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Investigation of some special tensor fields on space-times with holonomy algebras

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Abstract. This paper studies the concircular, projective and conharmonic curvature tensors on 4−dimensional Lorentzian manifolds known as space-times. We obtain some properties of these tensor fields by relating the known holonomy algebras for Lorentz signature $(+,+,+,-)$. For the space-times admitting special vector fields, such as parallel and recurrent vector fields, some theorems are proved. The eigenbivector structure of the investigated tensor fields is also examined in these spaces. These results obtained by considering the holonomy theory are associated with the algebraic classification of the Riemann curvature and Ricci tensors for Lorentz signature, and various examples related to the study are also given.

Keywords: concircular curvature tensor, projective curvature tensor, conharmonic curvature tensor, holonomy, Lorentz signature

MSC 2022 classification: primary 53B30, secondary 53C29

1 Introduction

Special transformations preserving some geometric structures have a crucial role in the literature (see, e.g., [13]). For this reason, the study of tensor fields that remain invariant under special transformations has attracted the interest of many researchers over the years. Three of the famous tensor fields

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that emerged in this way are concircular, projective and conharmonic curvature tensors each of which is studied not only by geometers but also by physicists. These tensor fields of type (1, 3) are invariant under concircular, projective and conharmonic transformations, respectively. More clearly, two special classes of conformal transformations are concircular and conharmonic transformations. Concircular curvature tensor is invariant under a concircular transformation, a conformal transformation which transforms a geodesic circle into a geodesic circle (see, [17, 18]). On the other hand, the harmonicity of a function does not have to be preserved under the conformal transformation. In this respect, a conformal transformation preserving this property is defined as a conharmonic transformation where the conharmonic curvature tensor remains invariant, [8]. A tensor that remains invariant under the projective transformation, which is another special transformation preserving geodesics, is the projective curvature tensor. Looking at the literature, various properties of these special tensor fields have been investigated on different manifolds (among them see, e.g., [1, 2, 4, 5, 8, 9, 11, 12, 13, 14, 17, 18]). However, in this study, we examine these tensor fields from a different perspective by using holonomy theory. In doing so, attention will be drawn to the properties of these special tensor fields on 4−dimensional Lorentzian manifolds.

Let (M, g) be a space-time, more explicitly, suppose that M is a smooth, connected manifold of dimension 4 admitting a smooth metric g (with components g_{ij}) of Lorentz signature $(+,+,+,-)$, which will be assumed to be not flat. The Levi-Civita connection of (M, g) and the tangent space to M at $m \in M$ will be denoted by ∇ and T_mM, respectively. For $u, v \in T_mM$, the inner product $g(u, v)$ arising from g at m will be written as $u \cdot v$. The causal character of a non-zero vector $u \in T_mM$ is either *spacelike*, *timelike* or *null* if it satisfies $u \cdot u > 0$, $u \cdot u < 0$ or $u \cdot u = 0$, respectively. A pseudo-orthonormal basis of mutually orthogonal vectors x, y, z, t for T_mM is given by

$$x \cdot x = y \cdot y = z \cdot z = -t \cdot t = 1.$$

One also has an associated null basis l, n, x, y so that l and n are null vectors satisfying $l \cdot n = 1$ and defined by $\sqrt{2}l = z + t$, $\sqrt{2}n = z - t$. The Riemann curvature tensor with components $R^i{}_{jkh}$ will be denoted by *Riem* where one gets the curvature tensor of type (0, 4) with components $R_{ijkh} = g_{im}R^m{}_{jkh}$. The Ricci tensor with components $R_{ij} = R^m{}_{imj}$ and the scalar curvature will be shown by *Ric* and r, respectively. The concircular, projective and conharmonic curvature tensors of type (1, 3) in 4−dimensional spaces are given, respectively, by

$$Z^i{}_{jkh} = R^i{}_{jkh} - \frac{r}{12}(\delta^i{}_h\, g_{jk} - \delta^i{}_k\, g_{jh}) \tag{1.1}$$

$$W^i{}_{jkh} = R^i{}_{jkh} - \frac{1}{3}(\delta^i{}_k \, R_{jh} - \delta^i{}_h \, R_{jk}) \tag{1.2}$$

$$L^i{}_{jkh} = R^i{}_{jkh} - \frac{1}{2}(\delta^i{}_h \, R_{jk} - \delta^i{}_k \, R_{jh} + g_{jk} R^i{}_h - g_{jh} R^i{}_k) \tag{1.3}$$

where Z, W, L denote the concircular, projective and conharmonic curvature tensors, respectively, and δ^i_j is the Kronecker delta.

The rest of the paper is structured as follows: Section 2 provides a brief summary of the bivectors and holonomy algebras on space-times which will be essential for the study. In Section 3, some results expressing the relationships of concircular, projective and conharmonic curvature tensor fields with holonomy algebras and space-times are proved. Accordingly, when the manifold admits a parallel or recurrent vector field, some properties of these special tensor fields are obtained. On the other hand, the eigenbivector structure of the related tensor fields is also investigated and studies are made by presenting several examples from holonomy algebras. Final remarks are given in Section 4 where the situation for positive definite metric signature is also mentioned very briefly.

2 Preliminaries

One of the important tools in this study is bivectors which are second order skew-symmetric tensor fields appearing also in the exterior algebra. Another important concept to be considered in the study is the theory of holonomy. A brief information on these topics is presented below.

2.1 Bivectors

Let $\Lambda_m M$ denote the 6−dimensional vector space of all bivectors at $m \in M$, which is a Lie algebra under matrix commutation [ ]. It is usually called that a non-zero bivector F is either *simple* (when the rank of F is 2) or *non-simple* (when the rank of F is 4) in 4−dimensional spaces. A simple bivector F with components $F^{ij} = -F^{ji}$ can be written as $F^{ij} = u^i v^j - v^i u^j$ for $u, v \in T_m M$ so that the 2−space spanned by u and v is uniquely determined by F and called the *blade* of F. A simple bivector F in a space-time is classified as *spacelike* (respectively, *timelike* or *null*) if the blade of F, which is denoted by $u \wedge v$, is a *spacelike* (each non-zero member of it is spacelike) (respectively, *timelike* (it contains exactly two distinct null directions) or *null* (it contains exactly one null direction)) 2−space at the point m. The detailed study of bivectors appears in the general theory of relativity and all canonical forms of the classified bivectors together with the corresponding Segre types can be found, e.g., in [4]. For a null basis l, n, x, y of $T_m M$, some examples of such bivectors are given by (a) $x \wedge y$

(simple, spacelike and Segre type $\{(11)z\bar{z}\}$), (b) $l \wedge n$ (simple, timelike and Segre type $\{11(11)\}$), (c) $l \wedge x$ or $l \wedge y$ (simple, null and Segre type $\{(31)\}$), (d) $\gamma(l \wedge n) + \eta(x \wedge y)$ (non-simple where $\gamma, \eta \in \mathbb{R}$, $\gamma \neq 0 \neq \eta$ and Segre type $\{11z\bar{z}\}$).

2.2 Holonomy algebras on space-times

The holonomy group Φ_m of a connection (more precisely of ∇) at a fix point $m \in M$ is the group defined by all parallel displacements along closed loops about m. It is known that the holonomy groups at any two points of M are isomorphic as M is path-connected, yielding the holonomy group Φ of (M, g). Moreover, Φ is a Lie group having a Lie algebra denoted by ϕ (see for details [10], [4]). As the metric g has Lorentz signature $(+,+,+,-)$, it follows that ϕ is a subalgebra of the orthogonal algebra of g, that is, $o(1,3)$. The important point of this fact for the study is that the bases of possible holonomy algebras can be represented by bivector notation which are given in [15]. Using the labellings R_1 (the case of flat which will be omitted), R_2, R_3, …, R_{15} (the general type) expressed in [15], these algebras are presented in Table 1. It is also useful to note that each potential holonomy algebra except type R_5, which will therefore not be considered, in Table 1 can take place as an actual holonomy algebra (see, e.g. [4]).

Type	Basis	Parallel vector fields	Recurrent vector fields	Type	Basis	Parallel vector fields	Recurrent vector fields
R_2	$l \wedge n$	$\langle x, y\rangle$	l, n	R_9	$l \wedge n$, $l \wedge x$, $l \wedge y$	—	l
R_3	$l \wedge x$	$\langle l, y\rangle$	—	R_{10}	$l \wedge n$, $l \wedge x$, $n \wedge x$	$\langle y\rangle$	—
R_4	$x \wedge y$	$\langle l, n\rangle$	—	R_{11}	$l \wedge x$, $l \wedge y$, $x \wedge y$	$\langle l\rangle$	—
R_5	$l \wedge n + \mu(x \wedge y)$	—	—	R_{12}	$l \wedge x$, $l \wedge y$, $l \wedge n + \mu(x \wedge y)$	—	l
R_6	$l \wedge n$, $l \wedge x$	$\langle y\rangle$	l	R_{13}	$x \wedge y$, $y \wedge z$, $x \wedge z$	$\langle t\rangle$	—
R_7	$l \wedge n$, $x \wedge y$	—	l, n	R_{14}	$l \wedge n$, $l \wedge x$, $l \wedge y$, $x \wedge y$	—	l
R_8	$l \wedge x$, $l \wedge y$	$\langle l\rangle$	—	R_{15}	$o(1,3)$	—	—

Table 1: Subalgebras of $o(1,3)$ are indicated. Here, $0 \neq \mu \in \mathbb{R}$ and the symbol $\langle\ \rangle$ is a spanning set. The bases of holonomy types are located in the second and sixth columns from which the dimension of these algebras can be easily understood. Possible (local) parallel and recurrent vector fields that may occur with these types are expressed, respectively, in the columns 3–7 and 4–8.

Recurrent and parallel vector fields have an important place in the theory of holonomy (for details, see [4]). Let $U \neq \emptyset$ be a connected, open subset of M. A vector field v is called *recurrent* on U if $\nabla v = \lambda \otimes v$ for some $1-$form λ, which will be called *recurrence* $1-$*form*, on M. Specially, if λ vanishes on U, that is if $\nabla v = 0$, then v is called *parallel* (also referred to as *covariantly constant*) on U. In the holonomy theory, it is known that if a non-zero tangent vector v is an eigenvector for *all* members of the related holonomy algebra of ϕ, then there exists a (local) smooth, recurrent vector field on some neighbourhood of $m \in M$ whose value at m is v. Additionally, if each eigenvalue for v is zero for all $F \in \phi$, then this vector field can be chosen to be parallel (for details see, [4]). Based on this information, parallel and recurrent vector fields are shown, respectively, in the columns 3–7 and 4–8 in Table 1. It is also clear from Table 1 that if a recurrent vector field occurs for a holonomy algebra, then it can only be null. Such a vector field will be called *properly recurrent* because of the fact that a non-null, recurrent $0 \neq v \in T_mM$ can always be rescaled to parallel. It is useful to note that if v is parallel, one has from the Ricci identity $v^h R_{hijk} = 0$. On the other hand, the Ricci identity for a nowhere-zero, recurrent vector field v is given by as follows:

$$(\nabla_k \nabla_j - \nabla_j \nabla_k) v_i = v^h R_{hijk} = (\nabla_k \lambda_j - \nabla_j \lambda_k) v_i. \tag{2.4}$$

It can be seen from equation (2.4) that if $v^h R_{hijk} \equiv 0$ on $U \subset M$, then λ is a gradient and so, v may be scaled to be parallel (for all details, we refer to [6]).

Another useful information for our study is related to the known algebraic classification of Riemann curvature and Ricci tensors in this metric signature. Firstly, it is remarked that *Riem* gives rise to a linear map $f : \Lambda_m M \to \Lambda_m M$, which is called the *curvature map*, defined by $F^{ij} \to R^{ij}{}_{kh} F^{kh}$ where $F \in \Lambda_m M$. One of the facts used in the theory of holonomy is that the range space of the curvature map, which is indicated by $\mathrm{rg} f$, is a subspace of the Lie algebra ϕ and that *Riem* can always be written as a symmetrized sum of products of bivectors of ϕ. Secondly, the rank of map f called the *curvature rank* at $m \in M$ leads to the algebraic classification of *Riem* for which five disjoint (and mutually exclusive) curvature classes occur. Namely, these classes are denoted by **A**, **B**, **C**, **D** and **O**. Finally, it is beneficial to note that for all metric signatures in $4-$dimensional manifolds, the algebraic classification of a second order symmetric tensor (and in particular, *Ric*) is known as the Jordan–Segre classification where the permitted Segre types for the Lorentzian case are $\{1, 111\}$ (where the comma separates off the eigenvalue corresponding to the timelike eigenvector from those associated with spacelike ones), $\{211\}$, $\{31\}$ and $\{11z\bar{z}\}$ together with their potential degeneracies (for all the details in this paragraph we refer to [4]).

3 Results and examples on special tensor fields with holonomy algebras

This section is devoted to state and prove the main results of the study. By the aid of (1.1), (1.2) and (1.3), it is obtained that the concircular, projective and conharmonic curvature tensors of type (0, 4) are respectively given by the following expressions:

$$Z_{ijkh} = R_{ijkh} - \frac{r}{12}(g_{ih}\, g_{jk} - g_{ik}\, g_{jh}), \tag{3.5}$$

$$W_{ijkh} = R_{ijkh} - \frac{1}{3}(g_{ik}\, R_{jh} - g_{ih}\, R_{jk}), \tag{3.6}$$

$$L_{ijkh} = R_{ijkh} - \frac{1}{2}(g_{ih}\, R_{jk} - g_{ik}\, R_{jh} + g_{jk}R_{ih} - g_{jh}R_{ik}). \tag{3.7}$$

Now, let us examine the relationships between the aforementioned tensor fields with parallel vector fields. Suppose that M admits a non-zero parallel vector field v. Then we have, $\nabla v = 0$, and so the Ricci identity and its contraction over g^{ik} yield that

$$R_{ijkh}v^h = 0\ , \quad R_{jh}v^h = 0. \tag{3.8}$$

For $v \in T_mM$, let us investigate the non-zero solutions of the following equations, respectively

$$\text{(a) } Z_{ijkh}v^h = 0, \qquad \text{(b) } W_{ijkh}v^h = 0, \qquad \text{(c) } L_{ijkh}v^h = 0. \tag{3.9}$$

• Firstly, consider the concircular curvature tensor Z together with the equation (3.9)(a). Contracting (3.5) by v^h and using (3.8), we get

$$Z_{ijkh}v^h = -\frac{r}{12}(v_i g_{jk} - v_j g_{ik}). \tag{3.10}$$

Substituting (3.9)(a) in (3.10), we obtain

$$r(v_i g_{jk} - v_j g_{ik}) = 0. \tag{3.11}$$

Multiplying (3.11) by g^{jk} gives that r must be zero as v is non-zero. It then follows from (3.5) that $Z = Riem$ (cf. [9]).

• Secondly, let us take into account the projective curvature tensor W together with the equation (3.9)(b). A contraction of (3.6) by v^h under the constraints of (3.8) reveals that

$$W_{ijkh}v^h = \frac{1}{3}v_i R_{jk}. \tag{3.12}$$

Plugging the equation (3.9)(b) into (3.12) gives

$$v_i R_{jk} = 0. \tag{3.13}$$

Thus, if (M, g) is not Ricci-flat, it then follows from (3.13) that there are no non-zero solutions of the equation (3.9)(b). Consequently, if M admits a parallel vector field v and if one investigates the non-zero solutions of the equation (3.9)(b), then (M, g) must be Ricci-flat and hence, $W = Riem$.

Remark 1. It can be seen from the equations (3.8) and (3.12) that if v is parallel, then one automatically has

$$W_{ijkh} v^j v^h = 0.$$

• Finally, consider the conharmonic curvature tensor L together with the equation (3.9)(c). Multiplying (3.7) by v^h, using the equations (3.8) and (3.9)(c) yield that

$$v_i R_{jk} - v_j R_{ik} = 0. \tag{3.14}$$

Contracting (3.14) by v^j, we get

$$(v_j v^j) R_{ik} = 0. \tag{3.15}$$

It follows from (3.15) that either (M, g) is Ricci-flat or v is null. Therefore, we have the following cases:

Case 1: If (M, g) is Ricci-flat, then we have from (3.7) that $L = Riem$.

Case 2: If v is null, contracting (3.14) by g^{jk} gives

$$r v_i - v^k R_{ik} = 0. \tag{3.16}$$

As v is non-zero and parallel, by using (3.8) and (3.16) it is achieved that the scalar curvature r must be zero. Moreover, the equation (3.14) yields that, the Segre type of Ricci tensor is $\{(211)\}$ with zero eigenvalue.

Combining the above results, we have proved the following theorem:

Theorem 1. *Let (M, g) be a space-time, and suppose that it admits a parallel vector field v on $U \subset M$. Then the following conditions hold:*

i. For the concircular curvature tensor Z; the non-zero solutions of the equation (3.9)(a) force the scalar curvature to be zero and so, $Z = Riem$.

ii. For the projective curvature tensor W; if (M, g) is not Ricci-flat, then there are no non-zero solutions of the equation (3.9)(b).

iii. *For the conharmonic curvature tensor L; if (M, g) is not Ricci-flat, the non-zero solutions of the equation (3.9)(c) force the vector field v to be null and the scalar curvature r to be zero. In this case, the Segre type of Ricci tensor is $\{(211)\}$ with zero eigenvalue.*

Example 1. Consider the holonomy type R_3 with the algebra $\langle l \wedge x\rangle$. As (M, g) is not flat, there exists a point $m \in M$ such that $Riem \neq 0$ at m. Thus, there is an open subset U such that $Riem$ is nowhere-zero on U. In that case, by restricting to this subset, we get a local expression for $Riem$ as in the following:

$$R_{ijkh} = \alpha(l_i x_j - x_i l_j)(l_k x_h - x_k l_h)$$

where $\alpha : U \to \mathbb{R}$ is a nowhere-zero smooth function. In this case, the Ricci tensor and the scalar curvature are calculated respectively as follows:

$$R_{jh} = \alpha l_j l_h, \quad r = 0. \tag{3.17}$$

From Table 1, since l is a parallel null vector field, the relations (3.8) are satisfied. In other words, we have $R_{ijkh}l^h = 0$ and so, $R_{jh}l^h = 0$. Therefore, it follows that $Z = Riem$ and $Z_{ijkh}l^h = 0$. Hence, the equation (3.9)(a) has at least one non-zero solution l and so, Theorem 1(i) is satisfied for holonomy type R_3. It is also noted that for this holonomy type, the curvature type is **D** and the Segre type of Ric is $\{(211)\}$ with eigenvalue zero. On the other hand, contracting (3.7) by l^h and then using (3.8) and (3.17), we obtain

$$\begin{aligned} L_{ijkh}l^h &= R_{ijkh}l^h - \frac{1}{2}(l_i R_{jk} - g_{ik}R_{jh}l^h + g_{jk}R_{ih}l^h - l_j R_{ik}) \\ &= -\frac{\alpha}{2}(l_i l_j l_k - l_j l_i l_k) \\ &= 0. \end{aligned} \tag{3.18}$$

Thus, it is achieved from (3.18) that the equation (3.9)(c) has at least one non-zero solution l which is null and hence, Theorem 1(iii) is also satisfied for holonomy type R_3.

Suppose now that M admits a non-zero recurrent vector field v. Then we have, $\nabla v = \lambda \otimes v$ for some $1-$form λ. Besides, the Ricci identity (2.4) yields that

$$R_{hijk}v^h = \theta_{jk}v_i \tag{3.19}$$

where $\theta_{jk} := \nabla_k \lambda_j - \nabla_j \lambda_k$. Multiplying (3.19) by g^{ik}, it is obtained that

$$R_{jh}v^h = \theta_{jh}v^h. \tag{3.20}$$

It is known that if v is a smooth recurrent vector field on a subset $U \subset M$ which is non-empty, open and connected, then v is an eigenvector of the Ricci tensor on U (see, Lemma 3 of [6] on page 271). It then follows that $R_{jh}v^h = \varphi v_j$ for some smooth function φ on $U \subset M$. By using this fact in (3.20), we obtain

$$\varphi v_j = R_{jh}v^h = \theta_{jh}v^h = -\theta_{hj}v^h. \tag{3.21}$$

Let us now investigate the relations of concircular, projective and conharmonic curvature tensors with recurrent vector fields.

∘ Assume that M admits a non-zero properly recurrent vector field v and let us consider the concircular curvature tensor Z. Contracting (3.5) by $v^j v^h$ and using (3.19), it can be seen that

$$Z_{ijkh}v^j v^h = \theta_{jk}v^j v_i - \frac{r}{12}(v_i v_k - v_j v^j g_{ik}). \tag{3.22}$$

The fact that v is a properly recurrent vector field yields it to be null. By the aid of this fact and (3.21), it can be achieved from (3.22) that

$$Z_{ijkh}v^j v^h = -\left(\varphi + \frac{r}{12}\right) v_i v_k$$

where v is also a null eigenvector of the Ricci tensor corresponding to eigenvalue φ.

∘ Now, let us do a similar investigation for the projective curvature tensor W. Contracting (3.6) by $v^j v^h$ and considering (3.19), after some calculations it follows that

$$W_{ijkh}v^j v^h = \theta_{jk}v^j v_i - \frac{1}{3}(\varphi v_j v^j g_{ik} - v_i R_{jk}v^j). \tag{3.23}$$

By using the equations (3.20) and (3.21) in (3.23), it can be seen that

$$W_{ijkh}v^j v^h = -\varphi v_i v_k - \frac{1}{3}(\varphi v_j v^j g_{ik} - \varphi v_i v_k). \tag{3.24}$$

Since v is null (as it is properly recurrent), from (3.24), it can be concluded that

$$W_{ijkh}v^j v^h = -\frac{2}{3}\varphi v_i v_k$$

where v is also a null eigenvector of the Ricci tensor corresponding to eigenvalue φ.

∘ We finally consider for the conharmonic curvature tensor L and perform similar steps as above. After a contraction (3.7) by $v^j v^h$ and using the relations (3.19) and (3.21), it can be derived that

$$L_{ijkh}v^j v^h = -2\varphi v_i v_k + \frac{1}{2}(\varphi v_j v^j g_{ik} + v_j v^j R_{ik}). \tag{3.25}$$

Furthermore, by using the fact that v is null, from (3.25), we obtain

$$L_{ijkh}v^j v^h = -2\varphi v_i v_k$$

where v is also an eigenvector of the Ricci tensor corresponding to eigenvalue φ.

Therefore, we proved the following theorem:

Theorem 2. *Let (M, g) be a space-time, and suppose that it admits a properly recurrent vector field v on $U \subset M$. Then*

i. The concircular curvature tensor Z satisfies

$$Z_{ijkh}v^j v^h = -\left(\varphi + \frac{r}{12}\right) v_i v_k.$$

ii. The projective curvature tensor W satisfies

$$W_{ijkh}v^j v^h = -\frac{2}{3}\varphi v_i v_k.$$

iii. The conharmonic curvature tensor L satisfies

$$L_{ijkh}v^j v^h = -2\varphi v_i v_k.$$

For all these conditions, v is a null eigenvector of the Ricci tensor corresponding to eigenvalue φ.

Remark 2. It is useful to note that each holonomy type except R_{10}, R_{13}, R_{15} given in Table 1 contains a recurrent or parallel vector field l (or n) which is null. In that case, this vector field is a repeated principal null direction of the Weyl curvature tensor, denoted by C, and the Petrov type is algebraically special at any $m \in M$ (for details, see [4], pages 220, 252–253).

Example 2. Consider the holonomy type R_2 with the algebra $\langle l \wedge n \rangle$. By applying similar steps performed in Example 1, one can calculate the local expressions for *Riem*, *Ric* and r, respectively, as follows:

$$R_{ijkh} = \alpha(l_i n_j - n_i l_j)(l_k n_h - n_k l_h), \tag{3.26}$$

$$R_{jh} = -\alpha(l_j n_h + n_j l_h), \tag{3.27}$$

$$r = -2\alpha \tag{3.28}$$

where $\alpha : U \to \mathbb{R}$ is a nowhere-zero smooth function. Plugging the equations (3.26) and (3.28) into (3.5), it follows that

$$Z_{ijkh} = \alpha(l_i n_j - n_i l_j)(l_k n_h - n_k l_h) + \frac{\alpha}{6}(g_{ih}g_{jk} - g_{jh}g_{ik}). \tag{3.29}$$

Contracting (3.29) by $l^j l^h$, we obtain

$$Z_{ijkh} l^j l^h = \frac{7\alpha}{6} l_i l_k. \tag{3.30}$$

On the other hand, from Table 1, it is known that l is a properly recurrent, null vector field. Moreover, it is easily seen from (3.27) that $R_{jh} l^h = -\alpha\, l_j$ and hence, l is an eigenvector of Ric corresponding to eigenvalue $\varphi = -\alpha$. In this case, by using the equation (3.28) we get that $-(\varphi + \frac{r}{12}) = \frac{7\alpha}{6}$ which is the coefficient of term $l_i l_k$ in (3.30). Therefore, Theorem 2(i) is satisfied for holonomy type R_2 with $\varphi = -\alpha$.

Similarly, for the projective and conharmonic tensor fields, we obtain

$$W_{ijkh} l^j l^h = \frac{2}{3}\alpha\, l_i l_k, \quad L_{ijkh} l^j l^h = 2\alpha\, l_i l_k$$

and so the relations in Theorem 2(ii) and (iii) are also satisfied for holonomy type R_2 with $\varphi = -\alpha$. It is noted that for this example, the vector field n could also be taken instead of l. Furthermore, from the equation (3.26), it can be seen that the curvature type is **D** and also, from the equation (3.27), one has in a basis l, n, x, y

$$R_{ij} l^j = -\alpha l_i,\ R_{ij} n^j = -\alpha n_i,\ R_{ij} x^j = 0,\ R_{ij} y^j = 0$$

which shows that the Segre type of Ric is $\{(11)(11)\}$.

Remark 3. As we proved in Theorem 2, if M admits a non-zero (properly) recurrent, null vector field v, one has the property $T_{ijkh} v^j v^h \propto v_i v_k$ where T can be one of the concircular, projective or conharmonic curvature tensors. It is noted that this special property does not have to be satisfied when one contracts the tensor fields in question to any arbitrary vector field. For example, consider the holonomy type R_4 with the algebra $\langle x \wedge y \rangle$. Then, one can compute $Riem$, Ric and r, respectively, as in the Examples 1 and 2. In this case, when one calculates the concircular curvature tensor Z from (3.5) and then contracting it by $x^j x^h$, it is found

$$Z_{ijkh} x^j x^h = \alpha y_i y_k - \frac{\alpha}{6}(x_i x_k - g_{ik})$$

where $\alpha : U \to \mathbb{R}$ is a nowhere-zero smooth function. This shows that $Z_{ijkh} x^j x^h$ is not proportional to $x_i x_k$. Therefore, Theorem 2 does not have to be implemented for vector fields that are not properly recurrent and null.

Next, let us discuss the link between eigenbivector structure of $Riem$ and the special tensor fields considered. The situation is known for the concircular curvature tensor, which was examined in [9] and where it was proven that a bivector $0 \neq F \in \Lambda_m M$ is an eigenbivector of the concircular curvature tensor corresponding to eigenvalue $\xi + \frac{r}{6}$ if and only if F is an eigenbivector of *Riem* corresponding to eigenvalue $\xi \in \mathbb{R}$ which is given by

$$R_{ijkh}F^{kh} = \xi F_{ij}. \tag{3.31}$$

Moreover, it was analysed in [9] that for 1−dimensional holonomy types R_2, R_3 and R_4, the concircular curvature tensor has a timelike, null and spacelike eigenbivectors, respectively, and for each of these cases, the curvature type is **D**.

Now, let us investigate this structure for the projective and conharmonic curvature tensors. Suppose that $0 \neq F \in \Lambda_m M$ is an eigenbivector of *Riem*. Then, the relation (3.31) is satisfied for some $\xi \in \mathbb{R}$. A contraction of (3.6) by $F^{kh}(= -F^{hk})$ gives

$$\begin{aligned} W_{ijkh}F^{kh} &= \xi F_{ij} - \frac{1}{3}(F_i^{\ h}\, R_{jh} - F^k{}_i\, R_{jk}) \\ &= \xi F_{ij} + \frac{2}{3}F^k{}_i\, R_{jk}. \end{aligned} \tag{3.32}$$

Similarly, contracting (3.7) by F^{kh}, it is obtained that

$$L_{ijkh}F^{kh} = \xi F_{ij} - F^k{}_i\, R_{jk} + F^k{}_j\, R_{ik}. \tag{3.33}$$

From the right hand sides of the equations (3.32) and (3.33), we conclude that an eigenbivector of the Riemann curvature tensor need not be an eigenbivector of either projective or conharmonic curvature tensors. Nevertheless, by looking at Table 1, it is also possible to find holonomy types where this situation may occur. Let us examine these situations in the following examples.

Example 3. Consider the 2−dimensional holonomy type R_6. Let P denote the bivector metric on $\Lambda_m M$ (for its definition, see [4]). For this holonomy type, a local expression for *Riem* is of the form

$$R_{ijkh} = \gamma F_{ij}F_{kh} + \delta G_{ij}G_{kh} + \eta(F_{ij}G_{kh} + G_{ij}F_{kh}) \tag{3.34}$$

where $F = l \wedge n$ (timelike) and $G = l \wedge x$ (null) are the generators and γ, δ, η are smooth functions. In this case, *Ric* can be written in the following form

$$R_{jh} = -\gamma(l_j n_h + n_j l_h) + \delta l_j l_h - \eta(l_j x_h + x_j l_h). \tag{3.35}$$

For the inner products between generators, we have

$$P(F, F) = F_{ij}F^{ij} = -2,\ P(F, G) = F_{ij}G^{ij} = 0,\ P(G, G) = G_{ij}G^{ij} = 0. \tag{3.36}$$

By evaluating the equations (3.34) and (3.36) together, one gets

$$R_{ijkh}F^{kh} = -2(\gamma F_{ij} + \eta G_{ij}), \quad R_{ijkh}G^{kh} = 0 \tag{3.37}$$

which shows that G is an eigenbivector of *Riem* corresponding to zero eigenvalue but F is not.

On the other hand, by using (3.32), (3.33), (3.35) and (3.37), it can be seen that

$$W_{ijkh}G^{kh} = \frac{2}{3}\left(-\gamma\ x_i l_j + \eta\ l_i l_j\right), \qquad L_{ijkh}G^{kh} = -\gamma G_{ij}. \tag{3.38}$$

It then follows from (3.38) that G is also an eigenbivector of the conharmonic curvature tensor L but this condition is not satisfied for the projective curvature tensor W.

Example 4. Let us now consider holonomy type R_8 with the algebra $\langle l \wedge x, l \wedge y\rangle$. For this type *Riem* is of the form (3.34) where $F = l \wedge x$ (null) and $G = l \wedge y$ (null). In this case, we have $R_{ijkh}F^{kh} = R_{ijkh}G^{kh} = 0$ and these indicate that both F and G are eigenbivectors of *Riem* with zero eigenvalue. Moreover, *Ric* is of the form $\psi l_i l_j$ for some smooth function ψ, in other words, its Segre type is $\{(211)\}$ with zero eigenvalue. With the help of (3.32) and (3.33), it is obtained that $W_{ijkh}F^{kh} = W_{ijkh}G^{kh} = 0$ and $L_{ijkh}F^{kh} = L_{ijkh}G^{kh} = 0$. According to these, the bivectors F and G are also eigenbivectors of projective and conharmonic curvature tensors corresponding to zero eigenvalue.

It can be seen from the Examples 3 and 4 that as the dimension of holonomy type increases, nothing definite can be said about the eigenbivector structure of the projective and conharmonic tensor fields. However, for $1-$dimensional holonomy types, by considering the generators it can be seen that the eigenbivector structure of these tensor fields in question shows a compatible behaviour in the usual null basis l, n, x, y. Indeed, for holonomy type R_2, by using (3.26) and (3.27) in (3.32) and (3.33), we respectively get

$$W_{ijkh}F^{kh} = -\frac{4}{3}\alpha F_{ij}, \quad L_{ijkh}F^{kh} = -4\alpha F_{ij} \tag{3.39}$$

where $F = l \wedge n$ is the generator of this holonomy type. It is achieved from the relations (3.39) that the timelike bivector F is an eigenbivector of projective and conharmonic curvature tensors corresponding to eigenvalues $-\frac{4}{3}\alpha$ and -4α, respectively. When similar calculations are made for holonomy type R_3, it can be seen that the null bivector $l \wedge x$ is an eigenbivector of the projective and conharmonic curvature tensors corresponding to zero eigenvalue. For holonomy type R_4, the local expressions for *Riem* and *Ric* are given, respectively, as follows:

$$R_{ijkh} = \alpha(x_i y_j - y_i x_j)(x_k y_h - y_k x_h), \quad R_{jh} = \alpha(x_j x_h + y_j y_h), \tag{3.40}$$

where $\alpha : U \to \mathbb{R}$ is a nowhere-zero smooth function. Then we have from (3.32), (3.33) and (3.40)

$$W_{ijkh}F^{kh} = \frac{4}{3}\alpha F_{ij}, \quad L_{ijkh}F^{kh} = 4\alpha F_{ij} \tag{3.41}$$

where $F = x \wedge y$ is the generator of this holonomy type. The relations (3.41) yield that the spacelike bivector F is an eigenbivector of projective and conharmonic curvature tensors corresponding to eigenvalues $\frac{4}{3}\alpha$ and 4α, respectively. Therefore, we proved the following theorem:

Theorem 3. *Let (M, g) be a space-time and assume that Riem does not vanish at $m \in M$. Then the following conditions are satisfied for $1-$dimensional holonomy types:*

i. For holonomy type R_2, the timelike bivector $l \wedge n$ is an eigenbivector of the projective and conharmonic curvature tensors corresponding to non-zero eigenvalue.

ii. For holonomy type R_3, the null bivector $l \wedge x$ is an eigenbivector of the projective and conharmonic curvature tensors corresponding to zero eigenvalue.

iii. For holonomy type R_4, the spacelike bivector $x \wedge y$ is an eigenbivector of the projective and conharmonic curvature tensors corresponding to non-zero eigenvalue.

For all of the above cases, Riem is of type **D**.

In the general theory of relativity, one of the best known family of exact solutions of Einstein's field equations are *pp-wave space-times* (see, e.g., [3, 16]) whose connection with our work is explained below.

Example 5. Consider a pp-wave space-time $M = \mathbb{R}^4$ with global coordinate system u, v, x, y whose metric tensor can be described in the following form

$$ds^2 = 2H(u, x, y)du^2 + 2dudv + dx^2 + dy^2 \tag{3.42}$$

where H is an arbitrary smooth function. For this metric, $l_i = u_{,i}$ is a null and parallel (co)vector field where the comma denotes the partial derivative. Moreover, *Ric* is expressed as

$$R_{ij} = (H_{xx} + H_{yy})l_i l_j \tag{3.43}$$

where H_{xx} and H_{yy} are second-order partial derivatives of H with respect to x and y. It is clear from (3.43) that if Ric is not zero, it is of Segre type $\{(211)\}$ with eigenvector l corresponding to zero eigenvalue. On the other hand, if the Laplacian of H, denoted by ΔH, with respect to x, y vanishes identically, that is, if $\Delta H = H_{xx} + H_{yy} = 0$, then (M, g) is Ricci-flat. In this case, (M, g) is known as a *vacuum pp-wave* in the theory of general relativity. Moreover, this implies that $r = 0$, and so $Riem = Z = W = L$ satisfying

$$R_{ijkh}l^h = Z_{ijkh}l^h = W_{ijkh}l^h = L_{ijkh}l^h = 0$$

as l is parallel. Thus, the equations (3.9) have a non-zero solution l and the holonomy group is of type R_8. Additionally, the null bivectors $F = l \wedge x$ and $G = l \wedge y$ are eigenbivectors of $Riem$ (and hence, of concircular, projective and conharmonic curvature tensors) corresponding to zero eigenvalue. It is also known that for the pp-wave space-times, the Weyl tensor C has Petrov type **N** with repeated principal null direction spanned by the vector field l or type **O** at each $m \in M$.

4 Final remarks and conclusion

In the present work, we found out some properties of the concircular, projective and conharmonic curvature tensors on space-times. The fact that these tensor fields have close connections with classification of the Riemann, Ricci and Weyl tensors has also come to the fore in this study. For holonomy types admitting parallel or recurrent vector fields, several conditions were obtained on these vector fields and the tensor fields in question. On the other hand, it is concluded that the eigenbivectors of Riemann curvature tensor do not have to be the eigenbivectors of the projective and conharmonic curvature tensors. Nevertheless, for all 1-dimensional holonomy types, the generator bivectors are the eigenbivectors of $Riem$, Z, W and L.

Finally, it will be helpful to make a few comments about the positive definite metric signature as well. In this case, g has signature $(+, +, +, +)$ and the Lie algebra ϕ is a subalgebra of the orthogonal algebra of g, that is, $o(4)$. For this signature, the holonomy types are given in [7], which are labelled as S_1, S_2, S_3, $\overset{+}{S}_3$, $\overset{+}{S}_4$ or S_6 (up to isomorphism). It can be checked that types S_1 and S_3 admit parallel vector fields whilst there is no properly recurrent vector field for any holonomy types in this signature. In addition, the conditions (i) and (ii) of Theorem 1 are also satisfied for this signature. However, for the conharmonic curvature tensor, the equation (3.15) implies (M, g) to be Ricci-flat as v cannot be null in that case and hence, $L = Riem$.

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