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$*$-Ricci soliton on GSSF with Sasakian metric

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Abstract. We study generalized Sasakian-space-forms (GSSF) $M^{2n+1}(k_1, k_2, k_3)$ with Sasakian metric admitting $*$-Ricci soliton. We obtain that either such GSSF has $k_1 = \frac{2n+1}{2n+2}$, $k_2 = k_3 = -\frac{1}{2n+2}$ and $*$-soliton is steady or $k_1 = 0$, $k_2 = k_3 = -1$ and $*$-soliton is expanding. Also, we provide some examples in support of results. Further, we give an example that GSSF with Sasakian metric with $k_1 \neq 0$ and $k_1 \neq \frac{2n+1}{2n+2}$ do not admit the $*$-Ricci soliton.

Keywords: $*$-Ricci soliton; Generalized Sasakian-space-forms; Sasakian manifolds; Positive-Sasakian; Null-Sasakian.

MSC 2020 classification: primary 53C15, secondary 53D15

1 Introduction

Hamilton [9] in 1988 introduced the concept of Ricci soliton as a generalization of an Einstein metric as well as a self-similar solution of the Ricci flow of Hamilton [6]. A triplet (g, V, ν) on a Riemannian manifold is a Ricci soliton if the Ricci tensor i.e. Ric satisfies the following equation [5]

$$Ric + \frac{1}{2}\mathcal{L}_V g = \nu g,$$

where $\mathcal{L}_V$ is the Lie-derivative along the potential vector field V, g is a Riemannian metric and ν a real scalar. The soliton is expanding, steady, or shrinking if ν is < 0, $= 0$ or > 0, respectively.

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The Ricci soliton on an almost contact metric(a.c.m) manifold was studied extensively by many geometers ([10, 11, 13, 15, 16]).

In 1959 Tachibana [17] introduced the notion of $*$-Ricci tensor and Hamada [8] studied the $*$-Ricci tensor and defined it on an a.c.m manifold M as follows:

$$S^*(W,U) = \frac{1}{2}\text{trace}(Z \mapsto R(W,\psi U)\psi Z),\ \forall\ W,U,Z \in TM, \tag{1.1}$$

where ψ is a $(1,1)$-tensor field and R is a Riemann curvature tensor. Kaimakamis and Panagiotidou in [12] studied $*$-Ricci soliton with $V = \xi$ and defined it as

$$Ric^* + \frac{1}{2}\mathcal{L}_V g = \nu g. \tag{1.2}$$

Further, the $*$-Ricci soliton with V belongs to principal curvature space and to the holomorphic distribution was studied by Chen [4]. Recently, Majhi et al. [14] and Ghosh and Patra [7] studied Sasakian manifold admitting (1.2).

The geometers [1] were interested to find the examples $M^{2n+1}(k_1, k_2, k_3)$ of GSSF with non-constant functions k_1, k_2, k_3. It is a significant project. In this paper, we study the existence and non-existence of $*$-Ricci soliton on GSSF with Sasakian structure and obtain the functions k_1, k_2, k_3 and nature of $*$-Ricci soliton i.e. whether $*$-soliton is expanding or steady or shrinking along with examples.

2 Preliminaries

A smooth manifold M^{2n+1} is called an a.c.m manifold if structure tensors (ψ, ξ, η, g) satisfies [2]:

$$\psi^2 = -I + \eta \otimes \xi,\ \eta(\xi) = 1, \tag{2.1}$$

$$\psi\xi = 0,\ \eta\ o\ \psi = 0,\ \eta(W) = g(W,\xi), \tag{2.2}$$

$$g(\psi W, \psi U) = g(W,U) - \eta(W)\eta(U), \tag{2.3}$$

where ψ is a $(1,1)$-tensor field, ξ a structure vector field, η a 1-form and g a Riemannian metric and W, $U \in TM$.

An a.c.m manifold $M^{2n+1}(\psi, \xi, \eta, g)$ is called a GSSF if there exists three smooth functions k_1, k_2, and k_3 on M such that the curvature tensor R is given by [1]

$$\begin{aligned} R(W,U)Z &= k_1\{g(U,Z)W - g(W,Z)U\} + k_2\{g(W,\psi Z)\psi U \\ &\quad -g(U,\psi Z)\psi W + 2g(W,\psi U)\psi Z\} + k_3\{g(W,Z)\eta(U)\xi \\ &\quad -g(U,Z)\eta(W)\xi + \eta(W)\eta(Z)U - \eta(U)\eta(Z)W\}, \end{aligned} \tag{2.4}$$

$\forall\ W,\ U,\ Z \in TM$. In particular M is a *Sasakian space form* if $k_1 = \frac{c+3}{4}$, $k_2 = k_3 = \frac{c-1}{4}$, *Kenmotsu space form* if $k_1 = \frac{c-3}{4}$, $k_2 = k_3 = \frac{c+1}{4}$, *cosymplectic space form* if $k_1 = k_2 = k_3 = \frac{c}{4}$.

Let $e_1, e_2, \ldots, e_{2n+1}$ are local orthonormal vector fields on M^{2n+1}, then Ricci curvature S is defined as [20]

$$S(W,U) = \sum_{i=1}^{2n+1} g(R(e_i, W)U, e_i),\ \forall\ W, U \in TM. \tag{2.5}$$

Definition 1. M is called η-Einstein if $S(W,U) = \delta g(W,U) + \mu\eta(W)\eta(U)$ $\forall\ W, U \in TM$, δ, $\mu \in \mathbb{R}$. Further, an η-Einstein Sasakian manifold with $\delta = -2$ and $\mu = 2n+2$ is called null-Sasakian and if $\delta > -2$ then positive-Sasakian [3].

Definition 2. [18] On a contact metric manifold M a vector field V is said to be an infinitesimal contact transformation if $\mathcal{L}_V\eta = f\eta$ for some $f \in C^\infty$ on M and an infinitesimal automorphism of the contact metric structure if it leaves η, ξ, g, ψ invariant.

The following commutation formulae will be useful to obtain our results. On a Riemannian manifold M [19], we have

$$(\nabla_Z \mathcal{L}_V g)(W,U) = g((\mathcal{L}_V\nabla)(Z,W),U) + g((\mathcal{L}_V\nabla)(Z,U),W), \tag{2.6}$$

$$(\mathcal{L}_V R)(W,U)Z = (\nabla_W \mathcal{L}_V \nabla)(U,Z) - (\nabla_U \mathcal{L}_V \nabla)(W,Z), \tag{2.7}$$

$\forall\ W,\ U,\ Z \in TM$.

3 $*$-Ricci Soliton on GSSF with Sasakian metric

Let $M^{2n+1}(k_1, k_2, k_3)$ be a GSSF with Sasakian metric, then [1]

$$\begin{aligned}
\nabla_W \xi &= (k_3 - k_1)\psi W, && (3.1)\\
(\nabla_W \psi)(U) &= (k_1 - k_3)\big(g(W,U)\xi - \eta(U)W\big), && (3.2)\\
(\nabla_W \eta)(U) &= g(\nabla_W \xi, U) = (k_3 - k_1)g(\psi W, U), && (3.3)\\
R(W,U)\xi &= (k_1 - k_3)(\eta(U)W - \eta(W)U), && (3.4)
\end{aligned}$$

$\forall\ W, U \in TM$.

We set following:

$$F_1 = k_1 + (1+2n)k_2;\ F_2 = 2nk_1 + 3k_2 - k_3;\ F_3 = 3k_2 + (2n-1)k_3. \tag{3.5}$$

Theorem 1. *[1] Let $M^{2n+1}(k_1, k_2, k_3)$ be a GSSF with Sasakian metric, then $k_2 = k_3 = k_1 - 1$.*

Theorem 2. *Let $M^{2n+1}(k_1, k_2, k_3)$ be a GSSF with Sasakian metric, then $\xi k_1 = 0$.*

Proof. Using (2.4) in (2.5), we obtain

$$S(W,U) = F_2 g(W,U) - F_3 \eta(W)\eta(U), \tag{3.6}$$

$\forall\ W, U \in TM$.

Using Theorem 1 in (3.6), we get

$$QW = ((2n+2)k_1 - 2)W - ((2n+2)k_1 - (2n+2))\eta(W)\xi. \tag{3.7}$$

Differentiating (3.7) along ξ and using (3.1), we find that

$$(\nabla_\xi Q)W = ((2n+2)\xi k_1)(W - \eta(W)\xi). \tag{3.8}$$

Also, since ξ is Killing on a Sasakian manifold, therefore we have [7]

$$\nabla_\xi Q = 0. \tag{3.9}$$

Comparing (3.8) and (3.9), we get $\xi k_1 = 0$, which completes the proof. QED

Theorem 3. *Let $M^{2n+1}(k_1, k_2, k_3)$ be a GSSF admitting $*$-Ricci soliton with Sasakian metric, then*

$$\nu = 2F_1 = 4(n+1)k_1 - 2(2n+1). \tag{3.10}$$

Moreover, k_1, k_2 and k_3 are constants and $k_1 \in \{0, \frac{2n+1}{2n+2}\}$.

Proof. Replacing U with ψU, Z with ψZ in (2.4) and using (1.1), we find

$$S^*(W,U) = F_1(g(W,U) - \eta(W)\eta(U)). \tag{3.11}$$

Using (3.11) in (1.2), we get

$$(\mathcal{L}_V g)(W,U) = 2\nu g(W,U) - 2F_1 g(\psi W, \psi U). \tag{3.12}$$

Differentiating (3.12) along $Z \in TM$, we find

$$\begin{aligned}(\nabla_Z \mathcal{L}_V g)(W,U) &= -2F_1(g((\nabla_Z\psi)W, \psi U) + g((\nabla_Z\psi)U, \psi W)) \\ &- 2(ZF_1)g(\psi W, \psi U).\end{aligned} \tag{3.13}$$

Using (2.6) in (3.13), we obtain

$$\begin{aligned}g((\mathcal{L}_V\nabla)(Z,W),U) + g((\mathcal{L}_V\nabla)(Z,U),W) &= -2F_1 g((\nabla_Z\psi)W, \psi U) \\ &-2F_1 g((\nabla_Z\psi)U, \psi W) - 2(ZF_1)g(\psi W, \psi U).\end{aligned} \tag{3.14}$$

Similarly, we have

$$g((\mathcal{L}_V\nabla)(W,U),Z)+g((\mathcal{L}_V\nabla)(W,Z),U)=-2F_1g((\nabla_W\psi)U,\psi Z) \\ -2F_1g((\nabla_W\psi)Z,\psi U)-2(WF_1)g(\psi U,\psi Z), \tag{3.15}$$

$$g((\mathcal{L}_V\nabla)(U,Z),W)+g((\mathcal{L}_V\nabla)(U,W),Z)=-2F_1g((\nabla_U\psi)Z,\psi W) \\ -2F_1g((\nabla_U\psi)W,\psi Z)-2(UF_1)g(\psi Z,\psi W). \tag{3.16}$$

Adding (3.15) and (3.16), then subtracting (3.14) we find that

$$\begin{aligned} g((\mathcal{L}_V\nabla)(W,U),Z) &= \\ & F_1g((\nabla_Z\psi)W,\psi U)+F_1g((\nabla_Z\psi)U,\psi W)+(ZF_1)g(\psi W,\psi U) \\ & -F_1g((\nabla_U\psi)Z,\psi W)-F_1g((\nabla_U\psi)W,\psi Z)-(UF_1)g(\psi Z,\psi W) \\ & -F_1g((\nabla_W\psi)U,\psi Z)-F_1g((\nabla_W\psi)Z,\psi U)-(WF_1)g(\psi U,\psi Z), \end{aligned} \tag{3.17}$$

$\forall\, W,\, U,\, Z \in TM$.

Putting $U=\xi$ in (3.17) and using (3.2), Theorem 1 and Theorem 2, we obtain

$$(\mathcal{L}_V\nabla)(W,\xi)=-2F_1\psi W. \tag{3.18}$$

Further, differentiating (3.18) along $U\in TM$ and using (3.1), (3.2), we find

$$\begin{aligned} (\nabla_U\mathcal{L}_V\nabla)(W,\xi) &= (\mathcal{L}_V\nabla)(W,\psi U)-2(UF_1)\psi W \\ & \quad -2F_1g(W,U)\xi+2F_1\eta(W)U. \end{aligned} \tag{3.19}$$

Using (3.19) in (2.7), we obtain

$$\begin{aligned} (\mathcal{L}_VR)(W,U)\xi &= (\mathcal{L}_V\nabla)(U,\psi W)-2(WF_1)\psi U+2F_1\eta(U)W \\ & - (\mathcal{L}_V\nabla)(W,\psi U)+2(UF_1)\psi W-2F_1\eta(W)U. \end{aligned} \tag{3.20}$$

Putting $U=\xi$ in (3.20) and using (2.1), (3.18) and Theorem 2, we obtain

$$(\mathcal{L}_VR)(W,\xi)\xi=4F_1(W-\eta(W)\xi). \tag{3.21}$$

Putting $U=\xi$, $Z=\xi$ in (2.4), we get

$$R(W,\xi)\xi=(k_1-k_3)(W-\eta(W)\xi). \tag{3.22}$$

Lie-differentiating (3.22) along V, we have

$$\begin{aligned} (\mathcal{L}_VR)(W,\xi)\xi &= V(k_1-k_3)(W-\eta(W)\xi) \\ &+ (k_1-k_3)(g(\mathcal{L}_V\xi,W)\xi-2\eta(\mathcal{L}_V\xi)W-(\mathcal{L}_V\eta)(W)\xi). \end{aligned} \tag{3.23}$$

Putting $U=\xi$ in (3.11), we obtain

$$S^*(W,\xi)=0. \tag{3.24}$$

Putting $U=\xi$ in (1.2) and using (3.24), we get

$$(\mathcal{L}_V g)(W,\xi)=2\nu\eta(W). \tag{3.25}$$

Lie-differentiating $\eta(W)=g(W,\xi)$ along V and using (3.25), we find

$$(\mathcal{L}_V\eta)(W)-g(\mathcal{L}_V\xi,W)-2\nu\eta(W)=0. \tag{3.26}$$

Taking Lie-derivative of $g(\xi,\xi)=1$ along V and using (3.25), we have

$$\eta(\mathcal{L}_V\xi)=-\nu. \tag{3.27}$$

Using (3.26), (3.27) and Theorem 1 in (3.23), we obtain

$$(\mathcal{L}_V R)(W,\xi)\xi=2\nu(W-\eta(W)\xi). \tag{3.28}$$

From (3.21) and (3.28), we get

$$\nu=2F_1. \tag{3.29}$$

Using (3.5) and Theorem 1 in (3.29), we obtain (3.10). Further, since ν is constant so k_1, k_2 and k_3 are constants. Moreover, from (3.5), we get F_1, F_2 and F_3 all are constants.

Now, using (3.2) and Theorem 1 in (3.17), we get

$$(\mathcal{L}_V\nabla)(W,U)=-2F_1\eta(W)\psi U-2F_1\eta(U)\psi W. \tag{3.30}$$

Differentiating (3.30) along Z and using (3.2), (3.3), Theorem 1 and Theorem 2, we obtain

$$\begin{aligned}(\nabla_Z\mathcal{L}_V\nabla)(W,U) &= -2F_1(g(\psi W,Z)\psi U+g(\psi U,Z)\psi W \\ &+ \eta(W)g(Z,U)\xi+\eta(U)g(Z,W)\xi-2\eta(W)\eta(U)Z).\end{aligned} \tag{3.31}$$

Using (3.31) in (2.7), we obtain

$$\begin{aligned}(\mathcal{L}_V R)(Z,W)U &= -2F_1(2g(\psi W,Z)\psi U+g(\psi U,Z)\psi W-g(\psi U,W)\psi Z \\ &+\eta(W)g(Z,U)\xi-\eta(Z)g(W,U)\xi+2\eta(Z)\eta(U)W-2\eta(W)\eta(U)Z),\end{aligned} \tag{3.32}$$

$\forall\, W,\, U,\, Z\in TM$.

Contracting (3.32) over Z, we find

$$(\mathcal{L}_V S)(W,U) = -4F_1(g(W,U) - (2n+1)\eta(W)\eta(U)). \tag{3.33}$$

Changing W to ψU and U to ψW in (3.33), we obtain

$$(\mathcal{L}_V S)(\psi U, \psi W) = -4F_1 g(\psi W, \psi U). \tag{3.34}$$

Lie-differentiating (3.6) along V and using (3.12) and (3.26), we have

$$\begin{aligned}(\mathcal{L}_V S)(W,U) = (V(F_2) + 2F_2\nu - 2F_2F_1)g(W,U) + (2F_2F_1 \\ -V(F_3) - 4\nu F_3)\eta(W)\eta(U) - F_3(g(\mathcal{L}_V\xi, W)\eta(U) + g(\mathcal{L}_V\xi, U)\eta(W)),\end{aligned} \tag{3.35}$$

$\forall\ W,\ U \in TM$.

Changing W to ψU and U to ψW in (3.35), we find

$$(\mathcal{L}_V S)(\psi U, \psi W) = (2F_2\nu - 2F_2F_1)g(\psi W, \psi U). \tag{3.36}$$

From (3.34) and (3.36), we get

$$F_2\nu - F_2F_1 = -2F_1. \tag{3.37}$$

Using (3.5), (3.29) and Theorem 1 in (3.37), we have

$$(2n+2)\big((2n+2)k_1 - (2n+1)\big)k_1 = 0,$$

which gives either $k_1 = \frac{2n+1}{2n+2}$ or $k_1 = 0$.

Thus proof is complete. $\boxed{QED}$

Corollary 1. *Let $M^{2n+1}(k_1, k_2, k_3)$ be a GSSF admitting $*$-Ricci soliton. If M^{2n+1} has Sasakian metric, then either $*$-soliton is steady or expanding. In the first case M is η-Einstein and positive-Sasakian with killing $*$-soliton vector field V. In second case M is null-Sasakian and V leaves ψ invariant.*

Proof. In view of Theorem 3:

Considering the case $k_1 = \frac{2n+1}{2n+2}$**.** Then, from (3.10), we get $\nu = 0$. Hence, $*$-soliton is steady. Using (3.5), Theorem 1 and $k_1 = \frac{2n+1}{2n+2}$ in (3.6) and in (1.2), we find

$$S(W,U) = (2n-1)g(W,U) + \eta(W)\eta(U),\ \ (\mathcal{L}_V g)(W,U) = 0,$$

$\forall\ W,\ U \in TM$. Hence M^{2n+1} is η-Einstein positive-Sasakian and V is killing.

Considering the case when $k_1 = 0$**.** Then (3.10) gives $\nu = -2(2n+1)$ and hence $*$-soliton is expanding. Using $k_1 = 0$ and Theorem 1 in (3.6), we obtain

$$S(W,U) = -2g(W,U) + 2(n+1)\eta(W)\eta(U),$$

$\forall$ W, $U \in TM$. Hence M^{2n+1} is null-Sasakian.

Using (3.11) in (1.2), we have

$$(\mathcal{L}_V g)(W,U) = (2\nu - 2F_1)g(W,U) + 2F_1\eta(W)\eta(U). \tag{3.38}$$

Now comparing (3.33) and (3.35), we get

$$\begin{aligned} V(F_2)g(W,U) + F_2(\mathcal{L}_V g)(W,U) - V(F_3)\eta(W)\eta(U) - F_3((\mathcal{L}_V\eta)(W)\eta(U) \\ +(\mathcal{L}_V\eta)(U)\eta(W)) = -4F_1(g(W,U) - (2n+1)\eta(W)\eta(U)). \end{aligned} \tag{3.39}$$

Using $k_1 = 0$, (3.38), Theorem 3 and putting $U = \xi$ and ψW in place of W in (3.39), we obtain $F_3(\mathcal{L}_V\eta)(\psi W) = 0$.

Since $F_3 \neq 0$, therefore, we get

$$(\mathcal{L}_V\eta)(\psi W) = 0. \tag{3.40}$$

By putting ψW in place of W in (3.40), we find

$$(\mathcal{L}_V\eta)(W) = \eta(W)(\mathcal{L}_V\eta)(\xi). \tag{3.41}$$

Using (3.26), (3.27) and $\nu = -2(2n+1)$ in (3.41), we get

$$(\mathcal{L}_V\eta)(W) = -2(2n+1)\eta(W). \tag{3.42}$$

Taking exterior derivative of (3.42), we get

$$(\mathcal{L}_V d\eta)(W,U) = -2(2n+1)d\eta(W,U). \tag{3.43}$$

As on Sasakian manifold

$$d\eta(W,U) = g(W,\psi U). \tag{3.44}$$

Using (3.44) in (3.43), we find

$$(\mathcal{L}_V d\eta)(W,U) = -2(2n+1)g(W,\psi U). \tag{3.45}$$

Lie-derivative of (3.44) along V gives

$$(\mathcal{L}_V d\eta)(W,U) = (\mathcal{L}_V g)(W,\psi U) + g(W,(\mathcal{L}_V\psi)U). \tag{3.46}$$

Replacing U by ψU and using $k_1 = 0$, Theorem 1, $\nu = 2F_1 = -2(2n+1)$ in (3.38), we obtain

$$(\mathcal{L}_V g)(W,\psi U) = -2(2n+1)g(W,\psi U). \tag{3.47}$$

From (3.45), (3.46) and (3.47), we get $\mathcal{L}_V\psi = 0$. Whereby proof is complete.

QED

Theorem 4. *Let $M^{2n+1}(k_1, k_2, k_3)$ be a GSSF admitting $*$-Ricci soliton with $V = b\xi$. If M admits Sasakian metric, then*
(i) $*$-Ricci soliton is steady and V is a constant multiple of ξ,
(ii) M is $*$-Ricci flat and k_1, k_2 and k_3 are constant.

Proof. (i) Suppose that $V = b\xi$, for some function b. Then, from (1.2), we obtain

$$bg(\nabla_W \xi, U) + (Wb)\eta(U) + bg(\nabla_U \xi, W) + (Ub)\eta(W) + 2S^*(W, U) - 2\nu g(W, U) = 0. \tag{3.48}$$

Putting $U = \xi$ in (3.48) and using (3.1), (3.24) and Theorem 1, we find

$$(Wb) + (\xi b)\eta(W) - 2\nu\eta(W) = 0. \tag{3.49}$$

Putting $W = \xi$ in (3.49), we get $\xi b = \nu$. Using it in (3.49), we find

$$(db)(W) - \nu\eta(W) = 0. \tag{3.50}$$

Taking exterior derivative of (3.50), we obtain $\nu d\eta = 0$. Which gives $\nu = 0$ as M^{2n+1} has Sasakian metric. Therefore, from (3.50), we get, $(db)(W) = 0$. Hence, b is a constant. Hence we have (i).

Taking $\nu = 0$ and $b =$ constant in (3.48), we obtain

$$bg(\nabla_W \xi, U) + bg(\nabla_U \xi, W) + 2S^*(W, U) = 0. \tag{3.51}$$

Using (3.1) in (3.51), we get $S^*(W, U) = 0$. Hence, M^{2n+1} is $*$-Ricci flat. Using this fact, (3.5) and Theorem 1 in (3.11), we get $k_1 = \frac{2n+1}{2n+2}$ and $k_2 = k_3 = \frac{-1}{2n+2}$. Thus proof is complete. QED

4 Examples of $*$-Ricci soliton on GSSF with Sasakian metric

Example 1. Consider $M = \{(x, y, z) \in R^3 : y \neq 0\}$ with

$$\begin{cases} \psi(e_1) = e_2,\ \psi(e_2) = -e_1,\ \psi(e_3) = 0, \quad \eta = -2ydx + dz, \\ e_3 = \xi = \frac{\partial}{\partial z}, \quad g = dx \otimes dx + dy \otimes dy + \eta \otimes \eta, \\ e_1 = \frac{\partial}{\partial y},\ e_2 = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial z}. \end{cases} \tag{4.1}$$

Moreover,

$$[e_l, e_3] = 0 \text{ for } l = 1, 2; \quad [e_1, e_2] = 2e_3. \tag{4.2}$$

The Koszul's formula $\forall\ W,\ U,\ Z \in TM$ with Riemannian connection ∇ is

$$\begin{aligned} 2g(\nabla_W U, Z) &= Wg(U,Z) + Ug(Z,W) - Zg(W,U) \\ &\quad -g(W,[U,Z]) - g(U,[W,Z]) + g(Z,[W,U]). \end{aligned} \tag{4.3}$$

From (4.2) and (4.3), we get

$$\begin{cases} \nabla_{e_1} e_1 = 0,\ \nabla_{e_2} e_1 = -e_3,\ \nabla_{e_3} e_1 = -e_2,\ \nabla_{e_1} e_2 = e_3,\ \nabla_{e_2} e_2 = 0, \\ \nabla_{e_3} e_2 = e_1,\ \nabla_{e_1} e_3 = -e_2,\ \nabla_{e_2} e_3 = e_1,\ \nabla_{e_3} e_3 = 0. \end{cases} \tag{4.4}$$

Using (4.1) and (4.4), we can see that M is a Sasakian manifold. Using (3.11), (4.2) and (4.4), we find

$$\begin{cases} R(e_l,e_3)e_3 = e_l,\ R(e_l,e_p)e_p = -3e_l,\ l \neq p,\ l,p = 1,2, \\ R(e_l,e_p)e_s = 0, l \neq p \neq s,\ l,p,s = 1,2,3,\ R(e_l,e_3)e_l = -e_3,\ l = 1,2, \\ S^*(e_l,e_l) = -3, l = 1,2,\ S^*(e_3,e_3) = S^*(e_l,e_p) = 0, l \neq p, l,p = 1,2,3. \end{cases} \tag{4.5}$$

Using (2.4) and (4.5), we find that M is a GSSF with $k_1 = 0$, $k_2 = k_3 = -1$.
The potential field on M is given by

$$\begin{aligned} V = (-3x - \frac{c_4}{2}y - \frac{c_3}{2})\frac{\partial}{\partial x} + (\frac{c_4}{2}x + \frac{c_1}{2} - 3y)\frac{\partial}{\partial y} \\ + (\frac{c_4}{2}x^2 + c_1 x - 6z - \frac{c_4}{2}y^2 + c_2)\frac{\partial}{\partial z}. \end{aligned}$$

Then, we have $[V,e_1] = 3e_1 + \frac{c_4}{2}e_2$, $[V,e_2] = -\frac{c_4}{2}e_1 + 3e_2$, $[V,e_3] = 6e_3$.
Now, we can see that

$$(\mathcal{L}_V g)(e_l,e_p) + 2S^*(e_l,e_p) = 2\nu g(e_l,e_p),$$

for $\nu = -6$ and $l,p = 1,2,3$.

Therefore $M^3(0,-1,-1)$ is the GSSF with Sasakian metric admitting *expanding $*$-Ricci soliton.*

Example 2. Consider $M = \{(x,y,z) \in R^3 : x,y \neq 0\}$ with

$$\begin{cases} \psi(e_1) = e_2,\ \psi(e_2) = -e_1,\ \psi(e_3) = 0,\ \eta = \frac{4}{4+3x^2+3y^2}(ydx - xdy) + dz, \\ e_3 = \xi = \frac{\partial}{\partial z},\ g = \frac{1}{(1+\frac{3x^2}{4}+\frac{3y^2}{4})^2}(dx \otimes dx + dy \otimes dy) + \eta \otimes \eta, \\ e_1 = (1 + \frac{3x^2+3y^2}{4})\frac{\partial}{\partial x} - y\frac{\partial}{\partial z}, e_2 = (1 + \frac{3x^2+3y^2}{4})\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}. \end{cases} \tag{4.6}$$

Moreover,

$$[e_l, e_3] = 0 \text{ for } l = 1,2;\ [e_1,e_2] = -\frac{3y}{2}e_1 + \frac{3x}{2}e_2 + 2e_3. \tag{4.7}$$

From (4.3) and (4.7), we get

$$\begin{cases}\nabla_{e_1}e_1=\frac{3y}{2}e_2, \nabla_{e_2}e_1=-\frac{3x}{2}e_2-e_3, \nabla_{e_3}e_1=-e_2, \nabla_{e_1}e_2=-\frac{3y}{2}e_1+e_3,\\ \nabla_{e_2}e_2=\frac{3x}{2}e_1, \nabla_{e_3}e_2=e_1,\ \nabla_{e_1}e_3=-e_2,\ \nabla_{e_2}e_3=e_1,\ \nabla_{e_3}e_3=0.\end{cases} \tag{4.8}$$

Using (4.6) and (4.8), we can see that M is a Sasakian manifold. Using (3.11), (4.7) and (4.8), we find

$$\begin{cases}R(e_l,e_p)e_s=0, l\neq p\neq s,\ l,p,s=1,2,3,\ R(e_l,e_3)e_3=e_l, l=1,2,\\ R(e_l,e_3)e_l=-e_3,\ l=1,2,\ R(e_l,e_p)e_p=0,\ l\neq p,\ l,p=1,2,\\ S^*(e_l,e_p)=0,\ l,p=1,2,3.\end{cases} \tag{4.9}$$

Using (2.4) and (4.9), we find that M is a GSSF with $k_1=\frac{3}{4}, k_2=k_3=-\frac{1}{4}$. The potential field V on M is given by

$$V=\frac{3}{32}\Big(-y^2+x^2+\frac{4}{3}\Big)\frac{\partial}{\partial x}+\frac{3xy}{16}\frac{\partial}{\partial y}+\frac{y}{8}\frac{\partial}{\partial z}. \tag{4.10}$$

Then, we have $[V,e_1]=-\frac{3}{16}ye_2$, $[V,e_2]=\frac{3}{16}ye_1$, $[V,e_3]=0$.
Now, we can see that

$$(\mathcal{L}_V g)(e_p,e_s)+2S^*(e_p,e_s)=2\nu g(e_p,e_s),$$

for $\nu=0$ and $p,s=1,2,3$.

Hence $M^3(\frac{3}{4},-\frac{1}{4},-\frac{1}{4})$ is the GSSF with Sasakian metric admitting steady ∗-Ricci soliton.

Now, we give an example of GSSF of arbitrary dimension with Sasakian metric admitting ∗-Ricci soliton:

Example 3. Consider $M=\{(x^p,y^p,z)\in R^{2n+1} : y^p\neq 0,\ p=1,\dots,n\}$ with

$$\begin{cases}\psi(e_p)=e_{n+p},\ \psi(e_{n+p})=-e_p,\ \psi(e_{2n+1})=0,\ \eta=-2\sum_{p=1}^{n}y^p dx^p+dz,\\ e_{2n+1}=\xi=\frac{\partial}{\partial z},\ g=\sum_{p=1}^{n}(dx^p\otimes dx^p+dy^p\otimes dy^p)+\eta\otimes\eta,\\ e_p=\frac{\partial}{\partial y^p},\ e_{n+p}=\frac{\partial}{\partial x^p}+2y^p\frac{\partial}{\partial z}.\end{cases} \tag{4.11}$$

We denote by $A=\{1,\dots,2n+1\}$, $B=\{1,\dots,2n\}$, $C=\{n+1,\dots,2n\}$. Then

$$\begin{cases}[e_p,e_{n+p}]=2e_{2n+1},\quad [e_p,e_s]=0,\ s\neq n+p, s\in A,\\ [e_q,e_s]=0,\ s\neq q-n, s\in A, q\in C.\end{cases} \tag{4.12}$$

From (4.3) and (4.12), we get

$$\begin{cases}\nabla_{e_p}e_{2n+1}=-e_{n+p}=\nabla_{e_{2n+1}}e_p, \nabla_{e_q}e_q=0,\ q\in A,\\ \nabla_{e_l}e_s=0,\ l,s\in B,\ s\notin\{l,l+n,l-n\},\\ \nabla_{e_p}e_{n+p}=e_{2n+1}=-\nabla_{e_{n+p}}e_p, \nabla_{e_{2n+1}}e_t=e_{t-n}=\nabla_{e_t}e_{2n+1}, t\in C.\end{cases} \tag{4.13}$$

Using (4.11) and (4.13), we can see that M is a Sasakian manifold. Using (3.11), (4.12) and (4.13), we find

$$\begin{cases}R(e_{2n+1},e_1)e_{2n+1}=-e_1, R(e_1,e_{n+1})e_1=3e_{n+1}, S^*(e_{2n+1},e_{2n+1})=0,\\ S^*(e_l,e_l)=-(2n+1),\ l\in B, S^*(e_q,e_s)=0,\ q\neq s,\ q,s\in A.\end{cases} \tag{4.14}$$

Using (2.4) and (4.14), we find that M is a GSSF with $k_1=0$, $k_2=-1$, $k_3=-1$.

The potential field is given by

$$V=\sum_{p=1}^{n}\Big(-(2n+1)y^p\frac{\partial}{\partial y^p}-(2n+1)x^p\frac{\partial}{\partial x^p}\Big)-2(2n+1)z\frac{\partial}{\partial z}.$$

Then, we have

$$[V,e_s]=(2n+1)e_s, s\in B,\ [V,e_{2n+1}]=2(2n+1)e_{2n+1}.$$

Now, we can see that

$$(\mathcal{L}_Vg)(e_q,e_s)+2S^*(e_q,e_s)=2\nu g(e_q,e_s),\ q,s\in A,$$

for $\nu=-2(2n+1)$. Hence $M^{2n+1}(0,-1,-1)$ is the GSSF with Sasakian metric admitting expanding $*$-Ricci soliton.

Now, we give an example of GSSF with Sasakian metric which does not admit $*$-Ricci soliton.

Example 4. Consider $M=\{(x,y,z)\in R^3 : x,y\neq 0\}$ with

$$\begin{cases}\psi(e_1)=e_2,\ \psi(e_2)=-e_1,\ \psi(e_3)=0,\ \eta=\frac{ydx-xdy}{4(1+x^2+y^2)}+\frac{1}{2}dz,\\ e_3=\xi=2\frac{\partial}{\partial z},\ g=\frac{1}{4(1+x^2+y^2)^2}(dx\otimes dx+dy\otimes dy)+\eta\otimes\eta,\\ e_1=2(1+x^2+y^2)\frac{\partial}{\partial x}-y\frac{\partial}{\partial z},\quad e_2=2(1+x^2+y^2)\frac{\partial}{\partial y}+x\frac{\partial}{\partial z}.\end{cases} \tag{4.15}$$

Moreover,

$$[e_1,e_3]=0,\ [e_1,e_2]=-4ye_1+4xe_2+2e_3,\ [e_2,e_3]=0. \tag{4.16}$$

From (4.3) and (4.16), we get

$$\begin{cases}\nabla_{e_1}e_1 = 4ye_2, \nabla_{e_2}e_1 = -4xe_2 - e_3,\ \nabla_{e_3}e_1 = -e_2, \nabla_{e_2}e_2 = 4xe_1,\\ \nabla_{e_1}e_2 = -4ye_1 + e_3, \nabla_{e_3}e_2 = e_1 = \nabla_{e_2}e_3, \nabla_{e_1}e_3 = -e_2, \nabla_{e_3}e_3 = 0.\end{cases} \tag{4.17}$$

Using (4.15) and (4.17), we can see that M is a Sasakian manifold. Using (4.16) and (4.17), we find

$$\begin{cases}R(e_1,e_2)e_1 = -13e_2, R(e_l,e_p)e_s = 0,\ l \neq p \neq s,\ l,p,s = 1,2,3,\\ R(e_1,e_2)e_2 = 13e_1,\ R(e_l,e_3)e_3 = e_l,\ R(e_l,e_3)e_l = -e_3,\ l = 1,2,\end{cases} \tag{4.18}$$

Using (2.4) and (4.18) we find that M is a GSSF with $k_1 = 4$, $k_2 = 3$, $k_3 = 3$.

Suppose that $M^3(4,3,3)$ admits ∗-Ricci soliton. Then, we can assume $V = \alpha e_1 + \beta e_2 + \gamma e_3$ locally with respect to orthonormal frame $\{e_1, e_2, e_3\}$ for some smooth functions. Using (3.11) and Theorem 3 in (1.2), we obtain

$$g(\nabla_W V, U) + g(W, \nabla_U V) - 26\eta(W)\eta(U) = 26g(W,U). \tag{4.19}$$

From (4.19), we get the following:

$$\begin{cases}e_1(\alpha) - 4\beta y = 13,\ e_2(\beta) - 4\alpha x = 13,\ e_2(\gamma) + e_3(\beta) = 2\alpha,\\ e_3(\gamma) = 26,\ e_1(\beta) + 4\alpha y + e_2(\alpha) + 4\beta x = 0,\ e_1(\gamma) + e_3(\alpha) = -2\beta.\end{cases} \tag{4.20}$$

We find that the system of equations (4.20) is inconsistent. Thus, M does not admit ∗-Ricci soliton.

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References

[1] P. Alegre, D. E. Blair, A. Carriazo: *Generalized Sasakian-space-forms*, Israel J. Math., **141** (2004), 157–183.

[2] D. E. Blair: Riemannian Geometry of Contact and Symplectic Manifolds, Prog. Math., **203**, Birkhäuser, Boston, 2010.

[3] C. P. Boyer, K. Galicki, P. Matzeu: *On eta-Einstein Sasakian geometry*, Commun. Math. Phys., **262** (2006), 177–208.

[4] X. Chen: *Real hypersurfaces with ∗-Ricci solitons of non-flat complex space forms*, Tokyo J. Math., **41** (2018), 433–451.

[5] J. T. Cho, M. Kimura: *Ricci solitons and real hypersurfaces in a complex space form*, Tohoku Math. J., **61** (2009), 205–212.

[6] B. Chow et al.: The Ricci Flow: Techniques and Applications, Math. Surveys Monogr., Amer. Math. Soc, **135**, 2007.

[7] A. Ghosh, D. S. Patra: *$*$-Ricci soliton within the frame-work of Sasakian and (k, μ)-contact manifold*, Int. J. Geom. Methods Mod. Phys. **15**, n. 7 (2018), Article no. 1850120.

[8] T. Hamada: *Real hypersurfaces of complex space forms in terms of Ricci $*$-tensor*, Tokyo J. Math., **25** (2002), 473–483.

[9] R. S. Hamilton: *The Ricci flow on surfaces*, Contemp. Math., **71** (1988), 237–262.

[10] C. He, M. Zhu: *Ricci solitons on Sasakian manifolds*, arxiv:1109.4407v2.2011, 2011.

[11] J.-B. Jun, I.-B. Kim, U. K. Kim: *On 3-dimensional almost contact metric manifolds*, Kyungpook Math. J., **34**, no. 2 (1994), 293–301.

[12] G. Kaimakamis, K. Panagiotidou: *$*$-Ricci solitons of real hypersurfaces in non-flat complex space forms*, J. Geom. Phys., **86** (2014), 408–413.

[13] P. Majhi, U. C. De: *On three dimensional generalized Sasakian-space-forms*, J. Geom., **108** (2017), 1039–1053.

[14] P. Majhi, U. C. De, Y. J. Suh: *$*$-Ricci solitons on Sasakian 3-manifolds*, Publ. Math. Debrecen, **93**, no. 1-2 (2018), 241–252.

[15] R. Sharma: *Certain results on K-contact and (κ, μ)-contact manofolds*, J. Geom., **89** (2008), 138—147.

[16] R. Sharma, A. Ghosh: *Sasakian 3-manifold as a Ricci soliton represents the Heisenberg group*, Int. J. Geom. Methods Mod. Phys., **8**, no. 1 (2011), 149–154.

[17] S. Tachibana: *On almost-analytic vectors in almost-Kählerian manifolds*, Tohoku Math. J.(2), **11**, no. 2 (1959), 247–265.

[18] S. Tanno: *Some transformations on manifolds with almost contact and contact metric structures, II*, Tohoku Math. J.(2), **15**, no. 4 (1963), 322–331.

[19] K. Yano: Integral Formulas in Riemannian Geometry, Marcel Dekker. Inc., New York, 1970.

[20] K. Yano, M. Kon: Structures on Manifolds, Series in Pure Mathematics, **3**, 1984.