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# Some consequences of restrictions on digitally continuous functions

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**Abstract.** We study the consequences of some restrictions on digitally continuous functions. One of our results modifies easily to yield an analogous result for topological spaces.

**Keywords:** digital topology, digital image, freezing set

**MSC 2020 classification:** 54H30, 54H25

## 1 Introduction

If $f : X \to Y$ is a continuous function between topological spaces, and $\emptyset \neq A \subset X$, it is often true that knowledge of $f|_A$ tells us little about $f|_{X\setminus A}$. A digital image is often a model of an object in Euclidean space, and the concept of a digitally continuous function is modeled on the "preservation of nearness" notion of a Euclidean continuous function; however, when we consider a continuous function $f : (X, \kappa) \to (Y, \lambda)$ between digital images, we often find that knowledge of $f|_A$ tells us much about $f|_{X\setminus A}$. In this paper, we continue the work of fixed point theory for digital images (see [24, 15, 18, 11, 12, 13, 14]) and coincidence theory for digital images (see [1]) by examining how restrictions placed on $f|_A$ limit $f|_{X\setminus A}$.

## 2 Preliminaries

Let $\mathbb{N}$ denote the set of natural numbers; $\mathbb{N}^* = \{0\} \cup \mathbb{N}$, the set of nonnegative integers; $\mathbb{Z}$, the set of integers; and $\mathbb{R}$, the set of real numbers. $\#X$ will be used for the number of members of a set $X$.

### 2.1 Adjacencies

Material in this section is largely quoted or paraphrased from [18].

A digital image is a pair $(X, \kappa)$ where $X \subset \mathbb{Z}^n$ for some $n$ and $\kappa$ is an adjacency on $X$. Thus, $(X, \kappa)$ is a graph for which $X$ is the vertex set and $\kappa$ determines the edge set. Usually, $X$ is finite, although there are papers that consider infinite $X$. Usually, adjacency reflects some type of "closeness" in $\mathbb{Z}^n$ of the adjacent points. When these "usual" conditions are satisfied, one may consider a subset $Y$ of $\mathbb{Z}^n$ (typically, an $n$-dimensional cube) containing $X$ as a model of a black-and-white "real world" image in which the black points (foreground) are represented by the members of $X$ and the white points (background) by members of $Y \setminus X$.

We write $x \leftrightarrow_\kappa y$, or $x \leftrightarrow y$ when $\kappa$ is understood or when it is unnecessary to mention $\kappa$, to indicate that $x$ and $y$ are $\kappa$-adjacent. Notations $x \leftrightarroweq_\kappa y$, or $x \leftrightarroweq y$ when $\kappa$ is understood, indicate that $x$ and $y$ are $\kappa$-adjacent or are equal.

The most commonly used adjacencies are the $c_u$ adjacencies, defined as follows. Let $X \subset \mathbb{Z}^n$ and let $u \in \mathbb{Z}$, $1 \le u \le n$. Then for points

$$x = (x_1, \dots, x_n) \neq (y_1, \dots, y_n) = y$$

we have $x \leftrightarrow_{c_u} y$ if and only if

- for at most $u$ indices $i$ we have $|x_i - y_i| = 1$, and
- for all indices $j$, $|x_j - y_j| \neq 1$ implies $x_j = y_j$.

The $c_u$-adjacencies are often denoted by the number of adjacent points a point can have in the adjacency. E.g.,

- in $\mathbb{Z}$, $c_1$-adjacency is 2-adjacency;
- in $\mathbb{Z}^2$, $c_1$-adjacency is 4-adjacency and $c_2$-adjacency is 8-adjacency;
- in $\mathbb{Z}^3$, $c_1$-adjacency is 6-adjacency, $c_2$-adjacency is 18-adjacency, and $c_3$-adjacency is 26-adjacency.

In this paper, we mostly use the $c_1$ and $c_2$ adjacencies in $\mathbb{Z}^2$.

Let $x \in (X, \kappa)$. We use the notations

$$N(X, x, \kappa) = \{ y \in X \mid y \leftrightarrow_\kappa x \}$$

and

$$N^*(X, x, \kappa) = \{ y \in X \mid y \leftrightarroweq_\kappa x \} = N(X, x, \kappa) \cup \{ x \}.$$

We say $\{ x_n \}_{n=0}^k \subset (X, \kappa)$ is a *$\kappa$-path* (or a *path* if $\kappa$ is understood) from $x_0$ to $x_k$ if $x_i \leftrightarroweq_\kappa x_{i+1}$ for $i \in \{ 0, \dots, k-1 \}$, and $k$ is the *length* of the path.

A subset $Y$ of a digital image $(X, \kappa)$ is *$\kappa$-connected* [24], or *connected* when $\kappa$ is understood, if for every pair of points $a, b \in Y$ there exists a $\kappa$-path in $Y$ from $a$ to $b$.

## 2.2 Digitally continuous functions

Material in this section is largely quoted or paraphrased from [18].

We denote by id or $\mathrm{id}_X$ the identity map $\mathrm{id}(x) = x$ for all $x \in X$.

**Definition 1.** [24, 4] Let $(X, \kappa)$ and $(Y, \lambda)$ be digital images. A function $f : X \to Y$ is $(\kappa, \lambda)$-*continuous*, or *digitally continuous* when $\kappa$ and $\lambda$ are understood, if for every $\kappa$-connected subset $X'$ of $X$, $f(X')$ is a $\lambda$-connected subset of $Y$. If $(X, \kappa) = (Y, \lambda)$, we say a function is $\kappa$-*continuous* to abbreviate "$(\kappa, \kappa)$-continuous."

**Theorem 1.** [4] *A function $f : X \to Y$ between digital images $(X, \kappa)$ and $(Y, \lambda)$ is $(\kappa, \lambda)$-continuous if and only if for every $x, y \in X$, if $x \leftrightarrow_\kappa y$ then $f(x) \leftrightarroweq_\lambda f(y)$.*

**Theorem 2.** [4] *Let $f : (X, \kappa) \to (Y, \lambda)$ and $g : (Y, \lambda) \to (Z, \mu)$ be continuous functions between digital images. Then $g \circ f : (X, \kappa) \to (Z, \mu)$ is continuous.*

**Definition 2.** Let $A \subset X$. A $\kappa$-continuous function $r : X \to A$ is a *retraction*, and $A$ *is a retract of* $X$, if $r(a) = a$ for all $a \in A$.

A function $f : (X, \kappa) \to (Y, \lambda)$ is an *isomorphism* (called a *homeomorphism* in [3]) if $f$ is a continuous bijection such that $f^{-1}$ is continuous.

We use the following notation. For a digital image $(X, \kappa)$,

$$C(X, \kappa) = \{\, f : X \to X \mid f \text{ is } \kappa\text{-continuous} \,\}.$$

Given $f \in C(X, \kappa)$, a point $x \in X$ is a *fixed point of* $f$ if $f(x) = x$. We denote by $\mathrm{Fix}(f)$ the set $\{\, x \in X \mid x \text{ is a fixed point of } f \,\}$. A point $x \in X$ is an *almost fixed point* [24, 26] or an *approximate fixed point* [15] of $f$ if $x \leftrightarroweq_\kappa f(x)$.

We use the projection functions $p_1, p_2 : \mathbb{Z}^2 \to \mathbb{Z}$ defined for $(x, y) \in \mathbb{Z}^2$ by $p_1(x, y) = x$, $p_2(x, y) = y$. These functions are $(c_1, c_1)$-continuous and $(c_2, c_1)$-continuous [22].

## 2.3 Freezing and cold sets

Material in this section is largely quoted or paraphrased from [11].

Knowledge of $\mathrm{Fix}(f)$ for $f \in C(X, \kappa)$ can tell us much about $f|_{X \setminus \mathrm{Fix}(f)}$. This motivates the study of freezing and cold sets.

**Definition 3.** [11] Let $(X, \kappa)$ be a digital image. We say $A \subset X$ is a *freezing set for* $X$ if given $g \in C(X, \kappa)$, $A \subset \mathrm{Fix}(g)$ implies $g = \mathrm{id}_X$. If no proper subset of a freezing set $A$ is a freezing set for $(X, \kappa)$, then $A$ is a *minimal freezing set*

**Definition 4.** [11] $A \subset X$ is a *cold set* for the connected digital image $(X, \kappa)$ if given $g \in C(X, \kappa)$ such that $g|_A = \mathrm{id}_A$, then for all $x \in X$, $g(x) \in N^*(X, x, \kappa)$.

**Remark 1.** [11] A freezing set is a cold set.

**Definition 5.** [12] Let $X \subset \mathbb{Z}^n$.

- The *boundary of X with respect to the $c_i$ adjacency*, $i \in \{1, 2\}$, is

$$Bd_i(X) = \{ x \in X \mid \text{there exists } y \in \mathbb{Z}^n \setminus X \text{ such that } y \leftrightarrow_{c_i} x \}.$$

  $Bd_1(X)$ is what is called the *boundary of X* in [23]. This paper uses both $Bd_1(X)$ and $Bd_2(X)$.

- The *interior of X* with respect to the $c_i$ adjacency is

$$Int_i(X) = X \setminus Bd_i(X).$$

**Theorem 3.** [11] *Let $X \subset \mathbb{Z}^n$ be finite. Then for $1 \le u \le n$, $Bd_1(X)$ is a freezing set for $(X, c_u)$.*

**Theorem 4.** [11] *Let $X = \Pi_{i=1}^n [0, m_i]_{\mathbb{Z}}$. Let $A = \Pi_{i=1}^n \{ 0, m_i \}$.*

- *Let $Y = \Pi_{i=1}^n [a_i, b_i]_{\mathbb{Z}}$ be such that $X \subset Y$. Let $f : X \to Y$ be $c_1$-continuous. If $A \subset \text{Fix}(f)$, then $X \subset \text{Fix}(f)$.*

- *A is a freezing set for $(X, c_1)$; minimal for $n \in \{1, 2\}$.*

**Theorem 5.** [11] *Let $X = \prod_{i=1}^n [0, m_i]_{\mathbb{Z}} \subset \mathbb{Z}^n$, where $m_i > 1$ for all i. Then $Bd_1(X)$ is a minimal freezing set for $(X, c_n)$.*

## 2.4 Digital disks and bounding curves

Material in this section is largely quoted or paraphrased from [12].

We say a finite $c_2$-connected set $S = \{ x_i \}_{i=1}^n \subset \mathbb{Z}^2$ is a *(digital) line segment* if the members of $S$ are collinear.

We say a segment with slope of $\pm 1$ is *slanted*. An *axis-parallel* segment is horizontal or vertical.

**Remark 2.** [12] A digital line segment must be axis-parallel or slanted.

A *closed curve* is a path $S = \{ s_i \}_{i=0}^m$ such that $s_0 = s_m$, and $0 < |i - j| < m$ implies $s_i \ne s_j$. If

$$N(S, x_0, \kappa) = N(S, x_m, \kappa) = \{ x_1, x_{m-1} \} \text{ and}$$

$$1 \le i < m \text{ implies } N(S, x_i, \kappa) = \{ x_{i-1}, x_{i+1} \},$$

$S$ is a *cycle*. We may also refer to a cycle as a *(digital) κ-simple closed curve*. For a simple closed curve $S \subset \mathbb{Z}^2$ we generally assume

- $m \ge 8$ if $\kappa = c_1$, and

- $m \geq 4$ if $\kappa = c_2$.

These requirements are necessary for the Jordan Curve Theorem of digital topology, below, as a $c_1$-simple closed curve in $\mathbb{Z}^2$ must have at least 8 points to have a nonempty finite complementary $c_2$-component, and a $c_2$-simple closed curve in $\mathbb{Z}^2$ must have at least 4 points to have a nonempty finite complementary $c_1$-component. Examples in [23] show why it is desirable to consider $S$ and $\mathbb{Z}^2 \setminus S$ with different adjacencies.

**Theorem 6.** [23] (Jordan Curve Theorem for digital topology) *Let $\{\kappa, \kappa'\} = \{c_1, c_2\}$. Let $S \subset \mathbb{Z}^2$ be a simple closed $\kappa$-curve such that $S$ has at least 8 points if $\kappa = c_1$ and such that $S$ has at least 4 points if $\kappa = c_2$. Then $\mathbb{Z}^2 \setminus S$ has exactly 2 $\kappa'$-connected components.*

One of the $\kappa'$-components of $\mathbb{Z}^2 \setminus S$ is finite and the other is infinite. This suggests the following.

**Definition 6.** [12] Let $S \subset \mathbb{Z}^2$ be a $c_2$-closed curve such that $\mathbb{Z}^2 \setminus S$ has two $c_1$-components, one finite and the other infinite. The union $D$ of $S$ and the finite $c_1$-component of $\mathbb{Z}^2 \setminus S$ is a *(digital) disk*. $S$ is a *bounding curve* of $D$. The finite $c_1$-component of $\mathbb{Z}^2 \setminus S$ is the *interior of* $S$, denoted $Int(S)$, and the infinite $c_1$-component of $\mathbb{Z}^2 \setminus S$ is the *exterior of* $S$, denoted $Ext(S)$.

Notes [12]:

- If $D$ is a digital disk determined as above by a bounding $c_2$-closed curve $S$, then $(S, c_1)$ can be disconnected. See Figure 1.

- There may be more than one closed curve $S$ bounding a given disk $D$. See Figure 2. When $S$ is understood as a bounding curve of a disk $D$, we use the notations $Int(S)$ and $Int(D)$ interchangeably.

- Since we are interested in finding *minimal* freezing or cold sets and since it turns out we often compute these from bounding curves, we may prefer those of minimal size. A bounding curve $S$ for a disk $D$ is *minimal* if there is no bounding curve $S'$ for $D$ such that $\#S' < \#S$.

- In particular, a bounding curve need not be contained in $Bd_1(D)$. E.g., in the disk $D$ shown in Figure 2(i), $(2, 2)$ is a point of the bounding curve; however, all of the points $c_1$-adjacent to $(2, 2)$ are members of $D$, so by Definition 5, $(2, 2) \notin Bd_1(D)$. However, a bounding curve for $D$ must be contained in $Bd_2(D)$.

- In Definition 6, we use $c_2$ adjacency for $S$ and we do not require $S$ to be simple. Figure 2 shows why these seem appropriate.

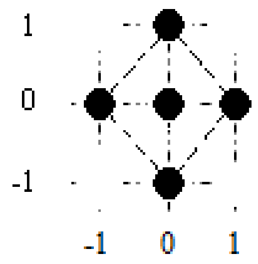


Figure 1. [12] The $c_1$-disk $D = \{ (x,y) \in \mathbb{Z}^2 \mid |x| + |y| < 2 \}$. The bounding curve $S = \{ (x,y) \in \mathbb{Z}^2 \mid |x| + |y| = 1 \} = D \setminus \{ (0,0) \}$ is not $c_1$-connected.

- The $c_2$ adjacency allows slanted segments in bounding curves and makes possible a bounding curve in subfigure (ii) with fewer points than the bounding curve in subfigure (i) in which adjacent pairs of the bounding curve are restricted to $c_1$ adjacency.
- Neither of the bounding curves shown in Figure 2 is a $c_2$-simple closed curve. E.g., non-consecutive points of each of the bounding curves, $(0,1)$ and $(1,0)$, are $c_2$-adjacent. The bounding curve shown in Figure 2(ii) is clearly also not a $c_1$-simple closed curve.

- A closed curve that is not simple may be the boundary $Bd_2$ of a digital image that is not a disk. This is illustrated in Figure 3.

More generally, we have the following.

**Definition 7.** [12] Let $X \subset \mathbb{Z}^2$ be a finite, $c_i$-connected set, $i \in \{1,2\}$. Suppose there are pairwise disjoint $c_2$-closed curves $S_j \subset X$, $1 \leq j \leq n$, such that

- $X \subset S_1 \cup Int(S_1)$;
- for $j > 1$, $D_j = S_j \cup Int(S_j)$ is a digital disk;
- no two of
$$S_1 \cup Ext(S_1), D_2, \ldots, D_n$$
are $c_1$-adjacent or $c_2$-adjacent; and
- we have
$$\mathbb{Z}^2 \setminus X = Ext(S_1) \cup \bigcup_{j=2}^{n} Int(S_j).$$

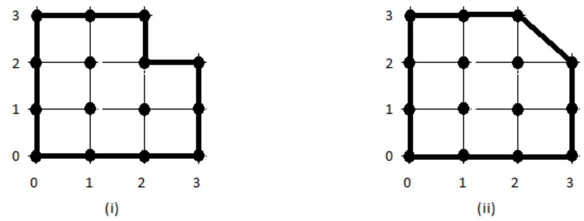


Figure 2. [12] Two views of $D = [0,3]_{\mathbb{Z}}^2 \setminus \{ (3,3) \}$, which can be regarded as a $c_1$-disk with either of the closed curves shown in dark as a bounding curve.
(i) The dark line segments show a $c_1$-simple closed curve $S$ that is a bounding curve for $D$. Note the point $(2,2)$ in the bounding curve shown. By Definition 5, $(2,2) \not\in Bd_1(D)$; however, $(2,2) \in Bd_2(D)$.
(ii) The dark line segments show a $c_2$-closed curve $S$ that is a minimal bounding curve for $D$.

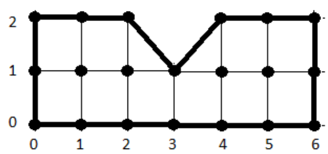


Figure 3. [12] $D = [0,6]_{\mathbb{Z}} \times [0,2]_{\mathbb{Z}} \setminus \{ (3,2) \}$ shown with a bounding curve $S$ in dark segments. $D$ is not a disk with either the $c_1$ or the $c_2$ adjacency, since with either of these adjacencies, $\mathbb{Z}^2 \setminus S$ has two bounded components, $\{ (1,1),(2,1) \}$ and $\{ (4,1),(5,1) \}$.

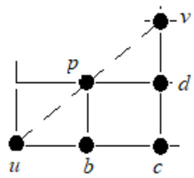


Figure 4. [12] $p \in \overline{uv}$ in a bounding curve, with $\overline{uv}$ slanted. Note $u \not\leftrightarrow_{c_1} p \not\leftrightarrow_{c_1} v$, $p \leftrightarrow_{c_2} c \not\leftrightarrow_{c_1} p$, $\{p, c\} \subset N(\mathbb{Z}^2, c_1, b) \cap N(\mathbb{Z}^2, c_1, d)$. If $X$ is slant-thick at $p$ then $c \in X$. (Not meant to be understood as showing all of $X$.)

Then $\{ S_j \}_{j=1}^n$ is a *set of bounding curves of* $X$.

Note: As above, a digital image $X \subset \mathbb{Z}^2$ may have more than one set of bounding curves.

## 2.5 Thickness

A notion of "thickness" in a digital image $X$, introduced in [12], means, roughly speaking, $X$ is "locally" like a disk.

Our definition of thickness depends on a notion of an "interior angle" of a disk. We have the following.

**Definition 8.** [12] Let $s_1$ and $s_2$ be sides of a digital disk $X \subset \mathbb{Z}^2$, i.e., maximal digital line segments in a bounding curve $S$ of $X$, such that $s_1 \cap s_2 = \{p\} \subset X$. The *interior angle of* $X$ *at* $p$ is the angle formed by $s_1$, $s_2$, and $Int(S)$.

**Definition 9.** [12] Let $X \subset \mathbb{Z}^2$ be a digital disk. Let $S$ be a bounding curve of $X$ and $p \in S$.

- Suppose $p$ is in a maximal slanted segment $\sigma$ of $S$ such that $p$ is not an endpoint of $\sigma$. Then $X$ is *slant-thick at* $p$ if there exists $c \in X$ such that (see Figure 4)
$$c \leftrightarrow_{c_2} p \not\leftrightarrow_{c_1} c, \qquad (2.1)$$
- Suppose $p$ is the vertex of a 90° ($\pi/2$ radians) interior angle $\theta$ of $S$. Then $X$ is *90°-thick at* $p$ if there exists $q \in Int(X)$ such that
  - if $\theta$ has axis-parallel sides then $q \leftrightarrow_{c_2} p \not\leftrightarrow_{c_1} q$ (see Figure 5(1));
  - if $\theta$ has slanted sides then $q \leftrightarrow_{c_1} p$ (see Figure 5(2)).
- Suppose $p$ is the vertex of a 135° ($3\pi/4$ radians) interior angle $\theta$ of $S$. Then $X$ is *135°-thick at* $p$ if there exist $b, b' \in X$ such that $b$ and $b'$ are in

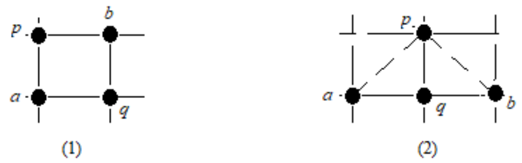


Figure 5. [12] (1) $\angle apb$ is a 90° ($\pi/2$ radians) angle of a bounding curve of $X$ at $p \in A_1$, with horizontal and vertical sides. If $X$ is 90°-thick at $p$ then $q \in Int(X)$. (Not meant to be understood as showing all of $X$.)
(2) $\angle apb$ is a 90° ($\pi/2$ radians) angle between slanted segments of a bounding curve. If $X$ is 90°-thick at $p$ then $q \in Int(X)$. (Not meant to be understood as showing all of $X$).

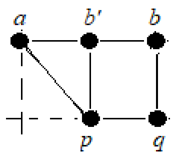


Figure 6. [12] $\angle apq$ is an angle of 135° degrees ($3\pi/4$ radians) of a bounding curve of $X$ at $p$, with $\overline{ap} \cup \overline{pq}$ a subset of the bounding curve. If $X$ is 135°-thick at $p$ then $b, b' \in X$. (Not meant to be understood as showing all of $X$.)

the interior of $\theta$ and (see Figure 6)

$$b \leftrightarrow_{c_2} p \not\leftrightarrow_{c_1} b \quad \text{and} \quad b' \leftrightarrow_{c_1} p.$$

**Definition 10.** [12, 14] Let $X \subset \mathbb{Z}^2$ be a digital disk. We say $X$ is *thick* if the following are satisfied. For some bounding curve $S$ of $X$,

- for every maximal slanted segment of $S$, if $p \in S$ is not an endpoint of $S$, then $X$ is slant-thick at $p$, and
- for every $p$ that is the vertex of a 90° ($\pi/2$ radians) interior angle $\theta$ of $S$, $X$ is 90°-*thick at* $p$, and
- for every $p$ that is the vertex of a 135° ($3\pi/4$ radians) interior angle $\theta$ of $S$, $X$ is 135°-thick at $p$.

### 2.6 Convexity

A set $X$ in a Euclidean space $\mathbb{R}^n$ is *convex* if for every pair of distinct points $x, y \in X$, the line segment $\overline{xy}$ from $x$ to $y$ is contained in $X$. The *convex hull of* $Y \subset \mathbb{R}^n$, denoted $hull(Y)$, is the smallest convex subset of $\mathbb{R}^n$ that contains $Y$. If $Y \subset \mathbb{R}^2$ is a finite set, then $hull(Y)$ is a single point if $Y$ is a singleton; a line segment if $Y$ has at least 2 members and all are collinear; otherwise, $hull(Y)$ is a polygonal disk, and the endpoints of the edges of $hull(Y)$ are its *vertices*.

A digital version of convexity can be stated for subsets of the digital plane $\mathbb{Z}^2$ as follows. A finite set $Y \subset \mathbb{Z}^2$ is *(digitally) convex* [12] if either

- $Y$ is a single point, or
- $Y$ is a digital line segment, or
- $Y$ is a digital disk with a bounding curve $S$ such that the endpoints of the maximal line segments of $S$ are the vertices of $hull(Y) \subset \mathbb{R}^2$.

## 3 Tools for determining fixed point sets

The following assertions will be useful in determining fixed point and freezing sets.

**Proposition 1.** (Corollary 8.4 of [18]) *Let $(X, \kappa)$ be a digital image and $f \in C(X, \kappa)$. Suppose $x, x' \in \text{Fix}(f)$ are such that there is a unique shortest $\kappa$-path $P$ in $X$ from $x$ to $x'$. Then $P \subseteq \text{Fix}(f)$.*

Lemma 1, below,

> ... can be interpreted to say that in a $c_u$-adjacency, a continuous function that moves a point $p$ also moves a point that is "behind" $p$. E.g., in $\mathbb{Z}^2$, if $q$ and $q'$ are $c_1$- or $c_2$-adjacent with $q$ left, right, above, or below $q'$, and a continuous function $f$ moves $q$ to the left, right, higher, or lower, respectively, then $f$ also moves $q'$ to the left, right, higher, or lower, respectively [11].

**Lemma 1.** [11] *Let $(X, c_u) \subset \mathbb{Z}^n$ be a digital image, $1 \leq u \leq n$. Let $q, q' \in X$ be such that $q \leftrightarrow_{c_u} q'$. Let $f \in C(X, c_u)$.*

*(1) If $p_i(f(q)) > p_i(q) > p_i(q')$ then $p_i(f(q')) > p_i(q')$.*

*(2) If $p_i(f(q)) < p_i(q) < p_i(q')$ then $p_i(f(q')) < p_i(q')$.*

**Remark 3.** [11] If $X \subset \mathbb{Z}^2$ is finite, then a set of bounding curves for $X$ is a freezing set for $(X, c_i)$, $i \in \{1, 2\}$.

In particular, we have:

**Theorem 7.** *Let $D$ be a digital disk in $\mathbb{Z}^2$. Let $S$ be a bounding curve for $D$. Then $S$ is a freezing set for $(D, c_1)$ and for $(D, c_2)$.*

The next two results form a dual pair.

**Theorem 8.** [12] *Let $X$ be a thick convex disk with a bounding curve $S$. Let $A_1$ be the set of points $x \in S$ such that $x$ is an endpoint of a maximal axis-parallel edge of $S$. Let $A_2$ be the union of slanted line segments in $S$. Then $A = A_1 \cup A_2$ is a minimal freezing set for $(X, c_1)$.*

**Theorem 9.** [12] *Let $X$ be a thick convex disk with a minimal bounding curve $S$. Let $B_1$ be the set of points $x \in S$ such that $x$ is an endpoint of a maximal slanted edge in $S$. Let $B_2$ be the union of maximal axis-parallel line segments in $S$. Let $B = B_1 \cup B_2$. Then $B$ is a minimal freezing set for $(X, c_2)$.*

The next two results form another dual pair, generalizing the previous pair.

**Theorem 10.** [13] *Let $V_i \subset X \subset \mathbb{Z}^2$, $i \in \{1, \ldots, n\}$ where each $V_i$ is a thick convex disk. Let $X' = \bigcup_{i=1}^n V_i$. Let $C_i$ be a bounding curve of $V_i$. Let $A_{1,i}$ be the set of endpoints of maximal horizontal or vertical segments of $C_i$. Let $A_{2,i}$ be the union of maximal slanted segments of $C_i$. Then $A = (X \setminus X') \cup \bigcup_{i=1}^n (A_{1,i} \cup A_{2,i})$ is a freezing set for $(X, c_1)$.*

**Theorem 11.** [13] *Let $V_i \subset X \subset \mathbb{Z}^2$, $i \in \{1, \ldots, n\}$ where each $V_i$ is a thick convex disk. Let $X' = \bigcup_{i=1}^n V_i$. Let $C_i$ be a bounding curve of $V_i$. Let $B_{1,i}$ be the union of maximal horizontal and maximal vertical segments of $C_i$. Let $B_{2,i}$ be the set of endpoints of maximal slanted segments of $C_i$. Then $B = (X \setminus X') \cup \bigcup_{i=1}^n (B_{1,i} \cup B_{2,i})$ is a freezing set for $(X, c_2)$ (the adjacency is misprinted as $c_1$ in* [13]*).*

# 4 Unifying sets

## 4.1 Definition and general properties

**Definition 11.** Let $(X, \kappa)$ be a digital image. Let $A \subset X$. Suppose whenever $f, g \in C(X, \kappa)$ are such that $f(A) = g(A) = A$ and $f|_A = g|_A$, we have $f = g$. Then we say $A$ is a *unifying set* for $(X, \kappa)$. $A$ is a *minimal unifying set* if $A$ is a unifying set and no proper subset of $A$ is a unifying set for $(X, \kappa)$.

**Remark 4.** Observe:

- By taking $g$ to be the identity function $\mathrm{id}_X$ in Definition 11, we see that a unifying set is a freezing set. We have not determined whether the converse is true.

- It is trivial that $X$ is a unifying set for $(X, \kappa)$. We are therefore interested in finding minimal unifying sets. In light of the above, a minimal freezing set is a "good candidate" for a minimal unifying set.

In the following, we study conditions for which a freezing set must be unifying.

The desirability of the requirement that $f(A) = g(A) = A$ in Definition 11 is illustrated in the following, in which this requirement is not met.

**Example 1.** Let $X = [0, m]_{\mathbb{Z}} \times [0, n]_{\mathbb{Z}}$ for $m \geq 2,\ n > 0$. Let $f, g : X \to X$ be the functions

$$f(x, y) = (0, y), \qquad g(x, y) = \begin{cases} (0, y) & \text{if } x \in \{0, m\}; \\ (1, y) & \text{if } 1 \leq x \leq m - 1, \end{cases}$$

We take

$$A = \{(0,0), (0,n), (m,0), (m,n)\}.$$

Note by Theorem 4, $A$ is a minimal freezing set for $(X, c_1)$. We see easily that $f, g \in C(X, c_1)$, $f|_A = g|_A$, $f(A) = g(A)$ is a proper subset of $A$, and $f \neq g$.

The following shows that unifying sets are preserved by isomorphism.

**Theorem 12.** *Let $(X, \kappa)$ and $(Y, \lambda)$ be digital images such that there exists an isomorphism $F : (X, \kappa) \to (Y, \lambda)$. If $A$ is a unifying set for $(X, \kappa)$ then $F(A)$ is a unifying set for $(Y, \lambda)$.*

*Proof.* Let $f, g \in C(Y, \lambda)$ such that $f(F(A)) = g(F(A)) = F(A)$ and $f|_{F(A)} = g|_{F(A)}$.

We have, by Theorem 2, $f' = F^{-1} \circ f \circ F,\ g' = F^{-1} \circ g \circ F \in C(X, \kappa)$, and for $a \in A$ we have $f \circ F(a) = g \circ F(a)$, so

$$f'(a) = F^{-1} \circ f \circ F(a) = F^{-1} \circ g \circ F(a), = g'(a).$$

Also, given $b = F(a)$ for $a \in A$, by assumption we have $f(b) = g(b)$, hence

$$f'(a) = F^{-1}(f(b)) = F^{-1}(g(b)) = g'(a).$$

Since $A$ is unifying, $f' = g'$. Therefore,

$$f = F \circ f' \circ F^{-1} = F \circ g' \circ F^{-1} = g,$$

so $F(A)$ is unifying for $(Y, \lambda)$. QED

We have the following generalization of Proposition 1.

**Proposition 2.** *Let $f, g : X \to Y$ such that $f$ and $g$ are both $(\kappa, \lambda)$-continuous. Suppose $x_0, x_1 \in X$ and there is a $\kappa$-path $P$ of length $n$ in $X$ from $x_0$ to $x_1$. Suppose $y_0 = f(x_0) = g(x_0)$, $y_1 = f(x_1) = g(x_1)$, and there is a unique shortest path $Q$ of length $n$ in $Y$ from $y_0$ to $y_1$. Then $f(P) = g(P) = Q$ and $f|_P = g|_P$.*

*Proof.* Since $f(P)$ and $g(P)$ must be $\lambda$-paths from $y_0$ to $y_1$, our uniqueness and length restrictions imply $f(P) = g(P) = Q$. Continuity implies $f|_P = g|_P$. QED

## 4.2 Cycles

**Theorem 13.** [11] *Let $n > 4$. Consider a digital cycle $C_n = \{ x_m \}_{m=0}^{n-1} \subset \mathbb{Z}^2$, where the members of $C_n$ are indexed circularly. Let $A = \{ x_i, x_j, x_k \}$ be a set of distinct members of $C_n$ such that $C_n$ is a union of unique shorter paths determined by these points. Then $A$ is a minimal freezing set for $C_n$.*

**Theorem 14.** *The set $A$ of Theorem 13 is a unifying set for $(C_n, \kappa)$, and any $f \in C(X, \kappa)$ such that $f(A) = A$ must be an isomorphism of $(X, \kappa)$.*

*Proof.* Let $\widehat{x_i x_j}$, $\widehat{x_i x_k}$, and $\widehat{x_j x_k}$ be the unique shorter paths in $C_n$ from $x_i$ to $x_j$, from $x_i$ to $x_k$, and from $x_j$ to $x_k$, respectively. Let $B = \{ \widehat{x_i x_j}, \widehat{x_i x_k}, \widehat{x_j x_k} \}$. Let $f, g \in C(C_n, \kappa)$ such that

$$f(A) = g(A) = A \text{ and } f|_A = g|_A. \tag{4.1}$$

Suppose $f \neq g$. Consider the following cases.

- The members of $B$ have distinct lengths. Without loss of generality,

$$length(\widehat{x_i x_j}) < length(\widehat{x_i x_k}) < length(\widehat{x_j x_k}). \tag{4.2}$$

  Since we have that both $f(\widehat{x_i x_j})$ and $g(\widehat{x_i x_j})$ are paths of length at most $length(\widehat{x_i x_j})$ from $f(x_i) = g(x_i)$ to $f(x_j) = g(x_j)$, from (4.2) and Proposition 2, $f(\widehat{x_i x_j}) = g(\widehat{x_i x_j})$ and $f|_{\widehat{x_i x_j}} = g|_{\widehat{x_i x_j}}$ is a bijection of $\widehat{x_i x_j}$. Indeed, we must have that $f$ and $g$ coincide with $\mathrm{id}_X$ on $\widehat{x_i x_j}$, for otherwise we would have $f(x_i) = g(x_i) = x_j$, $f(x_j) = g(x_j) = x_i$, $f(x_k) = g(x_k) = x_k$, so $f(\widehat{x_i x_k})$ is a $\kappa$-path from $x_j$ to $x_k$, contrary to (4.2). Then by (4.1) we have $f|_A = g|_A = \mathrm{id}_A$, and from Proposition 2 it follows that $f = g = \mathrm{id}_X$.

- Suppose two members, but not all three, of $B$ have the same length; without loss of generality, $length(\widehat{x_i x_j}) = length(\widehat{x_i x_k})$. Then either $f|_A = g|_A = \mathrm{id}_A$ or $f(x_i) = g(x_i) = x_i$, $f(x_j) = g(x_j) = x_k$, and $f(x_k) = g(x_k) = x_j$. Then much as above, $f = g$ is an isomorphism of $(X, \kappa)$.

- Suppose all three members of $B$ have the same length. Then $f|_A = g|_A$ is a permutation of $A$. Much as above, it follows that $f = g$ is an isomorphism of $(X, \kappa)$.

In all cases, we concluded that $f = g$ is an isomorphism of $(X, \kappa)$. Thus $A$ is a unifying set for $(X, \kappa)$. $\boxed{QED}$

### 4.3 Trees

A *tree* is a connected acyclic graph $(X, \kappa)$. By *acyclic* we mean lacking any closed curve of more than 2 points. The *degree* of a vertex $x$ in $X$ is the number of distinct vertices $y \in X$ such that $x \leftrightarrow y$.

**Theorem 15.** [11] *Let $(X, \kappa)$ be a digital image such that the graph $G = (X, \kappa)$ is a finite tree with $\#X > 1$. Let $A$ be the set of vertices of $G$ that have degree 1. Then $A$ is a minimal freezing set for $G$.*

**Theorem 16.** *Let $(X, \kappa)$ be a digital image such that the graph $G = (X, \kappa)$ is a finite tree with $\#X > 1$. Let $A$ be the set of vertices of $G$ that have degree 1. Then $A$ is a minimal unifying set for $G$. Also, if $f \in C(X, \kappa)$ such that $f(A) = A$, then $f$ is an isomorphism of $(X, \kappa)$.*

*Proof.* Let $a_0 \in A$. Since $X$ is finite, we have that $A$ is also finite - say, $A = \{ a_i \}_{i=0}^n$. Since $G$ is a tree, for $0 < i \leq n$ there is a unique shortest $\kappa$-path $P_i$ in $X$ from $a_0$ to $a_i$. Let $L = \{ \ell_j \}_{j=1}^m$ be the set of distinct lengths of the members of $\{ P_i \}_{i=1}^n$, with

$$\ell_1 < \ell_2 < \ldots < \ell_m.$$

Let $L_j = \{ P_i \mid \textit{length}(P_i) = \ell_j \}$. Let $f, g \in C(X, \kappa)$ be such that $f(A) = g(A) = A$ and $f|_A = g|_A$. Since $A$ is finite,

$$f|_A = g|_A : A \to A \text{ is a bijection.} \tag{4.3}$$

Every $P_k$ of length $\ell_1$ is the unique shortest $\kappa$-path in X from $a_0$ to some $a_k \in A \setminus \{ a_0 \}$. Since $f(P_k)$ is a path from $f(a_0) = g(a_0)$ to $f(a_k) = g(a_k)$, our choice of $\ell_1$ and Proposition 2 imply $f|_{P_k} = g|_{P_k}$, $f(P_k) = g(P_k)$ has length $\ell_1$, and from (4.3) that $f|_{L_1} = g|_{L_1}$ is a bijection of $L_1$. It follows easily that $f|_{L_1} = g|_{L_1}$ is an isomorphism. This provides the base case of an induction argument.

Suppose $u \in \mathbb{Z}$, $0 \leq u < m$; $f|_{P_k} = g|_{P_k}$ for every $P_k \in \bigcup_{j=1}^u L_j$; and

$$f|_{\bigcup_{j=1}^u L_j} = g|_{\bigcup_{j=1}^u L_j} \text{ is a bijection of } \bigcup_{j=1}^u L_j. \tag{4.4}$$

Now consider $P_k \in L_{u+1}$. $f(P_k)$ and $g(P_k)$ are $\kappa$-paths in $X$ from $f(a_0) = g(a_0)$ to $f(a_k) = g(a_k)$ of length at most $\ell_{u+1}$. By (4.3) and (4.4), $f(P_k)$ and $g(P_k)$ cannot have length less than $\ell_{u+1}$. Therefore, each of $f(P_k)$ and $g(P_k)$ belongs to $L_{u+1}$. By the uniqueness condition that defines $L_{u+1}$ it follows that $f|_{P_k} = g|_{P_k}$. By (4.3), $f|_{L_{u+1}} = g|_{L_{u+1}}$ is a bijection. It follows from the above that $f|_{\bigcup_{j=1}^{u+1} L_j} = g|_{\bigcup_{j=1}^{u+1} L_j}$ is a bijection of $\bigcup_{j=1}^{u+1} L_j$, and, further, an isomorphism.

This completes the induction. Since $X = \bigcup_{j=1}^{m} L_j$, we have $f = g$. Since $f$ was chosen arbitrarily, $A$ is a unifying set. Also, $f$ is an isomorphism.

To show the minimality of $A$, we see easily that for any $a \in A$ there is a $\kappa$-retraction $r : X \to X \setminus \{ a \}$, so $r$ and $\mathrm{id}_X$ are members of $C(X, \kappa)$ that coincide on $A \setminus \{ a \}$, $r(A \setminus \{ a \}) = \mathrm{id}_X(A \setminus \{ a \}) = (A \setminus \{ a \})$, but $r \neq \mathrm{id}_X$. QED

## 4.4 Complete graphs

**Theorem 17.** *Let $(X, \kappa)$ be a digital image that is a complete graph, where $\#X > 1$. Let $A \subset X$. Then the following are equivalent.*

*(1) $A = X$.*

*(2) $A$ is a unifying set for $(X, \kappa)$.*

*(3) $A$ is a freezing set for $(X, \kappa)$.*

*Proof.* 1) $\Rightarrow$ 2) $\Rightarrow$ 3): These implications are noted in Remark 4.

3) $\Rightarrow$ 1): Suppose otherwise. Then there exists $x_0 \in X \setminus A$. Let $x_1 \in X \setminus \{ x_0 \}$. Let $g : X \to X$ be defined by

$$g(x) = \begin{cases} x & \text{for } x \neq x_0; \\ x_1 & \text{for } x = x_0. \end{cases}$$

Since $(X, \kappa)$ is a complete graph, $g \in C(X, \kappa)$. Note $g|_A = \mathrm{id}_A$. But since $g(x_0) \neq x_0$, we have a contradiction of the assumption that $A$ is freezing. The contradiction gives us the desired conclusion. QED

## 4.5 Rectangles in $\mathbb{Z}^2$ with axis-parallel sides and $c_1$

In this section, we study unifying sets for digital rectangles with axis-parallel edges in $\mathbb{Z}^2$, using the $c_1$ adjacency.

**Proposition 3.** [14] *Let $X \subset \mathbb{Z}^2$. Let $S$ be a minimal bounding curve for $X$. Let $p_0$ be the vertex of an interior angle of $S$, formed by axis-parallel edges $E_1$ and $E_2$ of $S$, of measure $90°$ ($\pi/2$ radians). Let $A$ be any of a freezing set for $(X, c_1)$, a cold set for $(X, c_1)$, a freezing set for $(X, c_2)$, or a cold set for $(X, c_2)$. Let $X$ be $90°$-thick at $p_0$. Then $p_0 \in A$.*

**Proposition 4.** *Let $m > 1$, $n > 1$, and $X = [0,m]_{\mathbb{Z}} \times [0,n]_{\mathbb{Z}}$. Let $A \subset X$. Then $A$ is a freezing set for $(X, c_1)$ if and only if*

$$A' = \{ (0,0), (m,0), (0,n), (m,n) \} \subset A.$$

*Therefore, $A'$ is the only minimal freezing set for $(X, c_1)$.*

*Proof.* If $A$ is a freezing set, then by Proposition 3, $A' \subset A$. Since $A'$ is a freezing set by Theorem 10, it follows that $A'$ is unique as a minimal freezing set.

If $A' \subset A$ then, since $A'$ is a freezing set, $A$ is a freezing set [11]. QED

**Theorem 18.** *Let $X = [-m,m]_{\mathbb{Z}} \times [-n,n]_{\mathbb{Z}}$. Let*

$$A = \{ (-m,-n), (-m,n), (m,-n), (m,n) \}.$$

*Then $A$ is a unifying set for $(X, c_1)$. Further, every $f \in C(X, c_1)$ such that $f(A) = A$ is an isomorphism.*

*Proof.* Let $f, g \in C(X, c_1)$ be such that $f(A) = g(A) = A$ and $f|_A = g|_A$. Let $B, T, L, R$ be the bottom, top, left, and right edges, respectively:

$$B = [-m,m]_{\mathbb{Z}} \times \{ -n \}, \qquad T = [-m,m]_{\mathbb{Z}} \times \{ n \},$$

$$L = \{ -m \} \times [-n,n]_{\mathbb{Z}}, \qquad R = \{ m \} \times [-n,n]_{\mathbb{Z}}.$$

Consider the following cases.

- $m < n$. Since $f(A) = g(A) = A$, we have that $f(B)$, $g(B)$, $f(T)$, and $g(T)$ are $c_1$-paths of length at most 2m between distinct members of $A$, and since the closest distinct members of $A$ are joined by paths of length $2m$, $f(B)$, $g(B)$, $f(T)$, and $g(T)$ are paths of length $2m$. Therefore, $f(B \cup T) = g(B \cup T) = B \cup T$. Continuity implies that for all $(x,y) \in B \cup T$, one of the following holds:
  - $f(x,y) = g(x,y) = (x,y)$, or
  - $f(x,y) = g(x,y) = (-x,y)$, or
  - $f(x,y) = g(x,y) = (x,-y)$, or
  - $f(x,y) = g(x,y) = (-x,-y)$.

  Suppose the first case, $f(x,y) = g(x,y) = (x,y)$ for $(x,y) \in B \cup T$. Each $(x,y) \in X$ lies on the unique shortest $c_1$-path between $b = (x,-n)$ and $t = (x,n)$. Since $f(b) = g(b) = b$ and $f(t) = g(t) = t$, we must have $f(x,y) = g(x,y) = (x,y)$ by Proposition 2. Thus $f = g = \mathrm{id}_X$. Similarly, $f = g$ is an isomorphism of $(X, c_1)$ in the other cases.

- $m > n$. This case is similar to the case $m < n$, yielding the conclusion that $f = g$ is an isomorphism of $(X, c_1)$.

- $m = n$. In this case we have either $f(B \cup T) = g(B \cup T) = B \cup T$ or $f(B \cup T) = g(B \cup T) = L \cup R$. In the former case, $f|_{B \cup T}$ and $g|_{B \cup T}$ are given by one of the four possibilities listed above; in the latter case, one of the following holds. For $(x, y) \in B \cup T$,

  - $f(x, y) = g(x, y) = (y, x)$, or
  - $f(x, y) = g(x, y) = (y, -x)$, or
  - $f(x, y) = g(x, y) = (-y, x)$, or
  - $f(x, y) = g(x, y) = (-y, -x)$.

  An argument like that used above shows that in each of these cases, $f = g$ is an isomorphism of $(X, c_1)$.

Thus all cases lead to the conclusion that that $f = g$, hence $A$ is unifying; and that $f \in C(X, c_1)$ such that $f(A) = A$ implies $f$ is an isomorphism of $(X, c_1)$. QED

### 4.6 Rectangles in $\mathbb{Z}^2$ with slanted sides and $c_2$

In this section, we study unifying sets for digital rectangles with slanted edges in $\mathbb{Z}^2$, using the $c_2$ adjacency. Our assertions are dual to those of section 4.5 and have proofs with common elements.

**Proposition 5.** *Let $X$ be a digital rectangle in $\mathbb{Z}^2$ with slanted edges. Let $B \subset X$. Let $B'$ be the set of endpoints of edges of $X$. Then $B$ is a freezing set for $(X, c_2)$ if and only if $B' \subset B$. Therefore, $B'$ is the only minimal freezing set for $(X, c_2)$.*

*Proof.* By Theorem 12, there is no loss of generality in assuming

$$B' = \{\, (0,0), (m,m), (n,-n), (m+n, m-n) \,\} \text{ for some } m, n \in \mathbb{N}.$$

If $B$ is a freezing set, then by Proposition 3, $B' \subset B$. Since $B'$ is a freezing set by Theorem 11, it follows that $B'$ is unique as a minimal freezing set. QED

**Theorem 19.** *Let $X$ be the digital rectangle with endpoints of edges in the set*

$$B = \{\, (0,0), (m,m), (n,-n), (m+n, m-n) \,\}.$$

*Then $B$ is a unifying set for $(X, c_2)$. Further, every $f \in C(X, c_2)$ such that $f(B) = B$ is an isomorphism.*

*Proof.* Let $LR$ (lower right) be the edge of $X$ from $(n, -n)$ to $(m+n, m-n)$. Let $UL$ (upper left) be the edge of $X$ from $(0,0)$ to $(m,m)$. Let $LL$ (lower left) be the edge of $X$ from $(0,0)$ to $(n,-n)$. Let $UR$ (upper right) be the edge of $X$ from $(m,m)$ to $(m+n, m-n)$. For $m < n$, there are distinct isomorphisms $F_1, F_2, F_3, F_4 : S \to S$, where

$$S = LR \cup UL \cup LL \cup UR$$

is the bounding curve of $X$, where $F_1 = \mathrm{id}_S$, $F_2$ reverses the orientations of $UL$ and $LR$, $F_3$ interchanges $UL$ and $LR$ while preserving their orientations, and $F_4$ interchanges $UL$ and $LR$ and reverses their orientations.

Consider the following cases.

- $m < n$. Since $f(B) = g(B) = B$, we have that $f(UL)$, $g(UL)$, $f(LR)$, and $g(LR)$ are $c_2$-paths of length at most $m$ between distinct members of $B$, and since the closest distinct members of $B$ are joined by paths of length $m$, $f(UL)$, $g(UL)$, $f(LR)$, and $g(LR)$ are paths of length $m$. Therefore, $f(UL \cup LR) = g(UL \cup LR) = UL \cup LR$. Proposition 2 implies that for all $(x,y) \in UL \cup LR$, $f(x,y) = g(x,y) = F_i(x,y)$ for some index $i$.

  Suppose the first case,

  $$f(x,y) = g(x,y) = F_1(x,y) = (x,y) \text{ for } (x,y) \in UL \cup LR.$$

  Consider the following cases.

  - Suppose $(x,y) \in X$ lies on the unique shortest $c_2$-path (a slanted path) between some $d_1 \in UL$ and some $d_2 \in LR$. Since $f(d_j) = g(d_j) = d_j$ for $j \in \{1,2\}$, we must have $f(x,y) = g(x,y) = (x,y)$ by Proposition 2.
  - Otherwise, each of the points in

    $$W = \{(x-1,y), (x+1,y), (x,y-1), (x,y+1)\}$$

    is adjacent to $(x,y)$ and lies on a slanted unique shortest $c_2$-path between a point in $UL$ and a point in $LR$ (see Figure 8). By continuity and the previous case, $W \subset \mathrm{Fix}(f) \cap \mathrm{Fix}(g)$. By Lemma 1, it follows that $(x,y) \in \mathrm{Fix}(f) \cap \mathrm{Fix}(g)$

  Thus $f = g = \mathrm{id}_X$. Similarly, $f = g$ is an isomorphism of $(X, c_2)$ if

  $$f(x,y) = g(x,y) = F_2(x,y) \text{ for } (x,y) \in UL \cup LR,$$

  $$f(x,y) = g(x,y) = F_3(x,y) \text{ for } (x,y) \in UL \cup LR, \text{ or}$$

  $$f(x,y) = g(x,y) = F_4(x,y) \text{ for } (x,y) \in UL \cup LR.$$

- $m > n$. This case is similar to the case $m < n$, and we similarly conclude that $f = g$ is an isomorphism of $(X, c_2)$.

- $m = n$. Here, in addition to the isomorphisms $F_1, F_2, F_3, F_4$ discussed above, we also have isomorphisms $R_1, R_2, R_3, R_4$ of $(X, c_2)$ that rotate the edges of $X$ by 90° ($\pi/2$ radians) either clockwise or counterclockwise, either preserving or reversing the orientations of both $UL$ and $LR$. An argument like that used above shows that in each of these cases, $f = g$ is an isomorphism of $(X, c_2)$.

Thus all cases lead to the conclusion that $f = g$, hence $B$ is unifying; and that $f \in C(X, c_2)$ such that $f(B) = B$ implies $f$ is an isomorphism of $(X, c_2)$. QED

### 4.7 Generalized normal product

In this section, we consider unifying sets for Cartesian products of digital images using the normal product adjacency.

We have the following generalization of the normal product adjacency [2] for the Cartesian product of two graphs.

**Definition 12.** [25, 8] Let $u, v \in \mathbb{N}$, $1 \leq u \leq v$. Let $(X_i, \kappa_i)$ be digital images, $i \in \{1, \ldots, v)$. Let $x_i, y_i \in X_i$, $x = (x_1, \ldots, x_v)$, $y = (y_1, \ldots, y_v)$. Then $x \leftrightarrow y$ in the *generalized normal product adjacency* $NP_u(\kappa_1, \ldots, \kappa_v)$ if for at least 1 and at most $u$ indices $i$, $x_i \leftrightarrow_{\kappa_i} y_i$ and for all other indices $j$, $x_j = y_j$.

**Remark 5.** For $u = v = 2$, the generalized normal product adjacency coincides with the *normal product adjacency*. Sabidussi [25] uses *strong* for what we call the generalized normal product adjacency; we prefer the latter name, as "strong" also appears in the literature for what we call the normal product adjacency.

The following generalizes a result in [16, 7].

**Theorem 20.** [8] *Let $f_i : (X_i, \kappa_i) \to (Y_i, \lambda_i)$, $1 \leq i \leq v$. Then the product map*

$$f = \Pi_{i=1}^{v} f_i : (\Pi_{i=1}^{v} X_i, NP_v(\kappa_1, \ldots, \kappa_v)) \to (\Pi_{i=1}^{v} Y_i, NP_v(\lambda_1, \ldots, \lambda_v))$$

*given by $f(x_1, \ldots, x_v) = (f_1(x_1), \ldots, f_v(x_v))$ is continuous if and only if each $f_i$ is continuous.*

**Theorem 21.** *Let $\emptyset \neq A_i \subset X_i$, where $(X_i, \kappa_i)$ is a digital image, $1 \leq i \leq v \in \mathbb{N}$. Let $A = \Pi_{i=1}^{v} A_i$, $X = \Pi_{i=1}^{v} X_i$. If $A$ is a unifying set for $(X, NP_v(\kappa_1, \ldots, \kappa_v))$ then for each $i$, $A_i$ is a unifying set for $(X_i, \kappa_i)$.*

*Proof.* Suppose $A$ is a unifying set for $(X, NP_v(\kappa_1, \ldots, \kappa_v))$. For all $i$, let $f_i, g_i \in C(X_i, \kappa_i)$ be such that $f_i(A_i) = g_i(A_i) = A_i$ and $f_i|_{A_i} = g_i|_{A_i}$. Then by Theorem 20, $f = f_1 \times \cdots \times f_v$ and $g = g_1 \times \cdots \times g_v$ are members of $C(X, NP_v(\kappa_1, \ldots, \kappa_v))$. Further, given $a = (a_1, \ldots, a_v) \in A$, there exist $a'_i \in A_i$ such that $f_i(a'_i) = g_i(a'_i) = a_i$, and therefore we have $f(A) = g(A) = A$ and $f|_A = g|_A$. Since $A$ is unifying, we have $f = g$, and therefore $f_i = g_i$ for all $i$. Thus $A_i$ is unifying. QED

# 5 Shy maps that are retractions

Shy maps in digital topology were introduced in [5] and studied further in [6, 17, 7, 8, 9]. A version of shy maps for topological spaces was introduced in [10].

**Definition 13.** [5] Let $f : (X, \kappa) \to (Y, \lambda)$ be a continuous function of digital images. We say $f$ is *shy* if

- for each $y \in f(X)$, $f^{-1}(y)$ is connected, and
- for every $y_0, y_1 \in f(X)$ such that $y_0$ and $y_1$ are adjacent, $f^{-1}(\{ y_0, y_1 \})$ is connected.

We say a point $p$ of a connected graph $G = (X, \kappa)$ is an *articulation point* of $G$ if $(X \setminus \{ p \}, \kappa)$ is not connected.

**Theorem 22.** *Let $(X, \kappa)$ be a connected digital image. Let $\emptyset \neq R \subset X$. Let*

$$A = \left\{ \begin{array}{rl} p \in R \mid & p \text{ is an articulation point of } K \cup R \\ & \text{for some } \kappa\text{-component } K \text{ of } X \setminus R \end{array} \right\}. \tag{5.1}$$

*Then there is a unique function $r : X \to R$ that is a shy $\kappa$-retraction.*

*Proof.* For $x \in X \setminus R$, let $p_x \in A$ be the articulation point for the union of $R$ and the $\kappa$-component $K_x$ of $X \setminus R$ containing $x$. Let $r : X \to X$ be the function

$$r(x) = \left\{ \begin{array}{ll} x & \text{if } x \in R; \\ p_x & \text{if } x \in X \setminus R. \end{array} \right.$$

Clearly, $r(X) = R$ and $r|_R = \mathrm{id}_R$. It is easily seen that $r^{-1}(p_x) \setminus \{ p_x \} = K_x$ is a union of $\kappa$-component of $X \setminus R$ separated by $p_x$, and $r^{-1}(y) = \{ y \}$ for $y \in R \setminus A$. It follows that $r \in C(X, \kappa)$ and $r$ is a retraction of $X$ to $R$. By 5.1, $r^{-1}(p_x) = \{p_x\} \cup K_x$ is connected. It follows easily that $r$ is shy.

Suppose $f \in C(X, \kappa)$ is a shy retraction of $X$ to $R$. If there exists $x_0 \in X \setminus R$ such that $x_1 = f(x_0) \neq p_{x_0}$, then $p_{x_0}$ separates the points $x_0, x_1 \in f^{-1}(x_1)$, contrary to the assumption that $f$ is shy. The uniqueness of $r$ as a shy retraction follows. QED

**Corollary 1.** *Let $(X, \kappa)$ be a digital image that is a tree. Let $(R, \kappa)$ be a nonempty subtree of $(X, \kappa)$. Then there is a unique function $r : X \to R$ that is a shy $\kappa$-retraction.*

*Proof.* It is trivial that if $R = X$, we can take $r = \mathrm{id}_X$. Otherwise, we take $A$ as in (5.1). The assertion follows from Theorem 22. QED

For topological spaces, we have the following.

**Definition 14.** [10] Let $X$ and $Y$ be topological spaces and let $f : X \to Y$. Then $f$ is *shy* if $f$ is continuous and for every path-connected $Y' \subset f(X)$, $f^{-1}(Y')$ is a path-connected subset of $X$. □

By using an argument similar to the proof of Theorem 22, we get the following.

**Theorem 23.** *Let $X$ be a connected topological space. Let $\emptyset \neq A \subset R \subset X$ such that each $p \in A$ separates $R$ and a component of $X \setminus R$. Then there is a unique continuous function $r : X \to R$ that is a shy retraction.*

# 6 Approximate fixed points

Suppose $A \subset X$ and $A$ is a $\kappa$-freezing set for $X$. By definition, if $f \in C(X, \kappa)$ and $A \subset \mathrm{Fix}(f)$, then $f = \mathrm{id}_X$, i.e., $X = \mathrm{Fix}(f)$. If we weaken the hypothesis so that instead of assuming $A \subset \mathrm{Fix}(f)$ we assume every point of $A$ is an approximate fixed point of $f$, might we reach the weaker conclusion that every point of $X$ is an approximate fixed point of $f$? The answer is not generally affirmative; we give a counterexample below. We also examine basic examples for which an affirmative answer is shown.

## 6.1 Wedge of cycles

In this section, we show that a wedge of cycles $X$ can support a freezing set $A$ and a continuous self-map $f$ such that every point of $A$ is an approximate fixed point of $f$, but not every point of $X$ is an approximate fixed point of $f$.

**Theorem 24.** [11] *Let $C_m$ and $C_n$ be cycles, with $m > 4$, $n > 4$, where $C_m = \{ x_i \}_{i=0}^{m-1}$, $C_n = \{ x'_i \}_{i=0}^{n-1}$, with the members of $C_m$ and $C_n$ indexed circularly. Let $x_0 = x'_0$ be the wedge point of $X = C_m \vee C_n$. Let $x_i, x_j \in C_m$ and $x'_k, x'_p \in C_n$ be such that $C_m$ is the union of unique shorter paths determined by $x_i, x_j, x_0$ and $C_n$ is the union of unique shorter paths determined by $x'_k, x'_p, x'_0$. Then $A = \{ x_i, x_j, x'_k, x'_p \}$ is a freezing set for $X$.*

**Example 2.** Let $X = C_6 \vee C_m$, where $C_6 = \{ x_i \}_{i=0}^{5}$ and $C_n = \{ x'_i \}_{i=0}^{n-1}$ are $c_2$-simple closed curves in $\mathbb{Z}^2$, with the members of $C_6$ and $C_n$ indexed circularly.

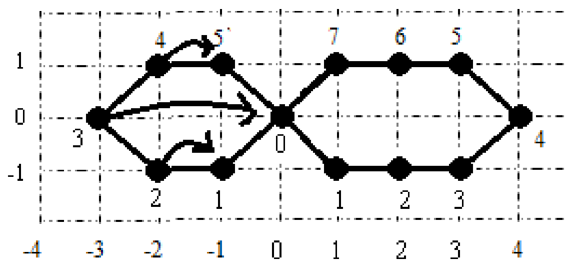


Figure 7. [12] The map $f$ of Example 2. Points are labeled by their indices as in the Example. The cycle with points $p = (x, y)$ for $x \leq 0$ represents $C_6$, for which $\{ x_0, x_2, x_4 \}$ is a $c_2$-freezing set; the cycle with points $p = (x, y)$ for $x \geq 0$ represents $C_m$ (here, $m = 8$, and $\{ x'_0, x'_3, x'_6 \}$ is a $c_2$-freezing set for $C_8$, so $A = \{ x_2, x_4, x'_3, x'_6 \}$ is a $c_2$-freezing set for $C_6 \vee C_8$). Arrows connect $p$ and $f(p)$ for points $p \notin \text{Fix}(f)$. Each point of $A$ is a $c_2$-approximate fixed point of $f$.

By Theorems 13 and 24, if $k$ and $p$ are chosen so that $\{ x'_0, x'_k, x'_p \}$ is a freezing set for $C_n$, then we can take $A = \{ x_2, x_4, x'_k, x'_p \}$ to be a freezing set for $(X, c_2)$. Now take $f : X \to X$ to be the function

$$f(x) = \begin{cases} x_0 & \text{if } x = x_3; \\ x_1 & \text{if } x = x_2; \\ x_5 & \text{if } x = x_4; \\ x & \text{otherwise.} \end{cases}$$

See Figure 7. One sees easily that $f \in C(X, c_2)$, that every member of $A$ is a $c_2$-approximate fixed point of $f$, but $x_3$ is not a $c_2$-approximate fixed point of $f$.

### 6.2 Disks in $(\mathbb{Z}^2, c_1)$

**Lemma 2.** *Let $q_0, q_1 \in X \subset \mathbb{Z}^2$. Suppose there is a horizontal or vertical $c_1$-path $P$ in $X$ from $q_0$ to $q_1$. Let $f : P \to X$ be $c_1$-continuous, such that $q_0$ and $q_1$ are $c_1$-approximate fixed points of $f$. Then every member of $P$ is a $c_1$-approximate fixed point of $f$.*

*Proof.* Without loss of generality, $P$ is horizontal, $q_0 = (0, 0)$, and $q_1 = (n, 0)$ for some $n \in \mathbb{N}$. Suppose there exists $q = (x, 0) \in P$ such that $q$ is not a $c_1$-approximate fixed point of $f$. Then $|x - p_1(f(q))| > 1$; or $|p_2(f(q))| > 1$; or $|x - p_1(f(q))| = 1$ and $|p_2(f(q))| = 1$.

If $|x - p_1(f(q))| > 1$ then either $p_1(f(q)) > x + 1$ or $p_1(f(q)) < x - 1$.

- Suppose $p_1(f(q)) > x+1$. Then by Lemma 1 we would have $p_1(f(q_0)) > 1$, contrary to the assumption that $q_0 = (0,0)$ is an approximate fixed point.
- If $p_1(f(q)) < x - 1$, then by Lemma 1 we would have $p_1(f(q_1)) < n - 1$, contrary to the assumption that $q_1 = (n, 0)$ is an approximate fixed point.

Suppose $|p_2(f(q))| > 1$. Without loss of generality, $p_2(f(q)) > 1$, as the case $p_2(f(q)) < 1$ can be handled similarly. Since $c_1$-adjacent points differ in only one coordinate and the $q_i$ as approximate fixed points implies $|p_2(f(q_i)| \leq 1$, $i \in \{0, 1\}$, there are at least 4 indices $j$ for which $p_2(f(x_j)) \neq p_2(f(x_{j+1}))$ and therefore at most $n - 4$ indices $j$ for which $p_1(f(x_j)) \neq p_1(f(x_{j+1}))$. This is a contradiction, since $x_0$ and $x_1$ being approximate fixed points implies $p_1(f(x_0)) \leq 1$ and $p_1(f(x_1) \geq n - 1$, so at least $n - 2$ indices $j$ would satisfy $p_1(f(x_j)) \neq p_1(f(x_{j+1}))$.

Suppose $|x - p_1(f(q))| = 1$ and $|p_2(f(q))| = 1$. Without loss of generality, $p_1(f(q)) = x + 1$ and $p_2(f(q)) = 1$. By the $c_1$-continuity of $f$ and Lemma 1 it follows that $p_1(f(q_0)) \geq 1$. Since $q_0$ is a $c_1$-approximate fixed point of $f$, $f(q_0) = (1, 0)$. Thus, $f(P)$ has length at least $x+1$, contrary to $P$ having length $x$.

Thus every case yields a contradiction brought about by assuming there is a point of $P$ that is not an approximate fixed point of $f$. The assertion follows. QED

**Theorem 25.** *Let $V_i \subset X \subset \mathbb{Z}^2$, $i \in \{1, \ldots, n\}$ where each $V_i$ is a thick convex disk. Let $X' = \bigcup_{i=1}^n V_i$. Let $C_i$ be a bounding curve of $V_i$. Let $A_{1,i}$ be the set of endpoints of maximal axis-parallel segments of $C_i$. Let $A_{2,i}$ be the union of maximal slanted segments of $C_i$.*

*(1) $A = (X \setminus X') \cup \bigcup_{i=1}^n (A_{1,i} \cup A_{2,i})$ is a freezing set for $(X, c_1)$.*

*(2) Suppose $f \in C(X, c_1)$ such that every point of $A$ is a $c_1$-approximate fixed point of $f$. Then every point of $X$ is a $c_1$-approximate fixed point of $f$.*

*Proof.* Assertion 1) is Theorem 10. To prove assertion 2), we argue as follows.

Let $S$ be a maximal digital segment of a bounding curve $C_i$ for $V_i$. If $S$ is horizontal or vertical, then by Lemma 2, every point of $S$ is a $c_1$-approximate fixed point of $f$. If $S$ is slanted, then $S \subset A$, so every point of $S$ is a $c_1$-approximate fixed point of $f$. Thus each point of $C_i$, is a $c_1$-approximate fixed point of $f$.

For $x \in X \setminus A$, there is a horizontal segment $P$ containing $x$ such that the endpoints of $P$ belong to $\bigcup_{i=1}^n C_i$, and therefore are approximate fixed points of $f$. By Lemma 2, every point of $P$ is a $c_1$-approximate fixed point of $f$. Thus, every point of $X$ is a $c_1$-approximate fixed point of $f$. QED

**Remark 6.** Theorems 10 and 25 simplify when $X' = X$, in which case $A = \bigcup_{i=1}^{n}(A_{1,i} \cup A_{2,i})$. They might be applied in this case when $i \neq j$ implies $V_i \cap V_j$ is empty, a single point, or a common edge of $V_i$ and $V_j$.

### 6.3 Disks in $(\mathbb{Z}^2, c_2)$

We show in this section that disks in $(\mathbb{Z}^2, c_2)$ yield results similar to those shown in section 6.2 for the $c_1$ adjacency.

**Lemma 3.** *Let $q_0, q_1 \in X \subset \mathbb{Z}^2$. Suppose there is a slanted $c_2$-path $P$ in $X$ from $q_0$ to $q_1$. Let $f : P \to X$ be $c_2$-continuous, such that $q_0$ and $q_1$ are $c_2$-approximate fixed points of $f$. Then every member of $P$ is a $c_2$-approximate fixed point of $f$.*

*Proof.* Without loss of generality, the slope of $P$ is 1. Without loss of generality, $q_0 = (0,0)$ and $q_1 = (n,n)$ for $n = \mathit{length}(P)$. Suppose there exists $p \in P$ that is not a $c_2$-approximate fixed point of $f$. Then $|p_1(f(p)) - p_1(p)| > 1$ or $|p_2(f(p)) - p_2(p)| > 1$.

- If $|p_1(f(p)) - p_1(p)| > 1$ then either $p_1(f(p)) - p_1(p) > 1$ or $p_1(p) - p_1(f(p)) > 1$.
  - If $p_1(f(p)) - p_1(p) > 1$ then by Lemma 1, $1 < p_1(f(q_0)) - p_1(q_0) = p_1(f(q_0))$, contrary to the assumption that $q_0$ is an approximate fixed point.
  - If $p_1(p) - p_1(f(p)) > 1$, then by Lemma 1, $1 < p_1(q_1) - p_1(f(q_1)) = n - p_1(f(q_1))$, or $p_1(f(q_1)) < n - 1$, contrary to the assumption that $q_1$ is an approximate fixed point.
- If $|p_2(f(p)) - p_2(p)| > 1$ then, similarly, we obtain contradictions.

Since all cases yield contradictions, the hypothesis of a $p \in P$ that is not a $c_2$-approximate fixed point of $f$ must be false. This completes the proof. QED

The following is a dual to Theorem 25.

**Theorem 26.** *Let $V_i \subset X \subset \mathbb{Z}^2$, $i \in \{1, \ldots, n\}$ where each $V_i$ is a thick convex disk. Let $X' = \bigcup_{i=1}^{n} V_i$. Let $C_i$ be a bounding curve of $V_i$. Let $B_{1,i}$ be the union of maximal horizontal and maximal vertical segments of $C_i$. Let $B_{2,i}$ be the set of endpoints of maximal slanted segments of $C_i$.*

*(1) $B = (X \setminus X') \cup \bigcup_{i=1}^{n}(B_{1,i} \cup B_{2,i})$ is a freezing set for $(X, c_2)$.*

*(2) Suppose $f \in C(X, c_2)$ such that every point of $B$ is a $c_2$-approximate fixed point of $f$. Then every point of $X$ is a $c_2$-approximate fixed point of $f$.*

*Proof.* Assertion 1) is Theorem 11. To prove assertion 2), we argue as follows.

By Lemma 3, every slanted segment of $C_i$ is made up entirely of $c_2$-approximate fixed points of $f$. Since $B$ by hypothesis consists of $c_2$-approximate fixed points of $f$, it follows that $C_i$ is made up entirely of $c_2$-approximate fixed points of $f$.

Lemma 3 lets us conclude that if $x \in X$ such that $x$ lies on a slanted segment $P$ that connects two points of $B$, then $x$ is a $c_2$-approximate fixed point of $f$.

This leaves us to consider points $p = (x_0, y_0) \in X$ such that $p$ does not lie either on an axis-parallel segment of $B$ or on a slanted segment $P$ that connects two points of $B$. Such a point must be in the interior of $X$ and therefore is $c_2$-adjacent to its 4 $c_1$-neighbors $q_1 = (x_0-1, y_0)$, $q_2 = (x_0+1, y_0)$, $q_3 = (x_0, y_0-1)$, and $q_4 = (x_0, y_0 + 1)$, each of which lies on a slanted segment joining members of $S$ (see Figure 8). Therefore, by Lemma 3, $q_1$, $q_2$, $q_3$, and $q_4$ are approximate fixed points of $f$.

Suppose $p$ is not a $c_2$-approximate fixed point of $f$. Then either

$$|p_1(f(p)) - x_0| > 1 \text{ or } |p_2(f(p)) - y_0| > 1.$$

- Suppose $|p_1(f(p)) - x_0| > 1$. Then either

$$p_1(f(p)) - x_0 > 1 \text{ or } x_0 - p_1(f(p)) > 1.$$

  - Suppose $p_1(f(p))-x_0 > 1$. Then by the continuity of $f$ and Lemma 1, $p_1(q_1) - p_1(f(q_1)) > 1$, contrary to $q_1$ being an approximate fixed point of $f$.
  - Suppose $x_0-p_1(f(p)) > 1$. Then by the continuity of $f$ and Lemma 1, $p_1(q_2) - p_1(f(q_2)) > 1$, contrary to $q_2$ being an approximate fixed point of $f$.

- Similarly, we obtain a contradiction if $|p_2(f(p)) - y_0| > 1$.

Since all cases yield a contradiction when we assume $p$ is not a $c_2$-approximate fixed point of $f$, this hypothesis must be incorrect. The assertion follows. QED

**Remark 7.** Theorems 11 and 26 simplify when $X' = X$, in which case $B = \bigcup_{i=1}^{n}(B_{1,i} \cup B_{2,i})$. They might be applied in this case when $i \neq j$ implies $V_i \cap V_j$ is empty, a single point, or a common edge of $V_i$ and $V_j$.

## 6.4 Trees

In this section, we use a result about freezing sets for trees to obtain a result about approximate fixed points for trees.

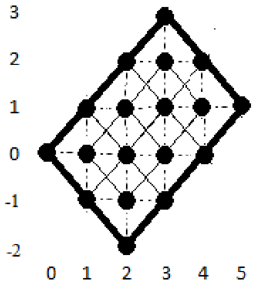


Figure 8. The point (1,0) in the digital image shown above does not lie on a slanted segment that joins 2 points of the boundary curve shown darkly.

**Theorem 27.** [11] *Let* $(X, \kappa)$ *be a digital image such that the graph* $G = (X, \kappa)$ *is a finite tree with* $\#X > 1$. *Let* $D_1$ *be the set of vertices of* $G$ *that have degree 1. Then* $D_1$ *is a minimal freezing set for* $G$.

**Lemma 4.** *Let* $(X, \kappa)$ *be a digital image such that the graph* $G = (X, \kappa)$ *is a finite tree. Let* $f \in C(X, \kappa)$. *Let* $a, b \in X$ *be such that* $a$ *and* $b$ *are* $\kappa$*-approximate fixed points of* $f$. *Let* $P$ *be the unique shortest path in* $G$ *from* $a$ *to* $b$. *Then* $f(P) \subset P \cup N(X, a, \kappa) \cup N(X, b, \kappa)$ *and every point of* $P$ *is a* $\kappa$*-approximate fixed point of* $f$.

*Proof.* Whithout loss of generality, $a \neq b$. Let $P = \{ x_i \}_{i=0}^{n}$ such that $x_0 = a$, $x_n = b$, and $x_i \leftrightarrow_\kappa x_j$ if and only if $|i - j| = 1$.

- Suppose $f(a) = a$. Let us show that

$$f(b) \in \{ x_{n-1}, b \} \subset P. \tag{6.1}$$

  We know that $f(b) \in N^*(X, b, \kappa)$. If (6.1) is false, then $f(P) = P \cup \{ f(b) \}$ is the unique shortest path in $G$ from $a = f(a)$ to $f(b)$. But $P \cup \{ f(b) \}$ has length $n+1$, and $\#P = n+1$ implies $length(f(P)) \leq n$, so we have a contradiction brought about by negating (6.1). Thus (6.1) is established.

  It follows that $f(P) \subset P$. Now suppose for some $k$ that $x_k$ is not an approximate fixed point of $f$. Then $f(x_k) = x_m$ for some $m$ such that $|k - m| > 1$. Without loss of generality, $m - k > 1$. Then by continuity and since $G$ is acyclic, $f(x_k)$ must "pull" [21] $f(a) = f(x_0)$ so that $f(a) = x_t$ for some $t > 1$, contrary to $a$ being an approximate fixed point of $f$. The contradiction establishes that each point of $P$ must be an approximate fixed point of $f$, and $f(P) \subset P$.

- Suppose $f(a) \not\in P$. Recall we are assuming $f(b) \in N^*(X,b,\kappa)$, so $f(b) \in \{x_{n-1}, b\}$ or $f(b) \not\in P$. We claim $f(b) = x_{n-1}$. For otherwise, $f(P) = \{f(a) \neq x_0, a = x_0, x_1, \ldots, x_n = b, f(b)\}$ where $f(b)$ may be equal to $b$, so $\#f(P) \in \{n+2, n+3\}$ while $\#P = n+1$, a contradiction. Therefore, $f(b) = x_{n-1}$. By the acyclicity of $G$, it follows that $f(P) = \{f(a)\} \cup \{x_i\}_{i=0}^m$, where $m \in \{n-1, n\}$. As in the case $f(a) = a$, it follows that every point of $P$ is an approximate fixed point of $f$, and $f(P) \subset P \cup N(X,a,\kappa)$.

- Suppose $f(a) \in P \setminus \{a\}$. Since $a$ is an approximate fixed point of $f$, it follows that $f(a) = x_1$. Since $f(b) \in N(X,b,\kappa)$, it follows as in the previous cases that every point of $P$ is an approximate fixed point of $f$, and $f(P) \subset P \cup N(X,b,\kappa)$.

This establishes the assertion. QED

**Theorem 28.** *Let $(X,\kappa)$ be a digital image such that the graph $G = (X,\kappa)$ is a finite tree with $\#X > 1$. Let $D_1$ be the set of vertices of $G$ that have degree 1. Then $D_1$ is a minimal freezing set for $(X,\kappa)$, and given a freezing set $A$ for $G$, we have $D_1 \subset A$.*

*Proof.* That $D_1$ is a freezing set comes from Theorem 27. The assertion is trivial for $\#X \in \{1,2\}$, so let us assume $\#X \geq 3$. Then $D_1 \neq \emptyset \neq X \setminus D_1$. Let $a \in D_1$ and let $x_0 \in D_1 \setminus \{a\}$. Consider $x_0$ as the root vertex of $X$. Then the function $f : X \to X$ given by

$$f(x) = \begin{cases} x & \text{for } x \neq a; \\ parent(a) & \text{for } x = a, \end{cases}$$

is easily seen to be a member of $C(X,\kappa)$. Further, if $A$ is any freezing set for $(X,\kappa)$, then $f|_{A\setminus\{a\}} = \mathrm{id}_{A\setminus\{a\}}$, so $A \setminus \{a\}$ is not a freezing set. Thus, $D_1$ is a minimal freezing set that is contained in every freezing set for $(X,\kappa)$. QED

**Theorem 29.** *Let $(X,\kappa)$ be a digital image such that the graph $G = (X,\kappa)$ is a finite tree. Let $A$ be a freezing set for $G$. Suppose $f \in C(X,\kappa)$ is such that for each $a \in A$, $a$ is an approximate fixed point of $f$. Then for all $x \in X$, $x$ is an approximate fixed point of $f$.*

*Proof.* The assertion is trivial for $\#X \in \{1,2\}$, so assume $\#X \geq 3$. Let $D_1$ be the set of vertices of $G$ that have degree 1. By Theorem 28, $D_1$ is a freezing set contained in $A$. Therefore, there is no loss of generality in assuming $A = D_1$.

Let $f \in C(X,\kappa)$ such that for each $d \in D_1$, $d$ is an approximate fixed point of $f$. We can choose $x_0 \in D_1$ as a root of $X$. Since $x \in X$ implies $x$ is on the unique shortest path in $G$ from $x_0$ to some $d \in D_1$, it follows from Lemma 4 that $x$ is an approximate fixed point of $f$. $\boxed{QED}$

### 6.5 Cycles

**Theorem 30.** *Let $(C_n, \kappa)$ be a digital cycle of $n$ distinct points, $n \in \mathbb{N}$, $n > 4$, with $C_n = \{ x_i \}_{i=0}^{n-1}$, such that $x_i \leftrightarrow_\kappa x_j$ if and only if $j = (i \pm 1) \bmod n$. Let $A = \{ x_u, x_v, x_w \}$ be a set of distinct members of $C_n$ such that $C_n$ is a union of unique shorter paths determined by these points. Let $f \in C(C_n, \kappa)$ be such that every member of $A$ is an approximate fixed point of $f$. Then every member of $C_n$ is an approximate fixed point of $f$, and $f$ is an isomorphism.*

*Proof.* Note by Theorem 13, $A$ is a minimal freezing set for $(C_n, \kappa)$.

First, we show that f must be a surjection. Without loss of generality, $0 \le u < v < w < n$. Suppose $B$ is the unique shorter path in $C_n$ from $x_u$ to $x_v$. Since we must have $\#f(B) \le \#B$ and $x_u$ and $x_v$ are approximate fixed points, we must have $f(x_u) \in \{ x_{u-1}, x_u, x_{u+1} \}$ and $f(x_v) \in \{ x_{v-1}, x_v, x_{v+1} \}$ (indices reduces $\bmod\, n$).

Suppose $f(x_u) = x_u$. We must have

$$\#f(B) \le \#B = v - u + 1 \le n/2,$$

so $f(x_v) \in \{ x_{v-1}, x_v \}$. If $f(x_v) = x_{v-1}$, then we must have $f(x_w) = x_{w-1}$, hence (proceeding with increasing indices, $\bmod\ n$,) $f(x_{u-1}) = x_{u-2}$, so $f$ would be discontinuous at the adjacent pair $x_{u-1}$ and $x_u$. Thus we would have $f(x_v) = x_v$ and $f(x_w) = x_w$. Thus $f|_A = \mathrm{id}_A$. Since $A$ is freezing, it follows that $f = \mathrm{id}_X$.

If $f(x_u) = x_{u-1}$ or $f(x_u) = x_{u+1}$, we can apply a rotation $r(x_i) = x_{(i-1) \bmod n}$ (respectively, $r(x_i) = x_{(i+1) \bmod n}$), which is an isomorphism. Then by the above, $r \circ f = \mathrm{id}_X$ is an isomorphism, so

$$f = r^{-1} \circ r \circ f = r^{-1} \circ \mathrm{id}_X = r^{-1}$$

is an isomorphism, with each member of $A$ an approximate fixed point.

Thus, in all cases, each member of $A$ is an approximate fixed point of $f$, which must be an isomorphism. $\boxed{QED}$

## 7 Further remarks

When a member of $C(X, \kappa)$ has restricted behavior on a subset $A$ of $X$, the restriction may have a powerful effect on the behavior of $f|_{X \setminus A}$. We have examined instances of this phenomenon with respect to freezing and cold sets, retractions, and shy maps, on a variety of basic digital images.

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