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Limiting sets in digital topology

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Abstract. Freezing sets and cold sets have been introduced as part of the theory of fixed points in digital topology. In this paper, we introduce a generalization of these notions, the limiting set, and examine properties of limiting sets.

Keywords: digital topology, digital image, freezing set, cold set, limiting set

MSC 2022 classification: primary 54H30, secondary 54H25

Introduction

The study of freezing sets and cold sets (see Definitions 1.8 and 2.1) stems from the fixed point theory of digital topology. Freezing and cold sets were studied in papers including [8, 9, 10, 12, 13, 14].

For a continuous function $f : (X, \kappa) \to (X, \kappa)$ and $A \subset X$, weaker restrictions on $(A, f|_A)$ than appear in the definitions of freezing and cold sets may yet result in interesting restrictions on f. In the current paper, we study how restrictions on how far f can move any member of A, can powerfully restrict how far f can move any member of X.

1 Preliminaries

We use $\mathbb{Z}$ for the set of integers, $\mathbb{N}$ for the set of natural numbers, and $\mathbb{N}^*$ for the set of nonnegative integers. Given $a, b \in \mathbb{Z}$, $a \leq b$,

$$[a, b]_{\mathbb{Z}} = \{ z \in \mathbb{Z} \mid a \leq z \leq b \}.$$

For a finite set X, the notation $\#X$ represents the number of distinct members of X.

Given a function $f : X \to Y$ and $A \subset X$, the notation $f|_A$ indicates the restriction of f to A. Also, we will occasionally use the notation fx to abbreviate $f(x)$.

1.1 Adjacencies

Much of this section is quoted or paraphrased from [15].

A digital image is a pair (X, κ) where $X \subset \mathbb{Z}^n$ for some n and κ is an adjacency on X. Thus, (X, κ) is a graph with X for the vertex set and κ determining the edge set. Usually, X is finite, although there are papers that consider infinite X, e.g., for covering spaces; we will consider infinite X in section 8. Usually, adjacency reflects some type of "closeness" in $\mathbb{Z}^n$ of the adjacent points. When these "usual" conditions are satisfied, one may consider the digital image as a model of a black-and-white "real world" digital image in which the black points (foreground) are the members of X and the white points (background) are members of $\mathbb{Z}^n \setminus X$.

We write $x \leftrightarrow_\kappa y$, or $x \leftrightarrow y$ when κ is understood or when it is unnecessary to mention κ, to indicate that x and y are κ-adjacent. Notations $x \leftrightarroweq_\kappa y$, or $x \leftrightarroweq y$ when κ is understood, indicate that $x \leftrightarrow_\kappa y$ or $x = y$.

The most commonly used adjacencies are the c_u adjacencies, defined as follows. Let $X \subset \mathbb{Z}^n$ and let $u \in \mathbb{N}$, $1 \leq u \leq n$. Then for points

$$x = (x_1, \ldots, x_n) \neq (y_1, \ldots, y_n) = y \text{ in } X$$

we have $x \leftrightarrow_{c_u} y$ if and only if

- for at most u indices i we have $|x_i - y_i| = 1$, and
- for all indices j, $|x_j - y_j| \neq 1$ implies $x_j = y_j$.

In low dimensions, the c_u-adjacencies are often denoted by the number of adjacent points a point can have in the adjacency. E.g.,

- in $\mathbb{Z}$, c_1-adjacency is 2-adjacency;
- in $\mathbb{Z}^2$, c_1-adjacency is 4-adjacency and c_2-adjacency is 8-adjacency;
- in $\mathbb{Z}^3$, c_1-adjacency is 6-adjacency, c_2-adjacency is 18-adjacency, and c_3-adjacency is 26-adjacency.

In this paper, we mostly use the c_1 and c_n adjacencies.

When (X, κ) is understood to be a digital image under discussion, we use the following notations. For $x \in X$,

$$N(x) = \{\, y \in X \mid y \leftrightarrow_\kappa x \,\},$$

$$N^*(x) = \{\, y \in X \mid y \leftrightarroweq_\kappa x \,\} = N(x) \cup \{\, x \,\}.$$

Definition 1.1. [22] Let $X \subset \mathbb{Z}^n$. The *boundary of* X is

$$Bd(X) = \{ x \in X \mid \text{there exists } y \in \mathbb{Z}^n \setminus X \text{ such that } x \leftrightarrow_{c_1} y \}.$$

Definition 1.2. A *digital κ-path P* in (X, κ) from $x \in X$ to $y \in X$ of *length* $= n$ is a subset $P = \{ x_i \}_{i=0}^{n}$ of X such that $x_0 = x$, $x_{i-1} \leftrightarrow_\kappa x_i$ for $i = 1, \ldots, n$, and $x_n = y$.

1.2 Digitally continuous functions

Much of this section is quoted or paraphrased from [15].

We denote by id or id_X the identity map $\mathrm{id}(x) = x$ for all $x \in X$.

Definition 1.3. [23, 5] Let (X, κ) and (Y, λ) be digital images. A function $f : X \to Y$ is (κ, λ)*-continuous*, or *digitally continuous* or just *continuous* when κ and λ are understood, if for every κ-connected subset X' of X, $f(X')$ is a λ-connected subset of Y. If $(X, \kappa) = (Y, \lambda)$, we say a function is κ*-continuous* to abbreviate "(κ, κ)-continuous."

Similar notions are referred to as *immersions*, *gradually varied operators*, and *gradually varied mappings* in [16, 17].

Theorem 1.4. [5] A function $f : X \to Y$ between digital images (X, κ) and (Y, λ) is (κ, λ)-continuous if and only if for every $x, y \in X$, if $x \leftrightarrow_\kappa y$ then $f(x) \leftrightarroweq_\lambda f(y)$.

Theorem 1.5. [5] Let $f : (X, \kappa) \to (Y, \lambda)$ and $g : (Y, \lambda) \to (Z, \mu)$ be continuous functions between digital images. Then $g \circ f : (X, \kappa) \to (Z, \mu)$ is continuous.

A function $f : (X, \kappa) \to (Y, \lambda)$ is an *isomorphism* (called a *homeomorphism* in [4]) if f is a continuous bijection such that f^{-1} is continuous.

For $X \in \mathbb{Z}^n$, the *projection to the i^{th} coordinate* is the function $p_i : X \to \mathbb{Z}$ defined by

$$p_i(x_1, \ldots, x_n) = x_i.$$

We denote by $C(X, \kappa)$ the set of κ-continuous functions $f : X \to X$.

Definition 1.6. ([5]; see also [20]) Let X and Y be digital images. Let $f, g : X \to Y$ be (κ, κ')-continuous functions. Suppose there is a positive integer m and a function $h : X \times [0, m]_{\mathbb{Z}} \to Y$ such that

- for all $x \in X$, $h(x, 0) = f(x)$ and $h(x, m) = g(x)$;
- for all $x \in X$, the induced function $h_x : [0, m]_{\mathbb{Z}} \to Y$ defined by

$$h_x(t) \;=\; h(x, t) \text{ for all } t \in [0, m]_{\mathbb{Z}}$$

  is $(c_1, \kappa')-$continuous. Thus, h_x is a path in Y.

- for all $t \in [0, m]_{\mathbb{Z}}$, the induced function $h_t : X \to Y$ defined by

$$h_t(x) \;=\; h(x, t) \text{ for all } x \in X$$

  is $(\kappa, \kappa')-$continuous.

Then h is a *digital* $(\kappa, \kappa')-$*homotopy between* f and g, and f and g are *digitally* $(\kappa, \kappa')-$*homotopic in* Y.

If $f \in C(X, \kappa)$ such that $f|_{f(X)} = \mathrm{id}_{f(X)}$, then f is a $(\kappa-)$*retraction of* X and $f(X)$ is a $(\kappa-)$*retract of* X.

1.3 Digital paths and simple closed curves

By Definition 1.2 and Theorem 1.4, $\{ x_i \}_{i=0}^{n} \subset X$ is a κ-path from x to y if and only if the function $f : [0, n]_{\mathbb{Z}} \to X$ given by $f(i) = x_i$ is (c_1, κ)-continuous. Such a function f is also called a κ-*path* from x to y.

If $i \neq j$ implies $x_i \neq x_j$ in a digital path P in (X, κ), we say P is an *arc* from $x \in X$ to $y \in X$.

Definition 1.7. [19] Let (X, κ) be a connected digital image. The *shortest path metric* for (X, κ) is

$$d_{(X,\kappa)}(x, y) = \min\{\mathit{length}(P) \mid P \text{ is a } \kappa\text{-arc in } X \text{ from } x \text{ to } y\}, \text{ for } x, y \in X.$$

Let

$$N^*(x, m) = \{ y \in X \mid d_{(X,\kappa)}(x, y) \le m \}$$

Notice $N^*(x, 1) = N^*(x)$.

Let (X, κ) be a finite connected digital image. The *diameter of* (X, κ) is

$$\mathit{diam}(X, \kappa) = \max\{ d_{(X,\kappa)}(x, y) \mid x, y \in X \}.$$

A *digital simple closed curve* of n points is a digital image (C_n, κ) such that $C_n = \{ c_i \}_{i=0}^{n-1}$, where $c_i \leftrightarrow_\kappa c_j$ if and only if $i = (j \pm 1) \mod n$. An indexing that satisfies these properties is a *circular ordering* or *circular indexing*. Often, we require $n \ge 4$. C_n may also be called a *cycle on* n *points*.

Let q and q' be distinct members of C_n. These points determine distinct arcs in C_n from q to q'. If one of these arcs is shorter than the other, the former is the *unique shortest arc in* C_n *from* q *to* q'.

1.4 Freezing sets

Definition 1.8. [8] Let (X,κ) be a digital image. We say $A \subset X$ is a *freezing set for* (X,κ) if given $f \in C(X,\kappa)$, $A \subset \mathrm{Fix}(f)$ implies $f = \mathrm{id}_X$. We say a freezing set A is *minimal* if no proper subset of A is a freezing set for (X,κ).

We recall the following.

Proposition 1.9. [15] Let (X,κ) be a digital image and $f \in C(X,\kappa)$. Suppose $x, x' \in \mathrm{Fix}(f)$ are such that there is a unique shortest κ-path P in X from x to x'. Then $P \subset \mathrm{Fix}(f)$.

Theorem 1.10. [8]; corrected proof in [12] Let $X = \Pi_{i=1}^n [r_i, s_i]_{\mathbb{Z}}$. Let $A = \Pi_{i=1}^n \{r_i, s_i\}$.

- Let $Y = \Pi_{i=1}^n [a_i, b_i]_{\mathbb{Z}}$ be such that $[r_i, s_i] \subset [a_i, b_i]_{\mathbb{Z}}$ for all i. Let $f : X \to Y$ be c_1-continuous. If $A \subset \mathrm{Fix}(f)$, then $X \subset \mathrm{Fix}(f)$.
- A is a freezing set for (X, c_1) that is minimal for $n \in \{1,2\}$.

Theorem 1.11. [8] Let $n > 4$. Let x_i, x_j, x_k be distinct members of C_n be such that C_n is a union of unique shorter arcs determined by pairs of these points. Let $f \in C(C_n,\kappa)$. Then $f = \mathrm{id}_{C_n}$ if and only if $\{x_i, x_j, x_k\} \subset \mathrm{Fix}(f)$; i.e., $\{x_i, x_j, x_k\}$ is a freezing set for C_n. Further, this freezing set is minimal.

2 n-limited sets

In this section, we introduce limiting sets and explore some of their basic properties.

2.1 Definition and general properties

Freezing sets and s-cold sets (the latter defined below) motivate this work.

Definition 2.1. [8] Given $s \in \mathbb{N}^*$, we say $A \subset X$ is an *s-cold set* for the connected digital image (X,κ) if given $g \in C(X,\kappa)$ such that $g|_A = \mathrm{id}_A$, then for all $x \in X$, $d_{(X,\kappa)}(x, g(x)) \leq s$. A *cold set* is a 1-cold set.

The notion of an s-cold set generalizes that of the freezing set, since a freezing set is 0-cold. Next, we introduce a generalization of a cold set.

Definition 2.2. Let X be a κ-connected digital image. Let $f \in C(X,\kappa)$ and let $A \subset X$. Let $m, n \in \mathbb{N}^*$. If for all $x \in A$ we have $d_{(X,\kappa)}(x, f(x)) \leq n$, we say $f|_A$ is an n-map. We say f is an n-map if $f|_X$ is an n-map. If for all $f \in C(X,\kappa)$, $f|_A$ being an m-map implies f is an n-map, then A is an *(m,n)-limiting set for* (X,κ) and (X,κ) is *(A,m,n)-limited*. Such a set is a *minimal (m,n)-limiting set for* (X,κ) if no proper subset A' of A is an (m,n)-limiting set for (X,κ).

Remark 2.3. The following are easily observed:

- $f \in C(X,\kappa)$ is a 0-map if and only if $f = \mathrm{id}_X$. Therefore, by Definitions 1.8 and 2.2, (X,κ) is $(A,0,0)$-limited if and only if A is a freezing set for (X,κ).
- More generally, (X,κ) is $(A,0,s)$-limited if and only if A is an s-cold set for (X,κ).
- $f \in C(X,\kappa)$ is a 1-map if and only if every $x \in X$ is an approximate fixed point of f, i.e., $f(x) \leftrightarrows_\kappa x$. Therefore, (X,κ) is $(A,1,1)$-limited if and only if for every $f \in C(X,\kappa)$, $f|_A$ being a 1-map implies every $x \in X$ is an approximate fixed point of f.
- If X is finite and κ-connected, then X is $(A,m,diam(X,\kappa))$-limited for $0 \le m \le diam(X,\kappa)$ and every nonempty $A \subset X$.
- If $f \in C(X,\kappa)$ such that $f|_A$ is an m-map and $A' \subset A$, then $f|_{A'}$ is an m-map.
- If (X,κ) is (A,m_0,n_0)-limited, $0 \le m_1 \le m_0$, and $n_0 \le n_1$, then (X,κ) is (A,m_1,n_1)-limited.
- If $f_i \in C(X,\kappa)$ are, respectively, n_i maps, $i \in \{1,2\}$, then $f_2 \circ f_1$ is an (n_1+n_2)-map.

Example 2.4. $\{0\}$ is a minimal $(0,1)$-limiting set for $([0,1]_{\mathbb{Z}},c_1)$, but is not $(0,0)$-limiting.

Proof. Both assertions follow from the observation that if $f : [0,1]_{\mathbb{Z}} \to [0,1]_{\mathbb{Z}}$ is the function $f(x) = 0$, then $f \in C([0,1]_{\mathbb{Z}},c_1)$. QED

Perhaps the significance of the property of being (A,m,n)-limited can be understood as follows. If n is much smaller than $diam(X,\kappa)$ and $f|_A$ is an m-map, then (X,κ) being (A,m,n)-limited implies f is an n-map, so f does not move any point of X by very much. Perhaps as a consequence, X and $f(X)$ will resemble each other, although such a conclusion will admit subjective exceptions.

The following generalizes the fact (that one can deduce from the second part of Theorem 1.10) that the set of endpoints of a digital interval is a freezing set for the interval.

Proposition 2.5. Let $X = [a,b]_{\mathbb{Z}}$. Let $m \in \mathbb{N}^*$. Let $A = \{a,b\}$. Then (X,c_1) is (A,m,m)-limited. Further, A is minimal if and only if $b - a > m$.

Proof. Let $f \in C(X,c_1)$ such that $f|_A$ is an m-map. Let $x \in X$, $a < x < b$.

- Suppose $f(x) > x + m$. Since $f(a) \leq a + m$ we have

$$f(x) - f(a) > x + m - (a + m) = x - a,$$

which is contrary to the continuity of f.

- We obtain a similar contradiction if $f(x) < x - m$.

Therefore, we must have $\mid f(x) - x \mid \leq m$, so f is an m-map.

Let $f_a, f_b : X \to X$ be given by $f_a(x) = a$, $f_b(x) = b$. These are c_1-continuous $(b-a)$-maps such that $f_a|_{\{a\}}$ is a 0-map, hence an m-map, and $f_b|_{\{b\}}$ is a 0-map, hence an m-map.

- If $b - a > m$, then A is minimal, since f_a and f_b are not m-maps.

- If $diam(X, c_1) = b - a \leq m$, then every member of $C(X, c_1)$ is an m-map, so A is not minimal.

QED

We have the following.

Proposition 2.6. Let (X, κ) be a connected digital image that is (A, m, n)-limited for some $\emptyset \neq A \subset X$ and $0 \leq m \leq n$. Then A is an n-cold set for (X, κ).

Proof. Let $f \in C(X, \kappa)$ such that $f|_A = \mathrm{id}_A$. Then $f|_A$ is a 0-map, hence an m-map. Therefore, f is an n-map. The assertion follows. QED

Proposition 2.7. [8] Let $m, n \in \mathbb{N}$. Let $X = [0, m]_{\mathbb{Z}} \times [0, n]_{\mathbb{Z}}$. Let $A \subset Bd(X)$ be such that no pair of c_1-adjacent members of $Bd(X)$ belong to $Bd(X) \setminus A$. Then A is a 1-cold set for (X, c_2).

Observe that the hypothesis for testing a set A for n-coldness, that $f|_A$ is a 0-map, is stricter than the hypothesis for testing (A, m, n)-limitedness, that $f|_A$ is an m-map. Therefore, it is perhaps not surprising that the converse of Proposition 2.6 is not generally true, as shown by the following.

Example 2.8. Let $X = [0, 2]_{\mathbb{Z}}^2$. Let

$$A = \{\, (0, 0), (0, 2), (2, 0), (2, 2) \,\} \subset X.$$

By Proposition 2.7, A is 1-cold for (X, c_2), i.e., (X, c_2) is $(A, 0, 1)$-limited. However, (X, c_2) is not $(A, 1, 1)$-limited.

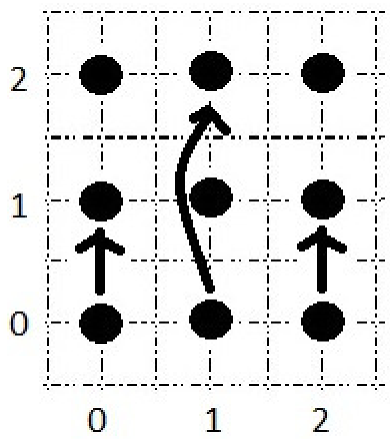

Figure 1: $X = [0,2]_{\mathbb{Z}}^2$. $f : X \to X$ is illustrated by arrows showing how points not fixed by f are mapped.

$$f(0,0) = (0,1), \quad f(1,0) = (1,2), \quad f(2,0) = (2,1).$$

This is the function of Example 2.8. One sees easily that $f \in C(X, c_2)$ and f is a 2-map, although for the corner set A, a 1-cold set of (X, c_2), $f|_A$ is a 1-map.

Proof. Let $f : X \to X$ be the function (see Figure 1)

$$f(x,y) = \begin{cases} (x,y) & \text{if } y > 0; \\ (x,1) & \text{if } (x,y) \in \{\,(0,0),(2,0)\,\}; \\ (1,2) & \text{if } (x,y) = (1,0). \end{cases}$$

It is easily seen that $f \in C(X, c_2)$ and $f|_A$ is a 1-map, but f is not a 1-map since $d_{(X,c_2)}((1,0), f(1,0)) = 2$. QED

We show that being (A, m, n)-limited is an isomorphism invariant.

Theorem 2.9. Let $F : (X, \kappa) \to (Y, \lambda)$ be an isomorphism of connected digital images. Let $\emptyset \neq A \subset X$. Let $m, n \in \mathbb{N}^*$. If (X, κ) is (A, m, n)-limited, then (Y, λ) is $(F(A), m, n)$-limited. Further, if A is a minimal (m, n)-limiting set for (X, κ), then $F(A)$ is a minimal (m, n)-limiting set for (Y, λ).

Proof. Let $f \in C(Y, \lambda)$ such that $f|_{F(A)}$ is an m-map. By Theorem 1.5, $G = F^{-1} \circ f \circ F \in C(X, \kappa)$.

Let $a \in A$. Then

$$m \geq d_{(Y,\lambda)}(Fa, fFa) = d_{(X,\kappa)}(F^{-1}Fa, F^{-1}fFa) = d_{(X,\kappa)}(a, Ga);$$

i.e., $G|_A$ is an m-map. It follows that G is an n-map.

Let $y = F(x)$ be an arbitrary point of Y. Then

$$d_{(Y,\lambda)}(y, f(y)) = d_{(X,\kappa)}(F^{-1}(y), F^{-1}(f(y))) =$$

$$d_{(X,\kappa)}(F^{-1}(F(x)), F^{-1}(f(F(x)))) = d_{(X,\kappa)}(x, G(x)) \le n.$$

Thus f is an n-map.

If A is minimal, suppose $F(A)$ is not. Then there is a proper subset B of $F(A)$ such that B is an (m,n)-limiting set for (Y,λ). Then $F^{-1}(B)$ is a proper subset of A and, by the above, is (m,n)-limiting for (X,κ). This is a contradiction of the minimality of A, which establishes that B is minimal. QED

Theorem 2.10. Let $\emptyset \neq A \subset X$ for a connected digital image (X,κ). Let $k \in \mathbb{N}^*$. Suppose for every $x \in X$ there exists $a_x \in A$ such that $d_{(X,\kappa)}(x, a_x) \le k$. Suppose $f \in C(X,\kappa)$. If $f|_A$ is an m-map, then f is an $(m+2k)$-map.

Proof. Given $x \in X$, let f and a_x be as described above. Then

$$d_{(X,\kappa)}(x, f(x)) \le d_{(X,\kappa)}(x, a_x) + d_{(X,\kappa)}(a_x, f(a_x)) + d_{(X,\kappa)}(f(a_x), f(x)) \le$$

$$k + m + k = m + 2k.$$

The assertion follows. QED

Corollary 2.11. Let $\emptyset \neq A \subset X$ for a connected digital image (X,κ). Let $k \in \mathbb{Z}$, $k \ge 0$. Suppose for every $x \in X$ there exists $a_x \in A$ such that $d_{(X,\kappa)}(x, a_x) \le k$. Then (X,κ) is $(A, m, m+2k)$-limited.

Proof. This follows from Theorem 2.10. QED

Recall A is a *dominating set for* (X,κ) if given $x \in X$ there exists $a \in A$ such that $x \leftrightarrows_\kappa a$.

Corollary 2.12. Let A be a dominating set for a connected digital image (X,κ).

- Suppose $f \in C(X,\kappa)$. If $f|_A$ is an m-map, then f is an $(m+2)$-map.
- If for every $f \in C(X,\kappa)$, $f|_A$ is an m-map, then (X,κ) is $(A, m, m+2)$-limited.

Proof. These assertions follow by taking $k = 1$ in Theorem 2.10 and Corollary 2.11. QED

We cannot in general replace $m+2$ by $m+1$ in Theorem 2.10 or in Corollary 2.11, as shown by the following.

Example 2.13. Let $X = [-1,1]_{\mathbb{Z}}$. Then $A = \{0\}$ dominates (X, c_1) and the function $f(x) = |\ x\ |$ is easily seen to be a member of $C(X, c_1)$ such that $f|_A$ is a 0-map, but f is a 2-map that is not a 1-map.

2.2 Diameter

Lemma 2.14. Let Y be a κ-connected subset of the connected digital image (X, κ). Let $y_0, y_1 \in Y$. Then

$$d_{(Y,\kappa)}(y_0, y_1) \geq d_{(X,\kappa)}(y_0, y_1).$$

Proof. We have $d_{(Y,\kappa)}(y_0, y_1) = length(P)$, where P is a shortest κ-path in Y from y_0 to y_1. Since $P \subset X$, the assertion follows. QED

That the inequality in Lemma 2.14 may be strict is shown in the following.

Example 2.15. Let (X, κ), $X = \{p_i\}_{i=0}^{m}$, be a digital simple closed curve of $m+1$ points that are circularly indexed, $m \geq 4$. Let $Y = X \setminus \{p_0\}$. Then

$$d_{(Y,\kappa)}(p_1, p_m) = m - 1 > 2 = d_{(X,\kappa)}(p_1, p_m).$$

Theorem 2.16. Let (X, κ) be a finite connected digital image. Let $f : X \to X$ be an m-map. Consider the inequality

$$diam(f(X)) \geq diam(X) - 2m \tag{2.1}$$

Inequality (2.1) is valid if $diam(X)$ is computed using $d_{(X,\kappa)}$ or using $d_{(f(X),\kappa)}$.

Proof. Since X is finite, $diam(X) = M < \infty$. Therefore, there exist $x, y \in X$ and a κ-path P in X of length M from x to y. We have

$$M = d_{(X,\kappa)}(x, y) \leq d_{(X,\kappa)}(x, f(x)) + d_{(X,\kappa)}(f(x), f(y)) + d_{(X,\kappa)}(f(y), y)$$

$$\leq m + d_{(X,\kappa)}(f(x), f(y)) + m, \qquad \text{or}$$

$$M - 2m \leq d_{(X,\kappa)}(f(x), f(y)).$$

Using Lemma 2.14, it follows that

$$M - 2m \leq diam_{(X,\kappa)}(f(X)) \leq diam_{(f(X),\kappa)}(f(X)).$$

QED

2.3 Hyperspace metrics

The *Hausdorff metric* [21] is often used as a measure of how similarly positioned two objects are in a metric space. Given nonempty subsets A and B of a metric space (X, d), the Hausdorff metric (based on d) for the distance between A and B, $H_d(A, B)$, is the smallest $\varepsilon \geq 0$ such that given $a \in A$ and $b \in B$, there exist $a' \in A$ and $b' \in B$ such that

$$\max\{\, d(a, b'), d(a', b)\,\} \leq \varepsilon.$$

Borsuk's *metric of continuity* [2, 3] based on a metric d has been adapted to digital topology [11] as follows. For digital images Y_0 and Y_1 in connected (X, κ), the metric of continuity $\delta_d(Y_0, Y_1)$ is the greatest lower bound of numbers $t > 0$ such that there are κ-continuous $f : Y_0 \to Y_1$ and $g : Y_1 \to Y_0$ with

$$d_{(X,\kappa)}(x, f(x)) \leq t \text{ for all } x \in Y_0 \text{ and } d_{(X,\kappa)}(y, g(y)) \leq t \text{ for all } y \in Y_1.$$

Proposition 2.17. [11] Given finite digital images (X, κ) and (Y, κ) in $\mathbb{Z}^n$ and a metric d for $\mathbb{Z}^n$, $H_d(X, Y) \leq \delta_d(X, Y)$.

Theorem 2.18. Let (X, κ) be a finite connected digital image. Let $f \in C(X, \kappa)$ be an m-map. Then

$$H_{d_{(X,\kappa)}}(X, f(X)) \leq \delta_{d_{(X,\kappa)}}(X, f(X)) \leq m.$$

Proof. That $H_{d_{(X,\kappa)}}(X, f(X)) \leq \delta_{d_{(X,\kappa)}}(X, f(X))$ comes from Proposition 2.17.

Let $x \in X$ and let $y = f(x) \in f(X)$. By choice of f, $d_{(X,\kappa)}(x, y) \leq m$.

Let $g : f(X) \to X$ be the inclusion function $g(y) = y$. Clearly g is a 0-map, hence an m-map. The assertion follows. QED

2.4 Rigid images

Definition 2.19. [18] A digital image (X, κ) is *rigid* if the only member of $C(X, \kappa)$ that is homotopic to id_X is id_X.

See Figure 2 for an example of a rigid digital image.

Proposition 2.20. Let (X, κ) be a connected digital image that is rigid. Then there is no 1-map in $C(X, \kappa)$ other than id_X.

Proof. Let f be a member of $C(X, \kappa)$ that is a 1-map. Then the function $h : X \times [0, 1]_{\mathbb{Z}} \to X$ defined by

$$h(x, 0) = x, \quad h(x, 1) = f(x)$$

is easily seen to be a homotopy from id_X to f. It follows from Definition 2.19 that $f = \mathrm{id}_X$. QED

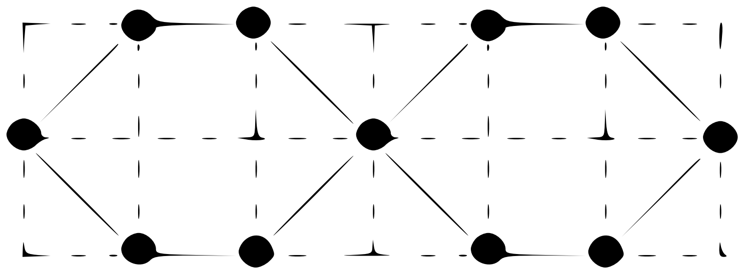

Figure 2: The digital image of Figure 1 of [15]. Example 3.11 of [15] shows this image, using c_2 adjacency, is rigid.

3 Trees

In this section, (T, κ) is a finite tree and A is the set of members of T that have degree 1, i.e., for $a \in T$, $\#N(a) = 1$ if and only if $a \in A$.

Theorem 3.1. [8] A is a minimal freezing set for (T, κ) when $\#T > 1$.

Theorem 3.2. Let (T, κ) be a finite tree and let A be the set of members of T that have degree 1. Then A is an (m, m)-limiting set for (T, κ), for $m \in \{0, 1\}$. For $m = 1$, A is minimal if and only if $\#T \neq 2$.

Proof. Trivially, when $\#T = 1$, $A = \emptyset$ is $(0, 0)$-limiting. By Theorem 3.1, A is $(0, 0)$-limiting when $\#T > 1$. Next we show that A is a $(1, 1)$-limiting set for (T, κ).

Let $x \in X$, $f \in C(T, \kappa)$ such that $f|_A$ is a 1-map. We show that the assumption

$$d_{(T,\kappa)}(x, f(x)) > 1 \tag{3.2}$$

leads to a contradiction.

There exist $y_0, y_1 \in A$ such that the unique shortest κ-path P in T from y_0 to y_1 contains $\{x, f(x)\}$.

Let $p : [0, k]_{\mathbb{Z}} \to P$ be an isomorphism with $p(0) = y_0$ and $p(k) = y_1$. Since $y_0, y_1 \in A$, we must have $\{f(y_0), f(y_1)\} \subset P$. Since $f|_A$ is a 1-map,

$$f(y_0) \in \{y_0 = p(0), p(1)\}, \qquad f(y_1) \in \{y_1 = p(k), p(k-1)\}$$

Let $x = p(t)$, $f(x) = p(t')$.

- Suppose $t \leq t'$. By (3.2), $t' > t + 1$, and $d_{(T,\kappa)}(y_0, x) = t$. Then

  $$d_{(T,\kappa)}(f(x), f(y_0)) \geq t' - 1 > t + 1 - 1 = t = d_{(T,\kappa)}(x, y_0),$$

  which is impossible, since f is continuous.

- A similar contradiction arises if $t > t'$.

Therefore, f is a 1-map, which establishes that A is $(1,1)$-limiting.

Suppose $\#T \neq 2$. As above, A is minimal for $\#T = 1$. So assume $\#T > 2$. Then $\#A > 1$, since (T, κ) is a tree. Let a and b be distinct members of A. Let P be the unique shortest κ-path in T from a to b. There is a (c_1, κ)-isomorphism $p : [0, length(P)]_{\mathbb{Z}} \to P$ such that $p(0) = a$. Then the function $f : T \to T$ given by

$$f(x) = \begin{cases} x & \text{if } x \neq a; \\ p(2) & \text{if } x = a, \end{cases}$$

is easily seen to belong to $C(T, \kappa)$; and f is a 2-map. Thus $A \setminus \{ a \}$ is not a $(1,1)$-limiting set, so A is a minimal $(1,1)$-limiting set for (T, κ).

For $\#T = 2$, $A \neq \emptyset$, so A is not minimal since in this case we see easily that $\emptyset$ is a minimal $(1,1)$-limiting set for (T, κ). QED

4 Rectangles with axis-parallel sides

In this section, we consider limiting sets for digital rectangles of the form $[a,b]_{\mathbb{Z}} \times [c,d]_{\mathbb{Z}} \subset \mathbb{Z}^2$.

4.1 c_1 (A, m, m)-limited rectangles

Theorem 4.1. Let $X = [0,j]_{\mathbb{Z}} \times [0,k]_{Z}$. Let A be the set of corners of X, i.e.,

$$A = \{ (0,0),\ (0,k),\ (j,0),\ (j,k) \}.$$

Then (X, c_1) is (A, m, m)-limited for all $m \in \mathbb{N}^*$.

Proof. Let $f \in C(X, c_1)$ such that $f|_A$ is an m-map. Suppose there exists $(a,b) \in X$ such that

$$d_{(X,c_1)}((a,b), f(a,b)) > m \tag{4.3}$$

First we consider the case in which f moves (a,b) away from the corner $(0,0)$, i.e., $p_1(f(a,b)) \geq a$, $p_2(f(a,b)) \geq b$. From (4.3) it follows that

$$p_1(f(a,b)) - a + p_2(f(a,b)) - b > m,$$

so

$$\begin{aligned} d_{(X,c_1)}((a,b),(0,0)) = a + b < p_1(f(a,b)) + p_2(f(a,b)) - m \leq \\ p_1(f(a,b)) + p_2(f(a,b)) - [p_1(f(0,0)) + p_2(f(0,0))] \\ = d_{(X,c_1)}(f(a,b), f(0,0)). \end{aligned}$$

which is impossible since f is continuous.

A similar argument can be made for each of the following cases.

- f moves (a,b) away from the corner $(j,0)$, i.e.,

$$p_1(f(a,b)) \leq a, \quad p_2(f(a,b)) \geq b.$$

  Here we obtain $d_{(X,c_1)}(f(a,b), f(j,0)) > d_{(X,c_1)}((a,b),(j,0))$. As above, the latter is a contradiction.

- f moves (a,b) away from the corner $(0,k)$, i.e.,

$$p_1(f(a,b)) \geq a, \quad p_2(f(a,b)) \leq b.$$

  Here we obtain $d_{(X,c_1)}(f(a,b), f(0,k)) > d_{(X,c_1)}((a,b),(0,k))$. As above, the latter is a contradiction.

- f moves (a,b) away from the corner (j,k), i.e.,

$$p_1(f(a,b)) \leq a, \quad p_2(f(a,b)) \leq b.$$

  Here we obtain $d_{(X,c_1)}(f(a,b), f(j,k)) > d_{(X,c_1)}((a,b),(j,k))$. As above, the latter is a contradiction.

Since each case yields a contradiction, the assertion is established. $\boxed{QED}$

4.2 c_n (A,m,m)-limited n-cubes

Theorem 4.2. Let $X = \prod_{i=1}^{n} [a_i, b_i]_{\mathbb{Z}}$. Let A be the set of corners of X, i.e., $A = \prod_{i=1}^{n} \{ a_i, b_i \}$. Let $m > 0$. Let $f \in C(X, c_n)$ such that $f|_A$ is m-limited. Then f is m-limited. Hence (X, c_n) is (A,m,m)-limited.

Proof. If otherwise, then for some $x = (x_1, \ldots, x_n) \in X$, $d_{(X,c_n)}(x, f(x)) > m$. Then for some index i, $\mid x_i - p_i(f(x)) \mid \; > m$. This leads to a contradiction as in the proof of Theorem 4.1. The assertion follows. $\boxed{QED}$

Proposition 4.3. [8] Let $X = \prod_{i=1}^{n} [0, m_i]_{\mathbb{Z}} \subset \mathbb{Z}^n$, where $m_i > 1$ for all i. Let $A \subset Bd(X)$ be such that A is not c_n-dominating in $Bd(X)$. Then A is not a cold set for (X, c_n).

Remark 4.4. We cannot include 0 as a value of m in Theorem 4.2 since, by Proposition 4.3, if for some index i we have $b_i - a_i > 2$ then A is not cold and therefore is not a freezing set for (X, c_n).

5 Simple closed curves

We consider limiting sets for digital simple closed curves. Throughout this section, C_v is a digital simple closed curve (a *cycle* of v points): $C_v = \{ x_i \}_{i=0}^{v-1}$ for some $v \geq 4$ with the members of C_v indexed circularly.

For $v \geq 4$, we define

$$D(v) = \begin{cases} \frac{v-2}{4} - 1 & \text{if } v-2 \text{ is a multiple of 4;} \\ \lfloor \frac{v-2}{4} \rfloor & \text{if } v-2 \text{ is not a multiple of 4.} \end{cases}$$

For $0 \leq d < v$, we say the function $r_d : C_v \to C_v$ given by

$$r_d(x_i) = x_{(i+d) \mod v}$$

is a *rotation.*

The function $f : C_v \to C_v$ given by $f(x_i) = x_{(v-i) \mod v}$ is the *flip map.*

Rotations and flip maps are isomorphisms of (C_v, κ).

Theorem 5.1. [15] Let $f \in C(C_v, \kappa)$. Then

- f is not surjective, or
- f is a rotation, or
- f is the composition of a flip map and a rotation.

Theorem 5.2. [6] Let $f : C_v \to \mathbb{Z}$ be (κ, c_1)-continuous. Assume v is even. If f is not surjective, there exists $x_u \in C_v$ such that $f(x_u) \leftrightarrows_{c_1} f(x_{u+v/2})$.

Proposition 5.3. Let $f \in C(C_v, \kappa)$. Suppose f is not surjective and v is even. Then there is an index j such that $d_{(C_v,\kappa)}(x_j, f(x_j)) \geq (v-2)/4$, so f is not a $D(v)$-map.

Proof. Since f is nonsurjective and continuous, $f(C_v)$ is (κ, c_1)-isomorphic to a digital interval. By Theorem 5.2, there exists $x_u \in C_v$ such that $f(x_u) \leftrightarrows_\kappa f(x_{u+v/2})$. Hence

$$v/2 = d_{(C_v,\kappa)}(x_u, x_{u+v/2}) \leq$$

$$d_{(C_v,\kappa)}(x_u, f(x_u)) + d_{(C_v,\kappa)}(f(x_u), f(x_{u+v/2})) + d_{(C_v,\kappa)}(f(x_{u+v/2}), x_{u+v/2}) \leq$$

$$d_{(C_v,\kappa)}(x_u, f(x_u)) + 1 + d_{(C_v,\kappa)}(f(x_{u+v/2}), x_{u+v/2})$$

or

$$\frac{v-2}{2} \leq d_{(C_v,\kappa)}(x_u, f(x_u)) + d_{(C_v,\kappa)}(f(x_{u+v/2}), x_{u+v/2})$$

Thus

$$\max\{\, d_{(C_v,\kappa)}(x_u, f(x_u)), d_{(C_v,\kappa)}(x_{u+v/2}, f(x_{u+v/2}))\,\} \ge \frac{v-2}{4} > D(v).$$

Thus f is not a $D(v)$-map. QED

Proposition 5.4. The rotation r_d is an m_d-map, for $m_d = \min\{\, d, v-d\,\}$. Indeed, $d_{(C_v,\kappa)}(x_i, r_d(x_i)) = m_d$ for all indices i.

Proof. Elementary and left to the reader. QED

Theorem 5.5. Let $A = \{\, x_i, x_j, x_k\,\} \subset C_v$ where C_v is a union of unique shorter arcs determined by pairs of these points. (Note that A is a freezing set for (C_v, κ), by Theorem 1.11.) Let R_{ij} be the unique shorter arc from x_i to x_j, R_{ik} be the unique shorter arc from x_i to x_k, and R_{jk} be the unique shorter arc from x_j to x_k. Let

$$0 \le m \le \min\{\, D(v),\ length(R_{ij})/2,\ length(R_{ik})/2,\ length(R_{jk})/2\,\}.$$

Then (C_v, κ) is (A, m, m)-limited.

Proof. By our choice of m and Proposition 5.3, every m-map in $C(C_v, \kappa)$ is a surjection.

By Proposition 5.4, if $f \in C(C_v, \kappa)$ is a rotation and $f|_A$ is an m-map, then f is an m-map.

Let ℓ be the flip map of C_v, $g = \ell \circ r_d : C_v \to C_v$, and $g' = r_d \circ \ell : C_v \to C_v$. By Theorem 1.5, $g, g' \in C(C_v, \kappa)$ and these functions are isomorphisms, hence distance-preserving with respect to $d_{(C_v,\kappa)}$. Let $G \in \{\, g,\ g'\,\}$.

Suppose $G(x_i) \in N^*(x_i, m)$. By our choice of m and the fact that G is orientation-reversing, $G(x_j) \notin N^*(x_j, m)$. Similarly, if $G(x_j) \in N^*(x_j, m)$ or $G(x_k) \in N^*(x_k, m)$, then $G(x_i) \notin N^*(x_i, m)$. Therefore G is not an m-map.

We conclude by Theorem 5.1 that if $f \in C(C_v, \kappa)$ such that $f|_A$ is an m-map, then f is a rotation and an m-map. Thus (C_v, κ) is $A(m, m)$-limited. QED

6 Limiting sets and retracts

We show how if X and Y are finite connected digital images such that Y is a retract of X, then a limiting set for Y is a limiting set for X, although not necessarily with the same (m, n) pair.

Theorem 6.1. Let $\emptyset \neq A \subset Y \subset X$ such that (X,κ) and (Y,κ) are finite and connected and Y is a κ-retract of X by a retraction $r \in C(X,\kappa)$ that is an ε-map. Let $h = H_{d_{(X,\kappa)}}(X,Y)$. Suppose Y is (A,m,n)-limited. Let $f \in C(X.\kappa)$ such that $f|_A$ is m-limited. Then f is an $(n+2h+\varepsilon)$-map; hence (X,κ) is $(A,m,n+2h+\varepsilon)$-limited.

Proof. Let $f \in C(X,\kappa)$ be such that $f|_A$ is an m-map. Then given $a \in A$, there is a κ-path $P \subset X$ from a to $f(a)$ of length at most m. Then $r \circ f|_Y \in C(Y,\kappa)$ and $r(P)$ is a κ-path in Y from $r(a) = a$ to $r(f(a))$ of length at most m. Therefore, $r \circ f|_A$ is an m-map, so $r \circ f|_Y$ is an n-map.

Let $x \in X$. There exists $y \in Y$ such that $d(x,y) \leq h$. Then

$$d(x,f(x)) \leq d(x,y) + d(y,r(f(y))) + d(r(f(y)),r(f(x))) + d(r(f(x)),f(x))$$

$$\leq h + n + d(y,x) + \varepsilon \leq n + 2h + \varepsilon.$$

Thus, f is an $(n+2h+\varepsilon)$-map; thus (X,κ) is $(A,m,n+2h+\varepsilon)$-limited. QED

The bound $n+2h+\varepsilon$ in Theorem 6.1 is not generally tight, as shown in the following.

Example 6.2. Let (X,κ) be a finited connected digital image. Let $x_0 \in X$ and $A = Y = \{x_0\}$. Let $h = H_{d(X,\kappa)}(X,Y)$. Clearly A is a freezing set for (Y,κ), i.e., (Y,κ) is $(A,0,0)$-limited. Since the function $r : X \to X$ given by $r(x) = x_0$ is an an h-map and a retraction of X to Y, Theorem 6.1 implies (X,κ) is $(A,0,3h)$-limited. However, (X,κ) is $(A,0,h)$-limited, as noted in Remark 2.3.

7 Cartesian products

Elementary properties of limiting sets for Cartesian products of digital images are discussed in this section.

Given digital images or graphs (X,κ) and (Y,λ), the *normal product adjacency* $NP(\kappa,\lambda)$ (also called the *strong adjacency* [25]) generated by κ and λ on the Cartesian product $X \times Y$ is defined as follows.

Definition 7.1. [1, 24] Let $x,x' \in X$, $y,y' \in Y$. Then (x,y) and (x',y') are $NP(\kappa,\lambda)$-adjacent in $X \times Y$ if and only if

- $x = x'$ and y and y' are λ-adjacent; or
- x and x' are κ-adjacent and $y = y'$; or
- x and x' are κ-adjacent and y and y' are λ-adjacent.

Definition 7.2. [24, 7] Let $u, v \in \mathbb{N}$, $1 \leq u \leq v$. Let (X_i, κ_i) be digital images, $i \in \{1, \ldots, v\}$. Let $x_i, y_i \in X_i$, $x = (x_1, \ldots, x_v)$, $y = (y_1, \ldots, y_v)$. Then $x \leftrightarrow y$ in the *generalized normal product adjacency* $NP_u(\kappa_1, \ldots, \kappa_v)$ if for at least 1 and at most u indices i, $x_i \leftrightarrow_{\kappa_i} y_i$, and for all other indices j, $x_j = y_j$.

Given a set of functions $f_i : X_i \to Y_i$ for $1 \leq i \leq n$, the *product function* $\Pi_{i=1}^n f_i : \Pi_{i=1}^n X_i \to \Pi_{i=1}^n Y_i$ is the function

$$(\Pi_{i=1}^n f_i)(x_1, \ldots, x_n) = (f_1(x_1), \ldots, f_n(x_n)), \text{ where } x_i \in X_i.$$

Theorem 7.3. [7] Let $f_i : (X_i, \kappa_i) \to (Y_i, \lambda_i)$, $1 \leq i \leq v$. Then the product map

$$\Pi_{i=1}^v f_i : (\Pi_{i=1}^v X_i, NP_v(\kappa_1, \ldots, \kappa_v)) \to (\Pi_{i=1}^v Y_i, NP_v(\lambda_1, \ldots, \lambda_v))$$

is continuous if and only if each f_i is continuous.

Theorem 7.4. Let $X = \prod_{i=1}^v X_i$. Let $\kappa = NP_v(\kappa_1, \ldots, \kappa_v)$. Let $\emptyset \neq A \subset X$. Let $A_i = p_i(A)$ for each index i. Suppose (X, κ) is (A, m, n)-limited. Then for each index i, (X_i, κ_i) is (A_i, m, n)-limited.

Proof. Suppose $f_i \in C(X_i, \kappa_i)$ such that $f_i|_{A_i}$ is an m-map. Then for each $x_i \in A_i$ there is a path $g_i : [0, m]_{\mathbb{Z}} \to X_i$ from x_i to $f_i(x_i)$. Consider the function $G : [0, m]_{\mathbb{Z}} \to X$ given by

$$G(t) = (g_1(t), \ldots, g_v(t)).$$

Clearly, G is (c_1, κ)-continuous. Also, by Theorem 7.3,

$$F = \prod_{i=1}^v f_i \in C(X, \kappa).$$

Thus, G is a κ-path of length at most m from $(x_1, \ldots, x_v) \in A$ to $F(x_1, \ldots, x_v)$. Hence $F|_A$ is an m-map. Therefore, F is an n-map. Therefore, given $x = (x_1, \ldots, x_v) \in X$, there is a path $H : [0, n]_{\mathbb{Z}} \to X$ from x to $F(x)$. Then $p_i \circ H$ is a κ_i-path of length at most n in X_i from x_i to $f_i(x_i)$. The assertion follows. QED

8 Infinite X

In this section, we give elementary properties of limiting sets for infinite digital images.

Proposition 8.1. Let X be an infinite subset of $\mathbb{Z}^n$. Then for $1 \leq u \leq v$, any $m, n \in \mathbb{N}^*$, and any finite $A \subset X$, (X, c_u) is not (A, m, n) limited.

Proof. Given any finite subset A of $\mathbb{Z}^n$, there exists a cube $Y = [-k,k]_{\mathbb{Z}}^v$ such that $A \subset Y$.

Let $r_1 : \mathbb{Z} \to \mathbb{Z}$ be the c_1-retraction

$$r_1(x) = \begin{cases} -k & \text{if } x \leq -k; \\ x & \text{if } -k \leq x \leq k; \\ k & \text{if } k \leq x. \end{cases}$$

Then the function $r_v : X \to Y$ given by

$$r_v(x_1, \ldots, x_v) = (r_1(x_1), \ldots, r_1(x_v))$$

is easily seen to be a c_u-retraction of $\mathbb{Z}^n$ to Y. Therefore $r|_A$ is a 0-map, hence an m-map.

However, since X is infinite and Y is finite, given any $m \in \mathbb{N}^*$, there exists $x \in X$ such that no member of Y is within m of x in the $d_{(X,c_u)}$ metric. The assertion follows. QED

Example 8.2. Let $m \in \mathbb{N}$. The set

$$m\mathbb{Z} = \{\, x \in \mathbb{Z} \mid x = mk \text{ for some } k \in \mathbb{Z}\,\}$$

is an (m,m)-limiting set for $(\mathbb{Z}, c_1)$.

Proof. If $m = 1$ then the assertion is trivial. Thus we assume $m > 1$.

Let $f \in C(\mathbb{Z}, c_1)$ such that $f|_{m\mathbb{Z}}$ is an m-map. Let $z \in \mathbb{Z} \setminus m\mathbb{Z}$. For some $k \in Z$ and $q \in [1, m-1]_{\mathbb{Z}}$,

$$z = mk + q.$$

We must show $|\, f(z) - z \,| \ \leq m$. Suppose this is false.

If $k \geq 0$ we proceed as follows.

- If $f(z) - z > m$ then

  $$f(z) > z + m = m(k+1) + q,$$

  and, since $f|_{m\mathbb{Z}}$ is an m-map, $f(mk) \leq m(k+1)$, so

  $$f(z) - f(mk) > m(k+1) + q - m(k+1) = q.$$

  Thus $f([mk,z]_{\mathbb{Z}})$ is a c_1- path in $\mathbb{Z}$ of length greater than q. But $[mk,z]_{\mathbb{Z}}$ is a c_1-path in $\mathbb{Z}$ of length q. This is impossible.

- Similarly, if $f(z) - z < -m$, we obtain a contradiction.

Similarly, if $k < 0$, we must obtain a contradiction. It follows that f is an m-map. Hence $m\mathbb{Z}$ is an (m,m)-limiting set for $(\mathbb{Z}, c_1)$. QED

9 Further remarks

The fixed point theory for digital images has led us to the study of limiting sets, in the sense that the notion of an (A, m, n)-limited digital image (X, κ) generalizes the notion of A being a freezing set for (X, κ). If m and n are small relative to $diam(X, \kappa)$, we may expect $f \in C(X, \kappa)$, such that $f|_A$ is an m-map, to move no point of X by very much (i.e., by more than n) and therefore, $f(X)$ might be expected to resemble X (although such a conclusion will be subjective and may subjectively admit of exceptions).

We have explored several basic properties of limiting sets, including some of their relationships with retractions, Cartesian products, and infinite cardinality. We have seen that often, if A is a freezing set for (X, κ) (i.e., (X, κ) is $(A, 0, 0)$-limited), then (X, κ) is (A, m, n)-limited for small m, n such that $(m, n) \neq (0, 0)$.

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