Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **43** (2023) no. 1, 57–65. doi:10.1285/i15900932v43n1p57

On a nonlocal boundary value problem for parabolic-hyperbolic type equation

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Received: 25.12.2022; accepted: 18.8.2023.

Abstract. We consider the mixed type equation with the nonlocal boundary condition and the initial condition. A uniqueness for the solvability of this boundary problem is shown.

Keywords: parabolic-hyperbolic equation, initial-boundary value problem, spectral method, unique solvability

MSC 2022 classification: 35J05, 35J08, 35J75

1 Introduction

In a rectangular domain $\mathfrak{D} = \{(\xi, t) : 0 < \xi < 1, \quad -\alpha < t < \beta\}$, we consider the following parabolic-hyperbolic type equation

$$\mathfrak{L}U \equiv \begin{cases} U_t - U_{\xi\xi} = 0, & t > 0 \\ U_{tt} - U_{\xi\xi} = 0, & t < 0 \end{cases}$$

where α, β are positive real numbers. In the domain $\mathfrak{D}$ we find the solution $U = U(\xi, t)$ that satisfies the conditions

$$U(\xi, t) \in \Omega = C(\overline{\mathfrak{D}}) \cap C^1(\mathfrak{D}) \cap C^1_\xi(\overline{\mathfrak{D}}) \cap C^2(\mathfrak{D}_-) \cap C^2_\xi(\mathfrak{D}_+),$$

$$\mathfrak{L}U(\xi, t) = 0, \quad (\xi, t) \in \mathfrak{D}_+ \cup \mathfrak{D}_-, \tag{1.1}$$

$$U(\xi, -\alpha) = \varphi(\xi), \quad 0 \le \xi \le 1, \tag{1.2}$$

$$\int_0^1 U(\xi, t) d\xi = 0, \quad -\alpha \le t \le \beta, \tag{1.3}$$

$$\int_0^1 \xi U(\xi, t) d\xi = 0, \quad -\alpha \le t \le \beta, \tag{1.4}$$

where $\mathfrak{D}_- = \mathfrak{D} \cap \{t < 0\}$, $\mathfrak{D}_+ = \mathfrak{D} \cap \{t > 0\}$ and $\varphi(\xi)$ is a given sufficiently smooth function such that

$$\int_0^1 \varphi(\xi)d\xi = 0, \quad \int_0^1 \xi\varphi(\xi)d\xi = 0.$$

The application of mixed type partial differential equations was first mentioned by Chaplygin [1]. It is well-known that the mixed type equations have many interesting applications in gas dynamics, in electromagnetic and fluid mechanics and others fields (see, e.g. [2]-[10]). The integral boundary conditions show that the physical process affects not only the point but also the entire object. This type of boundary conditions arise in plasma physics, in heat transfer [11], [12] and other fields. The integral boundary conditions of type (1.3), (1.4) may occur in econometric problems. Theory of boundary problems for various mixed type equations is one of the continuously and intensively developing theories of modern mathematics. These problems are generally solved using the maximum principle or integral equations for mixed type boundary problems.

In this study, we prove a theorem for the unique solvability of problem (1.1)-(1.4). The solution of this problem is constructed as the sum of a series of eigenfunctions corresponding to the spectral problems.

Firstly, in equation (1.1) fixing the variable t, then integrating with respect to ξ in the interval $[\epsilon, 1-\epsilon]$ such that $\epsilon > 0$ is a sufficiently small number, we get

$$\int_\varepsilon^{1-\varepsilon} \frac{\partial U}{\partial t} d\xi - \int_\varepsilon^{1-\varepsilon} \frac{\partial^2 U}{\partial \xi^2} d\xi = 0, \quad t > 0,$$
$$\int_\varepsilon^{1-\varepsilon} \frac{\partial^2 U}{\partial t^2} d\xi - \int_\varepsilon^{1-\varepsilon} \frac{\partial^2 U}{\partial \xi^2} d\xi = 0, \quad t < 0.$$

Here, when $\epsilon \to 0^+$ and keeping in mind (1.3), we have the pointwise boundary condition:

$$U_\xi(0,t) - U_\xi(1,t) = 0 \tag{1.5}$$

that is the integral condition (1.3) is reduced to the nonlocal condition (1.5).

In a similar way, from (1.4) we have the nonlocal condition

$$U_\xi(1,t) - U(1,t) + U(0,t) = 0. \tag{1.6}$$

Therefore, in this paper we investigate the solution of equation (1.1) satisfying the initial condition (1.2) and boundary conditions (1.5), (1.6) in domain Ω.

Let's apply the Fourier method to the boundary value problem (1.1), (1.2), (1.5), (1.6) and look for the solution in the form $U(\xi,t) = y(\xi)\omega(t) \neq 0$. Substituting the expression in equation (1.1) under conditions (1.5), (1.6), we obtain

$$y''(\xi) + \lambda y(\xi) = 0, \tag{1.7}$$

$$y'(0) - y'(1) = 0, \tag{1.8}$$

$$y'(1) + y(0) - y(1) = 0, \tag{1.9}$$

and differential equations for $\omega(t)$,

$$\omega'(t) + \lambda\omega(t) = 0, \quad 0 < t < \alpha, \tag{1.10}$$

$$\omega''(t) + \lambda\omega(t) = 0 \quad -\alpha < t < 0. \tag{1.11}$$

Here λ is a complex parameter. As known the boundary conditions (1.8), (1.9) are regular in the sense of Birkhoff and also strongly regular (see [13]).

In this case, all eigenvalues are simple except for a finite number of eigenvalues λ_n and the system of root functions $\{y_n(\xi)\}$ forms a Riesz basis in $L_2(0,1)$ (see [13]).

Lemma 1. *The boundary value problem (1.7)-(1.9) is self-adjoint.*

Proof of the lemma follows from the general theory of linear differential operators (see [13]).

We note that the characteristic equation of boundary value problem (1.7)-(1.9) is

$$\Delta(\mu) = -4i(2\cos\mu + \mu\sin\mu - 2)\left[1 + O\left(\frac{1}{\mu}\right)\right] = 0,$$

where $\lambda = \mu^2$. From this it follows that the eigenvalues of the boundary value problem (1.7)-(1.9) form two series λ_{n1}, λ_{n2}:

$$\lambda_{n1} = (2n\pi)^2, \quad n = 0, 1, 2, ...$$

$$\lambda_{n2} = [(2n+1)\pi]^2\left(1 + O\left(\frac{1}{n}\right)\right), \quad n \to \infty. \quad (\textit{see} \quad [13])$$

The corresponding eigenvalues have the following form

$$y_{n1}(\xi) = \cos 2n\pi\xi, \quad n = 0, 1, 2, ...,$$

$$y_{n2}(\xi) = \cos(2n+1)\pi\xi + O\left(\frac{1}{n}\right) \quad n \to \infty.$$

2 Uniqueness of the solution

Substituting $\lambda_k = \mu_k^2 \quad (Re\mu_k \geq 0)$ in equations (1.10),(1.11) we obtain

$$\omega_k(t) = \begin{cases} a_k e^{-\mu_k^2 t}, & t > 0, \\ b_k \sin\mu_k t + c_k \cos\mu_k t, & t < 0, \end{cases} \tag{2.12}$$

where λ_k are complex parameters for $k \geq 0$ and a_k, b_k, c_k are arbitrary constants.

Let's take the special solution of the problem in the Ω region in the form $U_k(\xi, t) = y_k(\xi)\omega_k(t)$ and choose the constants a_k, b_k, c_k so that the matching conditions

$$\omega_k(0+) = \omega_k(0-), \quad \omega_k'(0+) = \omega_k'(0-) \tag{2.13}$$

are satisfied. The function (2.12) satisfies conditions (2.13) if and only if $a_k = c_k$, $b_k = -a_k\mu_k$. In view of the last equalities the functions (2.12) take the form

$$\omega_k(t) = \begin{cases} c_k e^{-\mu_k^2 t}, & t > 0, \\ c_k \cos \mu_k t - c_k\mu_k \sin \mu_k t, & t < 0. \end{cases} \tag{2.14}$$

Let us consider the functions

$$\upsilon_k(t) = \int_0^1 U(\xi, t) y_k(\xi) d\xi, \quad k = 0, 1, 2, ... \tag{2.15}$$

where $U(\xi, t) \in \Omega$. According to the conditions (1.8), (1.9) we find the following equations

$$\upsilon_k'(t) + \mu_k^2 \upsilon_k(t) = 0, \quad t > 0, \tag{2.16}$$

$$\upsilon_k''(t) + \mu_k^2 \upsilon_k(t) = 0, \quad t < 0. \tag{2.17}$$

Therefore, the equations (2.16), (2.17) coincide with the equations (1.10), (1.11)) for $\lambda = \mu_k^2$ and $\omega_k(t) \equiv \upsilon_k(t)$ for $-\alpha < t < \beta$ i.e. $\upsilon_k(t)$ defined by formula (2.14):

$$\upsilon_k(t) = \begin{cases} c_k e^{-\mu_k^2 t}, & t > 0, \\ c_k \cos \mu_k t - c_k\mu_k \sin \mu_k t, & t < 0. \end{cases} \tag{2.18}$$

Now, to find the constants c_k, we use the initial condition (1.2):

$$\upsilon_k(-\alpha) = \int_0^1 U(\xi, -\alpha) y_k(\xi) d\xi = \int_0^1 \varphi(\xi) y_k(\xi) d\xi = \varphi_k. \tag{2.19}$$

Then, from (2.18) and (2.19) we obtain

$$c_k[\cos \mu_k\alpha + \mu_k \sin \mu_k\alpha] = \varphi_k. \tag{2.20}$$

From relation (2.20), when

$$d(k) = \cos \mu_k\alpha + \mu_k \sin \mu_k\alpha \neq 0, \tag{2.21}$$

we have

$$c_k = \frac{\varphi_k}{\cos \mu_k\alpha + \mu_k \sin \mu_k\alpha} = \frac{\varphi_k}{d(k)}. \tag{2.22}$$

Substituting (2.22) into the (2.18), the final form of the function

$$\upsilon_k(t) = \begin{cases} \frac{\varphi_k}{d(k)} e^{-\mu_k^2 t}, & t > 0, \\ \frac{\cos\mu_k t - \mu_k \sin\mu_k t}{d(k)} \varphi_k, & t < 0, \end{cases} \tag{2.23}$$

is obtained. Suppose that $\varphi(\xi) \equiv 0$, then $\varphi_k = 0$ and it follows from formulas (2.15) and (2.18) that

$$\int_0^1 U(\xi, t) y_k(\xi) d\xi = 0, \quad (k = 0, 1, 2, ...).$$

Since the system $\{y_k(\xi)\}$ forms a basis in $L_2\,(0, 1)$, it follows that it is also a complete system i.e. $U(\xi, t) \equiv 0$ almost everywhere for any $t \in [-\alpha, \beta]$. Since $U(\xi, t)$ is continuous in the closed $\overline{\mathfrak{D}}$ region, it follows that $U(\xi, t) \equiv 0$ in $\overline{\mathfrak{D}}$.

Assume that for some α and $k = p$ numbers, condition (2.21) doesn't hold: $d(p) = 0$. The homogeneous problem (1.1)-(1.4) (where $\varphi(\xi) \equiv 0$) has the nontrivial solution

$$U_p(\xi, t) = \begin{cases} c_p e^{-\mu_p^2 t} y_p(\xi), & t > 0, \\ c_p(\cos\mu_p t - \mu_p \sin\mu_p t) y_p(\xi), & t < 0, \end{cases} \tag{2.24}$$

where $c_p \neq 0$ is an arbitrary constant.

Thus, the following uniqueness theorem is proved.

Theorem 1. *If the solution of the boundary value problem (1.1)-(1.4) exists this solution is unique if and only if the condition (2.21) holds.*

3 Existence of the solution

We assume that $d(k) \neq 0$ and that there exists c_0 such that $|d(k)| \geq c_0 > 0$ holds. The solution of (1.1)-(1.4) can be written as

$$U(\xi, t) = \sum_{k=1}^{\infty} \upsilon_k(t) y_k(\xi). \tag{3.25}$$

It is obvious that $U_k(\xi, t) = \upsilon_k(t) y_k(\xi)$ fulfils the equation (1.1). To show that the series (3.25) is the solution of the boundary value problem (1.1)-(1.4), it is necessary to prove that this series converges uniformly in the region $\overline{\mathfrak{D}}$ and that it is differentiable term by term once with respect to ξ when $t < 0$ and twice with respect to ξ and t when $t > 0$.

Lemma 2. $\forall k \in \mathbb{N}^+$ *the following inequalities are true:*

$$|\upsilon_k(t)| \leq A_1 k\, |\varphi_k|\,, \quad \left|\upsilon_k'(t)\right| \leq A_2 k^2\, |\varphi_k|\,, \tag{3.26}$$

for $t \in [-\alpha, \beta]$ *and*

$$\left|\upsilon_k''(t)\right| \leq A_3 k^3 \left|\varphi_k\right|, \tag{3.27}$$

for $t \in [-\alpha, 0]$.

Proof. From (2.24) for $t \in [0, \beta]$

$$|\upsilon_k(t)| = \left|\frac{e^{-\mu_k^2 t}\varphi_k}{d(k)}\right| \leq \frac{1}{c_0}|\varphi_k| \leq \widetilde{A}_1 k \left|\varphi_k\right|,$$

$$\left|\upsilon_k'(t)\right| = \left|\frac{-\mu_k^2 e^{-\mu_k^2 t}\varphi_k}{d(k)}\right| \leq \frac{\mu_k^2}{c_0}\left|\varphi_k\right| \leq \widetilde{A}_2 k^2 \left|\varphi_k\right|$$

is found. Similarly, for $t \in [-\alpha, 0]$

$$|\upsilon_k(t)| = \left|\frac{\cos\mu_k t - \mu_k \sin\mu_k t}{d(k)}\varphi_k\right| \leq \frac{\sqrt{1+\mu_k^2}}{c_0}\left|\varphi_k\right| \leq \widetilde{A}_3 k \left|\varphi_k\right|,$$

$$\left|\upsilon_k'(t)\right| = \left|\frac{\sin\mu_k t + \mu_k \cos\mu_k t}{d(k)}\mu_k\varphi_k\right| \leq \frac{\sqrt{1+\mu_k^2}}{c_0}\mu_k\left|\varphi_k\right| \leq \widetilde{A}_4 k^2 \left|\varphi_k\right|,$$

$$\left|\upsilon_k''(t)\right| = \left|\frac{\cos\mu_k t - \mu_k \sin\mu_k t}{d(k)}\mu_k^2\varphi_k\right| \leq \mu_k^2|\upsilon_k(t)| \leq \widetilde{A}_3 k^3 \left|\varphi_k\right|,$$

is obtained. Here, $\widetilde{A}_j$ $(j = 1, 2, 3, 4)$ are positive constants. Then the equalities (3.26) and (3.27) are proved. $\boxed{QED}$

Using inequalities (3.26), (3.27), the series (3.25), first order derivative of the series (3.25) in $\overline{\mathfrak{D}}$ and second order derivative of the series (3.25) in $\overline{\mathfrak{D}}_+$ and $\overline{\mathfrak{D}}_-$ are majorized by the numerical series:

$$A_4 \sum_{k=1}^{\infty} k^3 \left|\varphi_k\right|. \tag{3.28}$$

Lemma 3. *If* $\varphi(\xi) \in C^4[0,1]$ *and* $\varphi^{(1+i)}(1) - \varphi^{(i)}(1) + \varphi^{(i)}(0) = 0 \quad \varphi^{(1+i)}(0) - \varphi^{(1+i)}(1) = 0, \quad i = 0, 2$ *then,*

$$\varphi_k = \frac{\varphi_k^{(4)}}{\mu_k^4}, \quad k \in \mathbb{Z}^+, \tag{3.29}$$

where

$$\varphi_k^{(4)} = \int_0^1 \varphi^{(4)}(\xi) y_k(\xi) d\xi, \quad \sum_{k=1}^{\infty}\left|\varphi_k^{(4)}\right|^2 \leq \left\|\varphi^{(4)}\right\|_{L_2(0,1)}. \tag{3.30}$$

Proof. Taking the integral (2.15) into account, we have

$$\upsilon_k(-\alpha) = \int_0^1 U(\xi,-\alpha)y_k(\xi)d\xi = \int_0^1 \varphi(\xi)y_k(\xi)d\xi = \varphi_k.$$

According to the (1.7), we get

$$\varphi_k = \int_0^1 \varphi(\xi)y_k(\xi)d\xi = -\frac{1}{\mu_k^2}\int_0^1 \varphi(\xi)y_k''(\xi)d\xi.$$

By integrating twice successively and taking into account the conditions of the lemma, we find

$$\varphi_k = -\frac{1}{\mu_k^2}\int_0^1 \varphi''(\xi)y_k(\xi)d\xi = -\frac{1}{\mu_k^2}\varphi_k^{(2)}. \tag{3.31}$$

As a result of the following similar process we obtain

$$\varphi_k^{(4)} = -\frac{1}{\mu_k^4}\int_0^1 \varphi^{(4)}(\xi)y_k(\xi)d\xi = -\frac{1}{\mu_k^4}\varphi_k^{(4)}. \tag{3.32}$$

Then we obtain formula (3.29). The validity of estimates (3.30) follows from the theory of Fourier series for the function $\varphi(\xi)$ and from the Bessel inequality with to eigenfunctions system $\{y_n(\xi)\}$. Then Lemma 3 is proved. $\boxed{QED}$

From Lemma 3, the series (3.28) is bounded by the following numerical series

$$A_5\sum_{k=1}^{\infty}\frac{1}{k}\left|\varphi_k^{(4)}\right|. \tag{3.33}$$

So, as usual, it is proved the uniform convergence of the series (3.25) in $\overline{\mathfrak{D}}$ and differentiable term by term once with respect to ξ when $t<0$ and twice with respect to ξ and t when $t>0$.

We note that, if $d_\alpha(p)=0$ for some α and $k=p=k_1,k_2,...,k_m$, where $1 \le k_1 < k_2 < ... < k_m \le k_0$, k_i, $i=1,2,...,m$, m being a fixed natural number, then to solve the boundary value problem (1.1)-(1.4), it is necessary and sufficient condition that

$$\varphi_k = \int_0^1 \varphi(\xi)y_k(\xi)d\xi = 0, \tag{3.34}$$

where $k=k_1,k_2,...,k_m$. In that case the solution is defined by the following form:

$$U(\xi,t) = \left(\sum_{k=1}^{k_1-1} +... \sum_{k=k_{m-1}+1}^{k_m-1} + \sum_{k=k_m+1}^{\infty}\right)\upsilon_k(t)y_k(\xi) + \sum_p B_pU_p(\xi,t). \tag{3.35}$$

In the last term the number p takes the values $k_1, k_2, ..., k_m$, B_p is an arbitrary constant, $U_p(\xi, t)$ is expressed as in formula (2.24) such as if at the left side of the (3.35) upper bounds are less than lower bounds, this term is equal to zero.

Theorem 2. *Assume that α is a rational number and $\varphi(\xi)$ satisfies the conditions of Lemma 3. If $d_\alpha(k) \neq 0$ for $k = \overline{1, k_0}$, then the boundary value problem (1.1)-(1.4) has a unique solution, which is defined by series (3.25). If $d_\alpha(k) = 0$ for some $k = k_1, k_2, ..., k_m \leq k_0$, the boundary value problem (1.1)-(1.4) is solvable only if the condition (3.34) holds and the solution is defined by series (3.25).*

Theorem 3. *Let $U(\xi, t) \in \Omega$ be the solution of the boundary value problem (1.1)-(1.4). Then the following inequality holds:*

$$\|U(\xi, t)\|_{C(\overline{\mathfrak{D}})} \leq M \left\|\varphi''(\xi)\right\|_{C[0,1]},$$

where $M > 0$ does not depend on $\varphi(\xi)$.

Proof is similar to Theorem 3 in [14].

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