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Further aspects of quasi statistical convergence of sequences

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Abstract. In this paper, we introduce the concept of quasi statistical supremum, quasi statistical infimum of a real-valued sequence $x = (x_k)$, and study some properties of the newly introduced notion. We also introduce the concept of quasi statistical monotonicity and establish the condition under which a quasi statistical monotonic sequence is quasi statistical convergent. We end up by giving a necessary and sufficient condition for the quasi statistical convergence of a real-valued sequence $x = (x_k)$.

Keywords: Quasi-density, quasi statistical convergence, quasi statistical monotonicity.

MSC 2022 classification: primary 40A35, secondary 40A05, 40G15

1 Introduction and background

The convergence of sequences plays a crucial role in various branches of mathematics and has many generalizations with the goal of providing deeper insights into summability theory. In 1951 Fast [5] and Steinhaus [20] introduced the idea of statistical convergence independently using the notion of natural density. Later on, it was further investigated from the sequence space point of view by Fridy [7, 8], Šalát [17], and many mathematicians across the globe. Following their work, several investigations and generalizations have been made by Altinok and Küçükaslan [1, 2], Hazarika and Esi [10], Mursaleen [14], and many others [3, 4, 11, 12, 18, 19, 21, 23]. Statistical convergence has become one of the most active areas of research due to its wide applicability in various branches of mathematics such as number theory, mathematical analysis, probability theory, etc.

In an attempt to generalize the notion of statistical convergence, in 2012 Ozguc and Yurdakadim [16] introduced the concept of quasi-statistical convergence in terms of quasi-density. They investigated the relationship of the newly introduced notion with statistical convergence. Very recently Ozguc [15] has introduced the notion of quasi statistical limit and cluster points and investigated a few properties. When studying some new notion of convergence of sequences, several closely related concepts occur quite naturally, such as supremum, infimum, monotonicity, etc. In this paper, we aim to introduce quasi statistical analogue of the above concepts and investigate a few fundamental properties along with some implication relations.

2 Definitions and main results

Let E be a subset of the set of all natural numbers $\mathbb{N}$ and suppose E_n denotes the set

$$E_n = \{k \in E : k \leq n\}.$$

The natural density [6] of E is denoted and defined by

$$\delta(E) = \lim_{n \to \infty} \frac{card(E_n)}{n},$$

provided that the limit exists. Here, $card(E_n)$ represents the cardinal number of the set E_n. Clearly, if $card(E) < \infty$, then $\delta(E) = 0$ and $\delta(\mathbb{N} \setminus E) = 1 - \delta(E)$, whenever the either sides exists.

A real-valued sequence $x = (x_k)$ is said to be statistically convergent [7] to x_0 if for each $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : |x_k - x_0| \geq \varepsilon\}) = 0.$$

In this case, x_0 is called the statistical limit of the sequence x and symbolically it is expressed as $x_k \xrightarrow{st} x_0$.

In [9], Fridy and Orhan defined the statistical boundedness of a real-valued sequence $x = (x_k)$ as follows:

A sequence $x = (x_k)$ is said to be statistically bounded [9] if there exists $B > 0$ such that

$$\delta(\{k \in \mathbb{N} : |x_k| > B\}) = 0.$$

In [22], the notion of statistical monotonicity was defined and a decomposition theorem was established. Further, a necessary and sufficient condition was given under which a statistically monotonic sequence becomes statistical convergent.

A real-valued sequence $x = (x_k)$ is said to be statistically monotonic increasing (decreasing) if there exists a set $M = \{m_1 < m_2 < \cdots < m_j < \cdots\} \subseteq \mathbb{N}$ such that $\delta(M) = 1$ and the subsequence (x_{m_j}) is monotonic increasing (decreasing).

In [16], the notion of natural density was extended to quasi density by involving a sequence $c = (c_n)$ satisfying the following properties:

$$c_n > 0\, \forall n \in \mathbb{N},\ \lim_{n\to\infty} c_n = \infty \ \text{ and } \ \limsup_n \frac{c_n}{n} < \infty. \tag{2.1}$$

The quasi-density of a set $E \subseteq \mathbb{N}$ is defined by $\delta_c(E) = \lim_{n\to\infty} \frac{|E_n|}{c_n}$, provided the limit exists. It should be noted that if $c_n = n$, then the above definition turns to the definition of natural density. Throughout the paper, we will use $c = (c_n)$ to denote sequences that satisfy (2.1).

In [16], Ozguc and Yurdakadim introduced the notion of quasi statistical convergence of real-valued sequences as follows:

A sequence (x_k) is said to be quasi statistical convergent to x_0 if for each $\varepsilon > 0$,

$$\delta_c(\{k \in \mathbb{N} : |x_k - x_0| \geq \varepsilon\}) = 0.$$

In this case, x_0 is called the quasi statistical limit of the sequence (x_k) and symbolically it is expressed as $x_k \xrightarrow{st_q} x_0$. They mainly studied the relationship of quasi statistical convergence and statistical convergence and show that the condition $\inf_n \frac{c_n}{n} > 0$ along with (2.1), plays a significant role for the equivalence of the concepts. Recently, in [15], Ozguc introduced the notion of quasi statistical boundedness of a real-valued sequence as follows:

A real-valued sequence (x_k) is said to be quasi statistical bounded if there exists $B > 0$ such that

$$\delta_c(\{k \in \mathbb{N} : |x_k| > B\}) = 0.$$

It should be noted that, if we choose $c_n = n$, $\forall n \in \mathbb{N}$, then the definition of quasi statistical convergence and quasi statistical boundedness turns to the definition of statistical convergence and statistical boundedness respectively.

Now we are ready to present some new definitions and the main results of the paper.

Definition 1. Let $x = (x_k)$ be a real-valued sequence.
(i) The real number l is said to be a quasi statistical lower bound of x, if

$$\delta_c(\{k \in \mathbb{N} : x_k < l\}) = 0 \,(\text{or } \delta_c(\{k \in \mathbb{N} : x_k \geq l\}) = \delta_c(\mathbb{N})).$$

(ii) The real number u is said to be a quasi statistical upper bound of x, if

$$\delta_c(\{k \in \mathbb{N} : x_k > u\}) = 0 \,(\text{or } \delta_c(\{k \in \mathbb{N} : x_k \leq u\}) = \delta_c(\mathbb{N})).$$

The set of all quasi statistical lower and upper bounds of the sequence $x = (x_k)$ is denoted by $L_q(x)$ and $U_q(x)$, respectively.

Definition 2. Let $x = (x_k)$ be a real-valued sequence.
(i) The real number i is said to be the quasi statistical infimum of the sequence $x = (x_k)$ if i is the supremum of the set $L_q(x)$. In other words,

$$st_q - \inf x = \sup L_q(x).$$

(ii) The real number s is said to be the quasi statistical supremum of the sequence $x = (x_k)$ if s is the infimum of the set $U_q(x)$. In other words,

$$st_q - \sup x = \inf U_q(x).$$

If we take $c_n = n$, $n \in \mathbb{N}$, then Definition 1 and Definition 2 coincides with Definitions given in [13] for natural density.

Theorem 1. *(i) If $L(x)$ denotes the set of all usual lower bounds of a sequence $x = (x_k)$, then*

$$L(x) \subset L_q(x);$$

(ii) If $U(x)$ denotes the set of all upper lower bounds of a sequence $x = (x_k)$, then

$$U(x) \subset U_q(x)$$

and both inclusions can be strict.

Proof. (i) Let $l \in L(x)$. Then, we have $\{k \in \mathbb{N} : x_k \geq l\} = \mathbb{N}$ and consequently,

$$\delta_c(\{k \in \mathbb{N} : x_k \geq l\}) = \delta_c(\mathbb{N}).$$

Hence, $l \in L_q(x)$, proving that $L(x) \subseteq L_q(x)$. To prove that the inclusion can be strict we construct a counterexample. Let A be a set such that $\delta_c(A) = 0$. Define a sequence $x = (x_k)$ as follows:

$$x_k = \begin{cases} 1, & k \notin A \\ (-1)^k k, & otherwise \end{cases}.$$

Then, $1 \in L_q(x)$ but $1 \notin L(x)$.
(ii) The proof is similar to that of (i), so omitted. QED

Theorem 2. *Let $x = (x_k)$ be a real-valued sequence. Then,*
(i) If $l \in L_q(x)$, then all real numbers smaller than l are quasi statistical lower bound of x.
(ii) If $u \in U_q(x)$, then all real numbers bigger than u are quasi statistical upper bound of x.

Proof. (i) Let $l \in L_q(x)$ and $l' < l$. Then, by definition

$$\delta_c(\{k \in \mathbb{N} : x_k \geq l\}) = \delta_c(\mathbb{N}).$$

Since, $l' < l$, so the inclusion

$$\{k \in \mathbb{N} : x_k \geq l\} \subseteq \{k \in \mathbb{N} : x_k \geq l'\}$$

holds and consequently $\delta_c(\{k \in \mathbb{N} : x_k \geq l'\}) = \delta_c(\mathbb{N})$. Hence, $l' \in L_q(x)$.
(ii) The proof is similar to that of (i), so omitted. QED

Remark 1. From Theorem 2, it is clear that for a real-valued sequence if $L_q(x) \neq \emptyset$ and $U_q(x) \neq \emptyset$, then

$$card(L_q(x)) = card(U_q(x)) = card(\mathbb{R}).$$

Theorem 3. *For any real-valued sequence $x = (x_k)$, following inequation*

$$\inf x \leq st_q - \inf x \leq st_q - \sup x \leq \sup x$$

holds.

Proof. From the definition of usual infimum we have

$$\delta_c(\{k \in \mathbb{N} : x_k \geq \inf x\}) = \delta_c(\mathbb{N}).$$

Therefore, $\inf x \in L_q(x)$ and consequently,

$$\inf x \leq st_q - \inf x. \tag{2.2}$$

In a similar way, one can prove that

$$\sup x \geq st_q - \sup x. \tag{2.3}$$

Now we will show that $st_q - \inf x \leq st_q - \sup x$. To prove this, it is sufficient to prove that $l \leq u$ for any $l \in L_q(x)$ and $u \in U_q(x)$.

If possible suppose there exists $l' \in L_q(x)$ and $u' \in U_q(x)$ such that $l' > u'$. Then, since $l' \in L_q(x)$, so by Theorem 2, $u' \in L_q(x)$, which is a contradiction on the assumption of u'. Hence, we must have $l \leq u$ for any $l \in L_q(x)$ and $u \in U_q(x)$. In other words,

$$st_q - \inf x \leq st_q - \sup x. \tag{2.4}$$

Combining (2.2), (2.3), and (2.4) we obtain the desired result. QED

Theorem 4. *For any real-valued sequence $x = (x_k)$, $x_k \xrightarrow{st_q} x_0$ if and only if $st_q - \inf x = st_q - \sup x = x_0$.*

Proof. Firstly we assume that $x_k \xrightarrow{st_q} x_0$ holds. Then, by definition of quasi statistical convergence for any $\varepsilon > 0$,

$$\delta_c(\{k \in \mathbb{N} : |x_k - x_0| \geq \varepsilon\}) = 0. \tag{2.5}$$

This implies that, for any $\varepsilon > 0$,

$$\delta_c(\{k \in \mathbb{N} : x_k \geq x_0 + \varepsilon\}) = 0 \quad \text{and} \quad \delta_c(\{k \in \mathbb{N} : x_k < x_0 + \varepsilon\}) = \delta_c(\mathbb{N}) \tag{2.6}$$

and

$$\delta_c(\{k \in \mathbb{N} : x_k \leq x_0 - \varepsilon\}) = 0 \quad \text{and} \quad \delta_c(\{k \in \mathbb{N} : x_k > x_0 - \varepsilon\}) = \delta_c(\mathbb{N}). \tag{2.7}$$

Now from (2.6) and (2.7), we obtain $x_0 + \varepsilon \in U_q(x)$ and $x_0 - \varepsilon \in L_q(x)$. Eventually, $U_q(x) = (x_0, \infty)$ and $L_q(x) = (-\infty, x_0)$ holds and we have $st_q - \inf x = st_q - \sup x = x_0$.

To prove the converse part, let $st_q - \inf x = st_q - \sup x = x_0$ i.e., $\sup L_q(x) = \inf U_q(x) = x_0$. Then, by definition of supremum and infimum, there exists at least one $l' \in L_q(x)$ and atleast one $l'' \in U_q(x)$ such that for any $\varepsilon > 0$, $x_0 - \varepsilon < l'$ and $x_0 + \varepsilon > l''$ holds. Consequently,

$$\{k \in \mathbb{N} : x_k \geq x_0 + \varepsilon\} \subset \{k \in \mathbb{N} : x_k \geq l'\}$$

and

$$\{k \in \mathbb{N} : x_k \leq x_0 + \varepsilon\} \subset \{k \in \mathbb{N} : x_k \leq l''\}.$$

Now since, $l' \in L_q(x)$ and $l'' \in U_q(x)$, so from the above inclusions we obtain $\delta_c(\{k \in \mathbb{N} : x_k \geq x_0 + \varepsilon\}) = 0$ and $\delta_c(\{k \in \mathbb{N} : x_k \leq x_0 - \varepsilon\}) = 0$ which altogether implies (2.5) and this completes the proof. QED

Corollary 1. *Let $x = (x_k)$ be a real-valued sequence. If $st_q - \inf x \neq st_q - \sup x$, then x is neither convergent nor quasi statistical convergent.*

Theorem 5. *Let $x = (x_k)$ and $y = (y_k)$ be two real-valued sequences such that $x - y = (x_k - y_k)$ is quasi statistical convergent to zero. Then,*

$$st_q - \inf x = st_q - \inf y \quad \textit{and} \quad st_q - \sup x = st_q - \sup y.$$

Proof. We only prove the first part i.e., $st_q - \inf x = st_q - \inf y$. The proof of the second part can be obtained by applying a similar technique.

Let the given conditions hold and suppose $l \in L_q(x)$ be arbitrary. Then, by definition

$$\delta_c(\{k \in \mathbb{N} : x_k < l\}) = 0.$$

Consequently,

$$\begin{aligned}\{k \in \mathbb{N} : y_k < l\} &= \{k \in \mathbb{N} : x_k - y_k \neq 0, y_k < l\} \cup \{k \in \mathbb{N} : x_k - y_k = 0, y_k < l\} \\ &\subseteq \{k \in \mathbb{N} : x_k - y_k \neq 0\} \cup \{k \in \mathbb{N} : x_k < l\}.\end{aligned}$$

From the above inclusion, it is clear that $\delta_c(\{k \in \mathbb{N} : y_k < l\}) = 0$ which implies $l \in L_q(y)$. This proves that

$$L_q(x) \subseteq L_q(y).$$

Similarly, one can establish $L_q(y) \subseteq L_q(x)$. Hence, $L_q(x) = L_q(y)$ holds and eventually $\sup L_q(x) = \sup L_q(y)$ i.e., $st_q - \inf x = st_q - \inf y$. QED

Remark 2. The converse of the above theorem is not necessarily true. Let $c_n = n$, $n \in \mathbb{N}$. Consider the sequences $x = (x_k)$ and $y = (y_k)$ defined by $x_k = (-1)^k$ and $y_k = (-1)^{k+1}$. Then, it is easy to verify that $st_q - \inf x = st_q - \inf y = 1$. But $\delta_c(\{k \in \mathbb{N} : x_k - y_k \neq 0\}) = \delta_c(\mathbb{N}) \neq 0$, i.e., $x - y$ is not quasi statistical convergent to zero.

Theorem 6. *Let $x = (x_k)$ be a real-valued sequence such that $x_k \xrightarrow{st_q} x_0$. Then, x_0 is uniquely determined.*

Proof. The proof is easy, so omitted. QED

Theorem 7. *Let $x = (x_k)$ and $y = (y_k)$ be two real-valued sequences such that $x_k \xrightarrow{st_q} x_0$ and $y_k \xrightarrow{st_q} y_0$. Then,*
(i) $x_k + y_k \xrightarrow{st_q} x_0 + y_0$ and (ii) $\lambda x_k \xrightarrow{st_q} \lambda x_0$, for $\lambda \in \mathbb{R}$.

Proof. The proof is easy, so omitted. QED

Definition 3. A real-valued sequence $x = (x_k)$ is said to be quasi statistically monotonic increasing (decreasing) if there exists a set $M = \{m_1 < m_2 < \cdots < m_j < \cdots\} \subseteq \mathbb{N}$ such that $\delta_c(M) = \delta_c(\mathbb{N})$ exists finitely and the subsequence (x_{m_j}) is monotonic increasing (decreasing).

If we take $c_n = n$, then Definition 3 coincides with the definition of statistically monotonic increasing (decreasing) sequence in [22].

Theorem 8. *Let $x = (x_k)$ be a real-valued sequence. Then,*
(i) x is quasi statistically monotonic increasing if and only if there exists two sequences $y = (y_k)$ and $z = (z_k)$ such that $x = y + z$ and (y_k) is monotonic increasing and $\delta_c(\{k \in \mathbb{N} : z_k \neq 0\}) = 0$;
(ii) x is quasi statistically monotonic decreasing if and only if there exists two

sequences $y = (y_k)$ and $z = (z_k)$ such that $x = y + z$ and (y_k) is monotonic decreasing and $\delta_c(\{k \in \mathbb{N} : z_k \neq 0\}) = 0$.

Proof. (i) Let x is quasi statistically monotonic increasing. Then by definition, there exists a set $M = \{m_1 < m_2 < \cdots < m_j < \cdots\} \subseteq \mathbb{N}$ such that $\delta_c(M) = \delta_c(\mathbb{N})$ exists finitely and the subsequence (x_{m_j}) is monotonic increasing. In other words, $x_{m_j} \leq x_{m_{j+1}}$ for any $j \in \mathbb{N}$. Let $n_k = \max\{m \in M : m \leq k\}$. Construct the sequences $y = (y_k)$ and $z = (z_k)$ defined by

$$y_k = \begin{cases} 0, & if\ k < m_1 \\ x_{n_k}, & otherwise \end{cases} \tag{2.8}$$

and

$$z_k = \begin{cases} x_k - y_k, & if\ k \notin M \\ 0, & if\ k \in M \end{cases}. \tag{2.9}$$

Then, it is clear from the construction that $x = y+z$, the sequence y is increasing and $\delta_c(\{k \in \mathbb{N} : z_k \neq 0\}) \leq \delta_c(\mathbb{N} \setminus M) = 0$.

The converse part is obvious, so omitted.

(ii) The proof is similar to that of (i), so omitted. *QED*

Theorem 9. *Let $x = (x_k)$ be a real-valued sequence. If x is quasi statistical bounded then it is statistical bounded.*

Proof. The proof is easy, so it is omitted. *QED*

The converse of the above theorem is not necessarily true. Following example illustrates the fact.

Example 1. Let (c_n) be a sequence satisfying $\lim\limits_{n\to\infty} c_n = \infty$ and $\lim\limits_{n\to\infty} \frac{\sqrt[3]{n}}{c_n} = \infty$. We can choose a subsequence (c_{n_p}) such that $c_{n_p} > 1$ for all $p \in \mathbb{N}$. Consider the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} c_k, & \text{if } k \text{ is a perfect cube and } c_k \in \{c_{n_p} : p \in \mathbb{N}\} \\ k, & \text{if } k \text{ is a perfect cube and } c_k \notin \{c_{n_p} : p \in \mathbb{N}\} \\ 0 & otherwise \end{cases}.$$

Then, it is easy to verify that x is statistical bounded but not quasi statistical bounded.

The following theorem gives a condition under which a statistical bounded sequence is also quasi statistical bounded.

Theorem 10. *Let $x = (x_k)$ be a real-valued statistical bounded sequence. Then x is quasi statistical bounded if $\inf\limits_n \frac{c_n}{n} > 0$.*

Proof. Since x is statistical bounded, so there exists $B > 0$ such that

$$\delta(\{k \in \mathbb{N} : |x_k| > B\}) = 0 \text{ i.e., } \lim_{n \longrightarrow \infty} \tfrac{1}{n} |\{k \in \mathbb{N} : |x_k| > B\}| = 0.$$

Now since the inequation

$$\frac{1}{n} |\{k \in \mathbb{N} : |x_k| > B\}| \geq \left(\inf_n \frac{c_n}{n} \right) \cdot \frac{1}{c_n} |\{k \in \mathbb{N} : |x_k| > B\}|$$

holds, so we must have

$$\lim_{n \longrightarrow \infty} \tfrac{1}{c_n} |\{k \in \mathbb{N} : |x_k| > B\}| = 0$$

i.e., x is quasi statistical bounded. QED

Theorem 11. *Let $x = (x_k)$ be a real-valued sequence. Then, x is quasi statistical bounded if and only if there exists a bounded sequence $y = (y_k)$ such that $\delta_c(\{k \in \mathbb{N} : x_k \neq y_k\}) = 0$.*

Proof. Firstly, let x is quasi statistical bounded. Then, by definition there exists $B > 0$ such that $\delta_c(A) = 0$, where $A = \{k \in \mathbb{N} : |x_k| > B\}$. Consider the seqence $y = (y_k)$ defined by

$$y_k = \begin{cases} x_k, & k \in \mathbb{N} \setminus A \\ 0, & otherwise \end{cases}.$$

Then, clearly y is bounded and $\delta_c(\{k \in \mathbb{N} : x_k \neq y_k\}) = 0$ because $\{k \in \mathbb{N} : x_k \neq y_k\} \subseteq A$.

For the converse part, let $\delta_c(\{k \in \mathbb{N} : x_k \neq y_k\}) = 0$. Since y is bounded, so there exists some $B > 0$ such that $|y_k| \leq B$ for all $k \in \mathbb{N}$. Then, the inclusion

$$\{k \in \mathbb{N} : x_k \neq y_k\} \supseteq \{k \in \mathbb{N} : |x_k| > B\}$$

holds and consequently, $\delta_c(\{k \in \mathbb{N} : |x_k| > B\}) = 0$. Hence, x is quasi statistical bounded. QED

Theorem 12. *Let $x = (x_k)$ be a real-valued sequence. Then,*
(i) If x is quasi statistical monotonic increasing sequence then x is quasi statistical convergent if and only if it is quasi statistical bounded;
(ii) If x is quasi statistical monotonic decreasing sequence then x is quasi statistical convergent if and only if it is quasi statistical bounded.

Proof. (i) Let x be quasi statistical monotonic increasing and $x_k \xrightarrow{st_q} x_0$. Then by Theorem 8 (i), there exists two sequences $y = (y_k)$ and $z = (z_k)$ such that $x = y + z$, where $\delta_c(\{k \in \mathbb{N} : z_k \neq 0\}) = 0$ i.e., $z_k \xrightarrow{st_q} 0$ and (y_k) is monotonic increasing. Consequently, $y = x - z$ and by Theorem 7, $y_k \xrightarrow{st_q} x_0$. Now the monotonicity of y implies that y is usual convergent and eventually bounded. Hence, x is quasi statistical bounded.

Conversly, suppose x is quasi statistical bounded. Then, there exists some $B > 0$ such that $\delta_c(N) = \delta_c(\mathbb{N})$, where $N = \{k \in \mathbb{N} : |x_k| \leq B\}$. Now we construct the decomposition $x = y + z$ in such a way that y is bounded. Let $M = \{m_1 < m_2 < \cdots < m_j < \cdots\} \subseteq \mathbb{N}$ be such that $\delta_c(M) = \delta_c(\mathbb{N})$ and $x_{m_j} \leq x_{m_{j+1}}$ for any $j \in \mathbb{N}$. Let $n_k = \max\{m \in M \cap N : m \leq k\}$ and define the sequences $y = (y_k)$ and $z = (z_k)$ by (2.8) and (2.9). Then, the rest of the proof can be easily obtained from Theorem 8.
(ii) The proof is similar to that of (i), so omitted. QED

Theorem 13. *Let $x = (x_k)$, $y = (y_k)$ and $z = (z_k)$ be three sequences such that $x_k \leq y_k \leq z_k$ for all $k \in M$ for some $M \subseteq \mathbb{N}$ with $\delta_c(M) = \delta_c(\mathbb{N})$. If $x_k \xrightarrow{st_q} x_0$ and $z_k \xrightarrow{st_q} x_0$, then $y_k \xrightarrow{st_q} x_0$.*

Proof. Let $\varepsilon > 0$ be given. Then the proof follows directly from the following inclusion:

$$\{k \in \mathbb{N} : |y_k - x_0| \geq \varepsilon\} \subseteq \{k \in \mathbb{N} : |x_k - x_0| \geq \varepsilon\} \cup \{k \in \mathbb{N} : |z_k - x_0| \geq \varepsilon\} \cup (\mathbb{N} \setminus M).$$

QED

Theorem 14. *A real-valued sequence $x = (x_k)$ is quasi statistical convergent if and only if for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $\delta_c(\{k \in \mathbb{N} : |x_k - x_{N_\varepsilon}| < \varepsilon\}) = \delta_c(\mathbb{N})$.*

Proof. Let $x_k \xrightarrow{st_q} x_0$. Then for any $\varepsilon > 0$, $\delta_c(B_\varepsilon) = \delta_c(\mathbb{N})$, where

$$B_\varepsilon = \{k \in \mathbb{N} : |x_k - x_0| < \tfrac{\varepsilon}{2}\}.$$

Fix $N_\varepsilon \in B_\varepsilon$. Then, for any $k \in B_\varepsilon$,

$$|x_k - x_{N_\varepsilon}| = |x_k - x_0 + x_0 - x_{N_\varepsilon}| \leq |x_k - x_0| + |x_{N_\varepsilon} - x_0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This implies that, $B_\varepsilon \subseteq \{k \in \mathbb{N} : |x_k - x_{N_\varepsilon}| < \varepsilon\}$ and consequently,

$$\delta_c(\{k \in \mathbb{N} : |x_k - x_{N_\varepsilon}| < \varepsilon\}) = \delta_c(\mathbb{N}).$$

Conversely, suppose that for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\delta_c(\{k \in \mathbb{N} : |x_k - x_{N_\varepsilon}| < \varepsilon\}) = \delta_c(\mathbb{N}).$$

Then, for any $\varepsilon > 0$, $\delta_c(C_\varepsilon) = \delta_c(\mathbb{N})$, where $C_\varepsilon = \{k \in \mathbb{N} : x_k \in [x_{N_\varepsilon} - \varepsilon, x_{N_\varepsilon} + \varepsilon]\}$. Denote the interval $[x_{N_\varepsilon} - \varepsilon, x_{N_\varepsilon} + \varepsilon]$ by I_ε. Fix an $\varepsilon > 0$. Then, $\delta_c(C_\varepsilon) = \delta_c(\mathbb{N})$ and $\delta_c(C_{\frac{\varepsilon}{2}}) = \delta_c(\mathbb{N})$ holds and consequently we have, $\delta_c(C_\varepsilon \cap C_{\frac{\varepsilon}{2}}) = \delta_c(\mathbb{N})$. But this implies that

$$I = I_\varepsilon \cap I_{\frac{\varepsilon}{2}} \neq \emptyset, \delta_c(\{k \in \mathbb{N} : x_k \in I\}) = \delta_c(\mathbb{N}), \quad \text{diam}\, I \leq \frac{1}{2} \text{diam}\, I_\varepsilon,$$

where diam I denote the length of the interval I. In this way, by induction, we can construct a sequence of closed intervals

$$I_\varepsilon = J_0 \supseteq J_1 \supseteq \cdots J_n \supseteq \cdots$$

such that

$$\text{diam}\, J_n \leq \tfrac{1}{2} \text{diam}\, J_{n-1} \text{ for } n = 2, 3, \cdots \text{ and } \delta_c(\{k \in \mathbb{N} : x_k \in J_n\}) = \delta_c(\mathbb{N}).$$

Then, there exists a $\eta \in \bigcap_{n \in \mathbb{N}} J_n$ and it is routine work to verify that $x_k \xrightarrow{st_q} \eta$. QED

Conclusion

In this paper, by defining the notion of quasi statistical lower and upper bound, we investigated how the notion of quasi statistical supremum, infimum are closely connected to the quasi statistical convergence of a real-valued sequence. Theorem 12 gives the essence of the quasi statistical boundedness for a quasi statistical monotonic sequence to become quasi statistical convergent. Theorem 14 gives the Cauchy condition for the quasi statistical convergence.

As a continuation of this work, one may investigate various properties of the space of all quasi statistical bounded sequence space:

$$l_\infty^{st_q} = \{x = (x_k) : \text{there exists } B > 0 \text{ such that } \delta_c(\{k \in \mathbb{N} : |x_k| > B\}) = 0\}.$$

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References

[1] M. Altinok, M. Küçükaslan: *A−statistical convergence and A−statistical monotonicity*, Appl. Math. E-Notes, **13** (2013), 249-260.

[2] M. Altinok, M. Küçükaslan: *A−statistical supremum-infimum and A−statistical convergence*, Azerb. J. Math., **4**, n. 2 (2014), 31-42.

[3] M. Altinok, M. Küçükaslan, U. Kaya: *Statistical extension of bounded sequence space*, Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat., **70**, n. 1 (2021), 82-99.

[4] M. Et, S. A. Mohiuddine, H. Sengul: *On lacunary statistical boundedness of order α*, Facta Univ. Ser. Math. Inform., **31**, n. 3 (2016), 707-716.

[5] H. Fast: *Sur la convergence statistique*, Cooloq. Math., **2** (1951), 241-244.

[6] A. R. Freedman, J. J. Sember: *Densities and summability*, Pacific J. Math., **95**, n. 2 (1981), 293-305.

[7] J. A. Fridy: *On statistical convergence*, Analysis, **5**, n. 4 (1985), 301-313.

[8] J. A. Fridy: *Statistical limit points*, Proc. Amer. Math. Soc., **118**, n. 4 (1993), 1187-1192.

[9] J. A. Fridy, C. Orhan: *Statistical limit superior and limit inferior*, Proc. Amer. Math. Soc., **125**, n. 12 (1997), 3625-3631.

[10] B. Hazarika, A. Esi: *On asymptotically Wijsman lacunary statistical convergence of set sequences in ideal context*, Filomat, **31**, n. 9 (2017), 2691-2703.

[11] B. Hazarika, S. A. Mohiuddine, M. Mursaleen: *Some inclusion results for lacunary statistical convergence in locally solid Riesz spaces*, Iran. J. Sci. Tech., **38**, n. A1 (2014), 61-68.

[12] B. Hazarika, E. Savas: *λ−statistical convergence in n−normed spaces*, An. Stiint. Univ. Ovidius Constanta Ser. Mat., **21**, n. 2 (2013), 141-153.

[13] M. Küçükaslan, M. Altinok: *Statistical supremum-infimum and statistical convergence*, Aligarh Bull. Math., **32**, n. 1-2 (2013), 39-54.

[14] M. Mursaleen: *λ−statistical convergence*, Math. Slovaca., **50**, n. 1 (2000), 111-115.

[15] I. Ozguc: *Results on quasi-statistical limit and quasi-statistical cluster points*, Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat., **69**, n. 1 (2020), 646-653.

[16] I. S. Ozguc, T. Yurdakadim: *On quasi-statistical convergence*, Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat., **61**, n. 1 (2012), 11-17.

[17] T. Šalát: *On statistically convergent sequences of real numbers*, Math. Slovaca., **30**, n. 2 (1980), 139-150.

[18] T. Šalát, B. C. Tripathy, M. Ziman: *On some properties of $\mathcal{I}$−convergence*, Tatra Mt. Math. Publ., **28** (2004), 279-286.

[19] E. Savas, P. Das: *A generalized statistical convergence via ideals*, Appl. Math. Lett., **24**, n. 6 (2011), 826-830.

[20] H. Steinhaus: *Sur la convergence ordinaire et la convergence asymptotique*, Colloq. Math., **2** (1951), 73-74.

[21] B. C. Tripathy: *On statistically convergent and statistically bounded sequences*, Bull. Malyasian Math. Soc., **20** (1997), 31-33.

[22] B. C. Tripathy: *On statistically convergent sequences*, Bull. Calcutta Math. Soc., **90** (1998), 259-262.

[23] B. C. Tripathy, M. Et: *On generalized difference lacunary statistical convergence*, Stud. Univ. Babes-Bolyai Math., **50**, n. 1 (2005), 119-130.