}\right)$*, we obtain a hyper-regulus replacement.*

(4) *The generalized André replacement of (3) is not André if and only if we choose at least two of the* $\lambda(i)$*'s to be distinct, which is possible if there are at least two distinct integers i,j, if and only if*

$$\left(s, \frac{(q^d-1)}{(q^e-1)}\right) \neq 1.$$

18 Corollary. *Choose any fixed integer* f *less than* d/e *such that* $(f, d/e) = 1$*. For each* i*,* $i = 1, \ldots, \left(s, \frac{(q^d-1)}{(q^e-1)}\right)$*, choose any integer* k_i *such that* $k_i d/e + f \leq ds$ *and let* $\lambda(i) = k_i d/e + f$*. Call the corresponding generalized André replacement* $\Sigma_{k_i,f}$*. If* $\left(s, \frac{q^d-1}{q^e-1}\right) \neq 1$ *and at least two of the integers* k_i *are distinct then we obtain a generalized André replacement which is not André. Furthermore, the translation plane obtained by this replacement of a single André net is not an André plane.*

19 Corollary. *If* A_α *is any* q^e*-André net, for* $\alpha \in \mathrm{GF}(q^e) - \{0\}$*, there is a generalized André replacement isomorphic to the replacement for* A_1*, considered as a* q^e*-André net.*

2.6 Multiple generalized André replacement

Let Σ be a Desarguesian affine plane of order $q^{ds} = q^{d_\alpha s_\alpha}$, for all $\alpha \in \mathrm{GF}(q)^*$, where $ds = d_\alpha s_\alpha$. There are $(q-1)$ mutually disjoint q-André nets A_α where

$$A_\alpha = \left\{ y = xm;\ m^{(q^{d_\alpha s_\alpha}-1)/(q-1)} = \alpha \right\},$$

for $\alpha \in \mathrm{GF}(q)^*$. Let $\left\{ \alpha_i;\ i = 1, \ldots, \left(s, \frac{(q^{d_\alpha}-1)}{(q-1)}\right) \right\}$, be a coset representative set for $F_d^{*s_\alpha(q-1)}$ in $F_d^{*(q-1)}$. Let

$$\mathcal{A}_\alpha^{q^{\lambda_\alpha(i)}} : \left\{ y = x^{q^{\lambda_\alpha(i)}} m b_\alpha;\ m^{(q^{d_\alpha s_\alpha}-1)/(q^{d_\alpha}-1)} \in \alpha_i F_{d_\alpha}^{*s_\alpha(q-1)} \right\},$$
$$\text{where } (d_a, \lambda_\alpha(i)) = 1 \quad \text{and} \quad (\lambda_\alpha(i) - \lambda_\alpha(j), d_\alpha) = d_\alpha.$$

20 Remark.

(1) The groups $F_{d_\alpha}^{*s_\alpha(q-1)}$ are not necessarily equal for all d_α as this depends on the integers d_α and s_α. The order of the group is $\frac{(q^{d_\alpha}-1)}{(q-1)} \Big/ \left(s_\alpha, \frac{(q^{d_\alpha}-1)}{(q-1)}\right)$.

(2) If $\lambda_\alpha(i) = \lambda_\alpha(j)$ for all i, j, then we have a standard André replacement.

(3) Let $\eta(ds)$ denote the number of divisors of ds not equal to ds but including 1. For each α, choose any divisor d_α of ds not equal to ds and let $s_\alpha = ds/d_\alpha$. There are at least $\left(s_\alpha, \frac{(q^{d_a}-1)}{(q-1)}\right)^{(s_\alpha-1)}$ possible sets

$$\left\{ \lambda_\alpha(i);\ i = 1, 2, \ldots, \left(s_\alpha, \frac{(q^{d_a}-1)}{(q-1)}\right) \right\},$$

leading to at least this number of distinct replacements for each André net.

Hence, there are at least

$$\left(s_\alpha, \frac{(q^{d_\alpha}-1)}{(q-1)}\right)^{(s_\alpha-1)\eta(ds)}$$

possible generalized André replacements for each André net.

Now it is possible to choose any number of the $(q-1)$ André nets to replace. If we replace exactly one André net, it is clear that the associated plane is isomorphic to the translation plane obtained from any other André net. However, except for such a choice, it is not completely clear what sorts of replacements provide isomorphic translation planes. Hence, we consider that there might be

$$1 + \sum_{k=2}^{q-1} \binom{q-1}{k}$$

possible ways of producing mutually non-isomorphic translation planes.

21 Theorem. *Let Σ be an affine Desarguesian plane of order q^{ds}. For each André net A_α for $\alpha \in \mathrm{GF}(q)^*$, choose a divisor $d_\alpha \neq ds$ of ds and let $s_\alpha = ds/d_\alpha$. Let*

$$\mathcal{S}_\alpha^{\lambda_a} = \bigcup \mathcal{A}_\alpha^{q^{\lambda_\alpha(i)}} : \left\{ y = x^{q^{\lambda_\alpha(i)}} m b_\alpha;\ m^{(q^{d_\alpha s_\alpha}-1)/(q^{d_\alpha}-1)} \in \alpha_i F_{d_\alpha}^{*s_\alpha(q-1)} \right\},$$

$$\text{where } (d_\alpha, \lambda_\alpha(i)) = 1 \quad \text{and} \quad (\lambda_\alpha(i) - \lambda_\alpha(j), d_\alpha) = d_\alpha,$$

$$i = 1, \ldots, \left(s_\alpha, \frac{(q^{d_\alpha}-1)}{(q-1)} \right), \quad b_\alpha^{(q^{ds}-1)/(q-1)} = \alpha.$$

(1) Then $\mathcal{S}_\alpha^{\lambda_a} \cup \{ x = 0, y = 0 \}$ is a spread for a generalized André plane.

(2) Let $\eta(ds) + 1$ equal the number of divisors of ds. There are at least

$$\left(s_\alpha, \frac{(q^{d_\alpha}-1)}{(q-1)} \right)^{(s_\alpha - 1)\eta(ds)k} \left(1 + \sum_{k=2}^{q-1} \binom{q-1}{k} \right)$$

different ways of choosing André nets and generalized André replacements that possibly lead to mutually non-isomorphic translation planes.

22 Corollary. *If $d_\alpha = d$ for all choices then the group F_d^* acts as a fixed-point-free collineation group of the associated translation plane. In this case, any such spread is a union of orbits of length either $(q^d - 1)/(q-1)$ or 1. Hence, the spread is a union of q-fans or components fixed by F_d^* (q^d-fans).*

The translation planes most under consideration belong to the class determined in the following theorem.

23 Theorem. *Let Σ be a Desarguesian affine plane of order q^{ds}. For each q-André A_α, $\alpha \in \mathrm{GF}(q)$, choose a divisor e_α of d and partition the components of A_α into $\frac{(q^{e_\alpha}-1)}{(q-1)}$ pieces each of cardinality $\frac{(q^{ds}-1)}{(q^{e_\alpha}-1)}$.*

(1) Each of the partition pieces will be q^{e_α}-André nets admitting the group F_d^*. Let $A_{\alpha,i}$, for $i = 1, \ldots, \frac{(q^{e_\alpha}-1)}{(q-1)}$, denote the set of q^{e_α}-André nets arising from A_α.

(2) Consider the subgroup $F_d^{*s(q^{e_\alpha}-1)}$ of $F_d^{*(q^{e_\alpha}-1)}$ and let

$$\left\{ \beta_{\alpha,i,j};\ j = 1, \ldots, \left(s, \frac{q^d - 1}{q^{e_\alpha} - 1} \right) \right\},\ i = 1, \ldots, \frac{(q^{e_\alpha}-1)}{(q-1)},\ \alpha \in \mathrm{GF}(q) - \{0\}$$

be a coset representative set for $F_d^{*s(q^{e_\alpha}-1)}$ of $F_d^{*(q^{e_\alpha}-1)}$, depending on α and i.

Let b_α be any element of F_{ds}^* such that $b_\alpha^{\frac{(q^{ds}-1)}{(q-1)}} = \alpha$. Let

$$A_{\alpha,i}^{e_\alpha \lambda_{\alpha,i}(j)} = \left\{ y = x^{q^{e_\alpha \lambda_{\alpha,i}(j)}} m b_\alpha;\ m^{(q^{ds}-1)/(q^d-1)} \in \beta_{\alpha,i,j} F_d^{*s(q^{e_\alpha}-1)} \right\},$$

where the functions $\lambda_{\alpha,i}$ are chosen to satisfy the conditions $(\lambda_{\alpha,i}(j), d/e_\alpha) = 1$ and $(\lambda_{\alpha,i}(j) - \lambda_{\alpha,i}(j), d/e_\alpha) = d/e_\alpha$.

Then $A_{\alpha,i}^{e_\alpha \lambda_{\alpha,i}(j)}$ is a union of $\left(\frac{(q^{ds}-1)}{(q^d-1)} \Big/ \left(s, \frac{(q^d-1)}{(q^{e_\alpha}-1)}\right)\right)$ F_d^*-orbits of length $\frac{(q^d-1)}{(q^{e_\alpha}-1)}$; a set of $\left(\frac{(q^{ds}-1)}{(q^d-1)} \Big/ \left(s, \frac{(q^d-1)}{(q^{e_\alpha}-1)}\right)\right)$ q^{e_α}-fans.

Hence, $\bigcup_{i=1}^{\frac{(q^{e_\alpha}-1)}{(q-1)}} A_{\alpha,i}^{e_\alpha \lambda_{\alpha,i}(j)}$ consists of $\frac{(q^{e_\alpha}-1)}{(q-1)} \left(\frac{(q^{ds}-1)}{(q^d-1)} \Big/ \left(s, \frac{(q^d-1)}{(q^{e_\alpha}-1)}\right)\right)$ q^{e_α}-fans.

(3) $$\left\{ x = 0, y = 0, \bigcup_{\alpha \in \mathrm{GF}(q)^*} \bigcup_{i=1}^{\frac{(q^{e_\alpha}-1)}{(q-1)}} A_{\alpha,i}^{e_\alpha \lambda_{\alpha,i}(j)} \right\} = S^{(e_\alpha;\ \alpha \in \mathrm{GF}(q)^*)}$$

is a spread defining a generalized André plane that admits F_d^* as a fixed-point-free collineation group. The spread is the union of two q^d-fans ($x = 0, y = 0$) and $\frac{(q^{e_\alpha}-1)}{(q-1)} \left(\frac{(q^{ds}-1)}{(q^d-1)} \Big/ \left(s, \frac{(q^d-1)}{(q^{e_\alpha}-1)}\right)\right)$ q^{e_α}-fans, for each of $q-1$ possible divisors e_α of d.

3 Planes admitting Fano configurations

It is now intended to show that any of the translation planes of odd square order described in the background section admit Fano configurations provided a certain restriction is placed on the numbers of replaced nets. The main construction technique revolves around the following general theorem. Note that all of the planes considered are obtained from a Desarguesian plane of order q^n, for n even, and are either obtained from multiple André replacement with nets of possible different degrees or from generalized André replacement of such nets. In particular, if the translation plane has kernel $\mathrm{GF}(q)$ and order q^n, then one of the nets, whether it is replaced with an André replacement or a generalized André replacement, must have degree $(q^n-1)/(q-1)$. Then the other nets may have varying degrees but the nets are decompositions of nets of degrees $(q^n-1)/(q-1)$. For example, if k divides n, a net of degree $(q^n-1)/(q-1)$ may be decomposed into $(q^k-1)/(q-1)$ nets of degree $(q^n-1)/(q^k-1)$. Hence, we could have a great many different André nets of varying degrees but there are at most $(q-1)$ nets of degree $(q^n-1)/(q-1)$ which form the base for the set of André nets.

24 Remark. We call nets of degree $(q^n-1)/(q-1)$ 'big nets' and assume that a big net remains big when considering replacements or is decomposed into $(q^k-1)/(q-1)$ nets of degrees $(q^n-1)/(q^k-1)$, for various divisors k of n. Note that all of the translation planes constructed are 'generalized André' planes. For purposes of reference, we call any translation plane under consideration a 'mixed generalized André plane' (since it may be constructed using a mixture of multiple André net replacements with generalized André replacement).

The main result shows that any mixed generalized André plane obtained by the replacement of a number less than or equal to approximately $1/4^{\text{th}}$ of the possible number of big nets must admit Fano configurations. The theorem that establishes this rests on the following general theorem.

25 Theorem. *Let K be a field isomorphic to $\mathrm{GF}(q^{2r=n})$, for q odd, and let Σ be the affine Desarguesian plane coordinatized by K. Let π be any mixed generalized André plane of kernel $\mathrm{GF}(q)$. Then we may assume that one of the André nets is big (of degree $(q^n-1)/(q-1)$) and with appropriate coordinates is*

$$A_1=\left\{\, y=xm;\ m^{(q^n-1)/(q-1)}=1\,\right\}.$$

Now assume that a set of λ big nets are considered in the replacement procedure. Without making any assumptions on the nature of the replacement, we may assume that $y=x^{q^i}$ is a component of a replacement net A_1^* for A_1, such that $(i,n)=1$. Choose c such that $c^q=-c$. We may identify λ with a subset of $\mathrm{GF}(q)^*$ with each element of $\mathrm{GF}(q)^*$ corresponding to a big net. Hence, $\mathrm{GF}(q)^*-\lambda$ will then be identified with the remaining big nets. Note that all big nets are disjoint from $x=0,y=0$.

Assume that for some c,

$$(*)\colon\ (1-c)^{-(q^n-1)/(q-1)}\quad\textit{and}\quad(1-c^{-1})^{-(q^n-1)/(q-1)}\quad\textit{are in }\mathrm{GF}(q)^*-\lambda.$$

We consider the quadrangle $ABCD$ as follows:
Let

$$A=(\infty),\ B=\left(0,\frac{c}{c-1}\right),\ C=(c,0),\ D=(1,0).$$

We note that

$$\begin{aligned}&AD\textit{ is } x=1,\ AB\textit{ is } x=0\quad\textit{and}\quad AC\textit{ is } x=c\,,\\&CD\textit{ is } y=0\,,\\&BD\textit{ is } y=-x\left(\frac{1}{1-c^{-1}}\right)+\frac{c}{c-1}\quad\textit{and}\\&BC\textit{ is } y=-x\left(\frac{1}{c-1}\right)+\frac{c}{c-1}\,.\end{aligned}$$

Furthermore,

$$\begin{aligned} AC \cap BD &= P = (c, -c), \\ AD \cap BC &= Q = (1, 1), \\ AB \cap CD &= R = (0, 0). \end{aligned}$$

We note that $y = x^{q^i}$ *contains the points* P, Q, R, *since* $(i, n) = 1$.

Then, the projective extension of the André plane π *is a projective plane of odd order that admits a Fano configuration.*

Proof. The points P, Q, R are collinear in π since $y = x^{q^i}$ is a line. We claim that the lines AB,AC,AD,BC,BD,CD of the quadrangle $ABCD$ are still lines in the André plane. Since (∞) and (0) are parallel classes that are not altered in the construction of the mixed generalized André plane π, it remains to check that BC and BD are still lines of π. Hence, we need to show that these lines are not among the ones that are replaced to construct the mixed generalized André plane π. This is true if and only if

$$\left\{\frac{1}{1-d}\right\}^{(q^n-1)/(q-1)} \in \mathrm{GF}(q)^* - \lambda, \quad \text{for } d \in \left\{c, c^{-1}\right\}.$$

Since

$$\left\{\frac{1}{1-d}\right\}^{(q^n-1)/(q-1)} = (1-d)^{-(q^n-1)/(q-1)},$$

we have the proof. $\boxed{QED}$

3.1 The 1/4-Theorem for mixed generalized André planes

26 Theorem. *Let* π *be any mixed generalized André plane of odd order* q^n *and kernel containing* $\mathrm{GF}(q)$.

(1) Assume that n *is even and that the number of big nets is*

$$\leq \left[\frac{(q-1)}{4} \Big/ \left(\frac{q-1}{2}, \frac{n}{2}\right)\right].$$

Then the projective extension of π *admits a Fano configuration.*

(2) Also, if $(n/2, (q-1)/2) = 1$ *then*

(a) If $q \equiv -1 \bmod 4$ *assume that the number of big nets is*

$$\leq (q-1)/4$$

and

(b) if $q \equiv 1 \bmod 4$ *assume that the number of big nets is*

$$\le [(q+1)/4]\,.$$

Then the projective extension of π *admits a Fano configuration.*

PROOF. We need to show that there exist a pair

$$\left\{\left(\frac{1}{1-c}\right)^{(q^n-1)/(q-1)}, \left(\frac{-c}{1-c}\right)^{(q^n-1)/(q-1)}\right\}$$

such that this set is a subset of $\mathrm{GF}(q)^* - \lambda$. Note that $c^q + c = 0$. Hence,

$$\left\{\left(\frac{1}{1-c}\right)^{(q^n-1)/(q-1)}, \left(\frac{-c}{1-c}\right)^{(q^n-1)/(q-1)}\right\}$$
$$= \left\{\left(\frac{1}{1+c^{q+1}}\right)^{\sum_{j=1}^{(n/2-1)} q^{2j}}, \left(\frac{1}{1+c^{-(q+1)}}\right)^{\sum_{j=1}^{(n/2-1)} q^{2j}}\right\}.$$

Hence,

$$\left\{\left(\frac{1}{1-c}\right)^{(q^n-1)/(q-1)}, \left(\frac{-c}{1-c}\right)^{(q^n-1)/(q-1)}\right\}$$

is

$$\left\{\left(\frac{1}{1+c^{q+1}}\right)^{n/2}, \left(\frac{1}{1-c^{-(q+1)}}\right)^{n/2}\right\}.$$

Note that $1+c^{q+1}$ takes on exactly $(q-1)/2$ non-zero elements. Hence, there are at most $\frac{(q-1)}{2} \,/\, \left(\frac{q-1}{2}, n/2\right)$ elements. Suppose that we have k big nets. Then we have a Fano configuration unless each of these nets is either a $\left(\frac{1}{1+c^{q+1}}\right)^{n/2}$-net or a $\left(\frac{1}{1+c^{-(q+1)}}\right)^{n/2}$-net. In order to ensure that we miss the remaining big nets (other than A_1), if $k-1 < \left[\frac{(q-1)}{2} \,/\, \left(\frac{q-1}{2}, n/2\right)/2\right]$, our big nets would miss at least one pair. Hence, if $k \le \left[\frac{(q-1)}{4} \,/\, \left(\frac{q-1}{2}, n/2\right)\right]$, this is ensured. This proves the first part. Now assume that $(n/2, (q-1)/2) = 1$. The remaining part of the theorem now follows immediately. $\boxed{QED}$

27 Corollary. *Under the conditions of the previous theorem, consider any mixed generalized André plane of square order q^n and kernel* $\mathrm{GF}(q)$*. Choose any c-pair. Then there are at most two big nets which are either c-nets or c^{-1}-nets. We may replace any subset of λ, for any any size λ, while to obtain a mixed generalized André plane, we restrict replacement. Then any subset of $\lambda - \{c, c^{-1}\}$ will produce a mixed generalized André plane whose projective extension admits Fano configurations.*

4 Infinite characteristic

There are certainly infinite planes which may be considered André planes. The question is, do any of these admit Fano configurations. For example, do the infinite Hall planes—those obtained by replacement of a single regulus net in a Pappian affine plane—admit Fano configurations. Note that Neumann [6] answered this question in the finite case.

In this section, we show that the infinite Hall planes always admit Fano configurations. Similar results are valid for larger-dimension André planes, but we shall not be concerned with such extensions.

Let π be a Pappian plane coordinatized by a field F which admits a quadratic extension $K = F[\theta]$ such that $\theta^2 = \gamma$, a nonsquare in F.

28 Lemma. *In π, whose coordinates are given by $x = 0, y = xm$ for all $m \in K$, the net*

$$D\colon y = xm;\ m^{\sigma+1} = 1 \quad \textit{for}\ \sigma \in Gal(K/F)$$

is a derivable net.

Proof. See Johnson [4]. $\boxed{QED}$

29 Theorem. *Derivation of any Pappian spread coordinatized by a field $K = F[\theta]$ such that $\theta^2 = \gamma$ is a nonsquare in F produces a Hall plane whose projective extension contains a Fano subplane.*

Proof. We merely need to check that derivation of the net D above produces a translation plane (this is proved in Johnson [3]) and that the general setup in the theorem in the finite case still holds in the more general case.

We therefore need to check that

$$(e(1-e))^{1+\sigma} \neq 1 \quad \text{and} \quad (1-e)^{\sigma+1} \neq 1.$$

Note that

$$(e/(1-e)) = (e(e+1)/((1-e)(1+e))) = (\gamma + e)/(1-\gamma).$$

Hence,

$$(e/(1-e))^{1+\sigma} = ((\gamma+e^{\sigma})(\gamma+e))/(1-\gamma)^2.$$

However, $e^{\sigma} = -e$, as $e^{\sigma}+e \in F$ and $e^{\sigma} = \alpha e + \beta$ for $\alpha, \beta \in F$ clearly implies that $\beta = 0$. Thus, $\alpha = -1$.

So,

$$((\gamma+e^{\sigma})(\gamma+e))/(1-\gamma)^2 = (\gamma^2-\gamma)/(1-\gamma)^2 = \gamma/(1-\gamma) \neq 1.$$

Furthermore,

$$(1/(1-e))^{\sigma+1} = 1 \text{ if and only if } (1-e)^{\sigma+1} = 1 = (1+e)(1-e),$$

which is $1-\gamma$. $\boxed{QED}$

The reader will have no trouble finding multiply derived versions of the above result and similarly there are André-type replacement procedures which will still work in the infinite case.

5 The general theorem

We list here the requisite result for replaceable translation planes. This is not the most general result which could be listed, but the details are messy for arbitrary coordinate ternary rings. On the other hand, if the coordinatization is linear then the same result holds. So, for Cartesian groups, the result is exactly as below. We might point out that Rahilly [7] used coordinates to show that there exist generalized Hall planes of odd order admitting Fano subplanes. Recall that a generalized Hall plane is one which may be constructed by the derivation of a semifield plane of order q^2 which is coordinatized by a semifield with middle nucleus isomorphic to GF(q).

Although Rahilly showed this to be true in special generalized Hall planes, the approach taken in that article was to simply verify the existence of a Fano subplane within the projective extension of the affine generalized Hall plane. Our method would be then to show that any semifield with middle nucleus (which is then derivable) constructs a Fano-like configuration in the semifield plane. Note that if the reader wants to try to prove this, using the situation depicted below, it would be necessary to recoordinatize the picture so as to use the semifield coordinate structure, since $x = 0$ is normally taken within the derivable net, not external to it.

Likewise, a semifield of order q^2 with right nucleus isomorphic to GF(q) is derivable and probably derives a plane whose projective extension contains a Fano subplane. These derived planes are called generalized Hall planes of type 2 (the generalized Hall planes then are of type 1).

This brings up an interesting problem:

Let S be a spread in a projective space $\mathrm{PG}(V,K)$, where K is a field, and let σ be a polarity of the projective space. Then $S\sigma$ is a dual spread which may or may not be a spread. It must be a spread in the finite case. If it is a spread then the corresponding translation plane is called the 'transposed' plane of the original. This is because the component matrices of the 'lines' of the new affine plane are matrix transposes of the component matrices of the original lines.

Open problem: Show that if the projective extension of a translation plane π has a Fano subplane then so does the projective extension of the transposed translation plane (assuming it exists).

Now a generalized Hall plane of type i transposes to a generalized Hall plane of type $i \bmod 2$, so it is probable that from Rahilly's result, we can at least guarantee that there are additional translation planes whose projective extensions contain Fano subplanes.

30 Theorem. *Let π be a translation plane with coordinatizing left quasifield $(Q,+,\cdot)$. Assume that π admits a replaceable net N with the following properties:*

(i) There is a subspace L of a replacement net containing $(0,0)$, $(1,1)$ and $(c, c\cdot(-b) - c\cdot(1-b))$, where $c\cdot(1-b)+b=0$,

(ii) the net N does not contain the parallel classes (∞), (0), $(-b)$ or (n) where $c\cdot n = b$.

Then $A=(\infty)$, $B=(0,b)$, $C=(c,0)$ and $D=(1,0)$ generate a Fano configuration in the translation plane obtained by the replacement of N.

Proof. $AC\cap BD = (c,(c\cdot(-b) - c\cdot(1-b))) = P$, $AD\cap BC = Q = (1,1)$ and $AB\cap CD = R = (0,0)$. Since all of the lines of the quadrangle except possibly for PQR are lines of the constructed translation plane, it follows that we obtain a quadrangle which generates a Fano configuration either in the projective extension of the constructed plane or the projective extension of the original plane. $\boxed{QED}$

6 Final comments

In a previous article [1], Fisher and Johnson have shown that a great variety of subregular translation planes of odd order admit Fano configurations. When the order is p^2, for p a prime, we have also shown that essentially all of the prime-square André planes where -1 is a non-square in $\mathrm{GF}(p)$ always admit Fano configurations. For example,

31 Corollary (Fisher and Johnson [1]). *Any non-Desarguesian André plane of order*

$$p^2 \in \left\{ 3^2, 5^2, 7^2, 11^2, 19^2, 23^2, 43^2, 47^2, 59^2, 67^2, 71^2, 79^2, 83^2 \right\}$$

admits a Fano configuration.

Furthermore, all nearfield subregular planes of order q^2 admit Fano configurations. In general, for subregular planes of order q^2, these always admit Fano configurations if the number of replaced regulus nets is $< \frac{(q+1)}{4}$ (the $1/4$-Theorem). Indeed for subregular André planes, the more general $3/8$-Theorem is proved.

Here a more general construction procedure for generalized André planes has been considered and the $1/4$-Theorem for these translation plane has been proved. In this setting, there are an enormous number of mutually non-isomorphic translation planes admitting Fano configurations. The focus has not been particularly on the André planes of order q^n, obtained by replacing nets of degree $(q^n-1)/(q-1)$; however, it is probably true that a $3/8$-Theorem may be proven by methods similar to the $n = 2$ case. This is left to the interested reader to verify. Furthermore, there are many other generalized André planes that may be constructed from a Desarguesian affine plane of order q^n by some sort of net replacement procedure and where one of the replacement lines involves the subspace $y = x^q m$. Since this device will allow our methods to work, it is clear that any such generalized André plane of order q^n, obtained from a Desarguesian affine plane by a net replacement procedure involving a number less than or equal to $1/4$ of the 'big' nets will admit a Fano configuration. However, $1/4$ of the big nets is not much of a restriction since it is possible to have a great number of replaceable small nets obtained by partitioning of the big nets. For example, a net of degree $(q^n-1)/(q-1)$ may be partitioned into $(q^k-1)/(q-1)$ nets of degree $(q^n-1)/(q^k-1)$ for k dividing n. Hence, when n has many factors, the number of possibilities for constructions is quite large and each such plane will admit Fano configurations.

Indeed, there are no known translation planes of odd order which are known not to admit Fano configurations. In fact, this is also true for finite projective planes of odd order—none exist which do not admit Fano configurations.

We end with an open problem.

32 Problem. Find a projective plane of odd order which does not admit a Fano configuration.

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