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On the Radio k-chromatic Number of Paths

Niranjan P. K.
Department of Mathematics, RV College of Engineering, Mysuru Road, Bengaluru-560059, India.
niranjanpk704@rvce.edu.in

Srinivasa Rao Kola
Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka, Surathkal, Mangaluru-575025, India.
srinukola@nitk.edu.in

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Abstract. A radio k-coloring of a graph G is an assignment f of positive integers (colors) to the vertices of G such that for any two vertices u and v of G, the difference between their colors is at least $1 + k - d(u, v)$. The span $rc_k(f)$ of f is $\max\{f(v) : v \in V(G)\}$. The radio k-chromatic number $rc_k(G)$ of G is $min\{rc_k(f) : f \ is \ a \ radio \ k\text{-}coloring \ of \ G\}$. In this paper, in an attempt to prove a conjecture on the radio k-chromatic number of path, we determine the radio k-chromatic number of paths P_n for $k + 5 \leq n \leq \frac{7k-1}{2}$ if k is odd and $k + 4 \leq n \leq \frac{5k+4}{2}$ if k is even.

Keywords: radio k-coloring, radio k-chromatic number , radio coloring , radio number

MSC 2020 classification: 05C15, 05C78, 05C12

1 Introduction

All graphs considered in this paper are simple connected graphs. We use standard graph theory terminology according to [10]. The channel assignment problem is the problem of assigning frequencies to transmitters in some optimal manner. Chartrand et al. [1] have introduced radio k-coloring of graphs as a variation of channel assignment problem. A radio k-coloring of a graph G is an assignment f of positive integers to the vertices of G such that $|f(u) - f(v)| \geq 1 + k - d(u, v)$ for every pair u and v of vertices in G. The span of f is the largest integer assigned by f and is denoted by $rc_k(f)$. The radio k-chromatic number $rc_k(G)$ of G is the minimum among the spans of all radio k-colorings of G. A radio k-coloring having span $rc_k(G)$ is called a minimal radio k-coloring of G. If k is the diameter d of G, then f is called a radio coloring of G and the radio d-chromatic number is called the radio number of G, denoted by $rn(G)$. A radio $(d-1)$-coloring and the corresponding chromatic number are said to be an antipodal coloring and the antipodal number $ac(G)$ of G, respectively. A radio

$(d-2)$-coloring and the radio $(d-2)$-chromatic number are referred as a nearly antipodal coloring and the nearly antipodal number $ac'(G)$ of G, respectively.

For any path P_{k+1} $(k \geq 1)$, Liu and Zhu [9] have determined the radio number as $\frac{k^2+3}{2}$ if k is odd and $\frac{k^2+6}{2}$ if k is even. Khennoufa and Togni [5] have shown that $ac(P_{k+2})$ is $\frac{k^2+5}{2}$ for an odd $k > 2$ and $\frac{k^2+6}{2}$ for an even $k > 3$. Kola and Panigrahi [6] have determined the nearly antipodal number of P_{k+3} as $\frac{k^2+7}{2}$ for an odd $k > 4$ and $\frac{k^2+8}{2}$ for an even $k > 5$. Also, in [7], they have found the radio k-chromatic number of P_{k+4} as $\frac{k^2+9}{2}$ for an odd $k > 6$ and given an upper bound for the same as $\frac{k^2+10}{2}$ for an even $k > 7$. Even though radio k-coloring of a graph G is defined for $k \leq diam(G)$, it is studied for $k > diam(G)$ as it is useful in determining the radio k-chromatic number of larger graphs. For any $k \geq n$, Kchikech et al. [4] have proved that $rc_k(P_n) = (n-1)k - \frac{1}{2}n(n-2) + 1$ if n is even and $rc_k(P_n) = (n-1)k - \frac{1}{2}(n-1)^2 + 2$ if n is odd.

For any path P_n and an integer k, $0 < k < n$, Chartrand et al. [2] have given an upper bound for $rc_k(P_n)$ as below.

Theorem 1. [2] *For* $0 < k < n-1$,

$$rc_k(P_n) \leq \begin{cases} \frac{k^2+2k+1}{2} & if\ k\ is\ odd, \\ \frac{k^2+2k+2}{2} & if\ k\ is\ even. \end{cases}$$

Kchikech et al. [4] have proposed the following conjecture.

Conjecture 1. [4] *For* $k \geq 5$,

$$\lim_{n\to\infty} rc_k(P_n) = \begin{cases} \frac{k^2+2k+1}{2} & \text{if } k \text{ is odd}, \\ \frac{k^2+2k+2}{2} & \text{if } k \text{ is even}. \end{cases}$$

In an attempt to prove Conjecture 1, Kola and Panigrahi [8] have given upper bounds of $rc_k(P_n)$ for different intervals of n as below.

Theorem 2. [8] *For* $k \geq 7$ *and* $4 \leq s \leq \lfloor \frac{k+1}{2} \rfloor$

$$rc_k(P_{k+s}) \leq \begin{cases} \frac{k^2+2s+1}{2} & if\ k\ is\ odd, \\ \frac{k^2+2s+2}{2} & if\ k\ is\ even. \end{cases}$$

Theorem 3. [8] *For any even* $k \geq 6$,

$$rc_k(P_n) \leq \begin{cases} \frac{k^2+k+2}{2} & if\ n = \frac{3k+2}{2}, \\ \frac{k^2+k+2s+4}{2} & if\ \frac{(3+2s)k+2s+4}{2} \leq n \leq \frac{(5+2s)k+2s+4}{2}, \end{cases}$$

where $s = 0, 1, 2, \ldots, \frac{k-4}{2}$.

Theorem 4. [8] *For any odd $k \geq 5$,*

$$rc_k(P_n) \leq \begin{cases} \frac{k^2+k+2}{2} & if\ \frac{3k+1}{2} < n \leq \frac{5k-1}{2}, \\ \frac{k^2+k+2s+4}{2} & if\ \frac{(5+2s)k+1}{2} \leq n \leq \frac{(7+2s)k-1}{2},\ s = 0, 1, 2, \ldots, \frac{k-5}{2}. \end{cases}$$

Further, Kola and Panigrahi [8] have re-conjectured Conjecture 1 as below.

Conjecture 2. [8] *For any integer $k \geq 5$ and $n \geq n_0$, $rc_k(P_n) = n_0$, where $n_0 = \frac{k^2+2k+2}{2}$ if k is even and $n_0 = \frac{k^2+2k+1}{2}$ if k is odd.*

In this article, we prove that the upper bounds given in Theorem 2 are exact. Also, we show that the bounds in Theorem 3 when $\frac{3k+2}{2} \leq n \leq \frac{5k+4}{2}$ and the bounds in Theorem 4 when $\frac{3k+1}{2} \leq n \leq \frac{7k-1}{2}$, are exact.

2 Preliminaries

To obtain lower bounds for the radio k-chromatic number of the paths, we use the lower bound technique for radio k-coloring given by Das et al. [3]. For a subset S of the vertex set of a graph G, let $N(S)$ be the set of all vertices of G adjacent to at least one vertex of S.

Theorem 5. [3] *If f is a radio k-coloring of a graph G, then*

$$rc_k(f) \geq |D_k| - 2p + 2\sum_{i=0}^{p-1} |L_i|(p-i) + \alpha + \beta, \tag{2.1}$$

where D_k and L_i's are defined as follows. If $k = 2p+1$, then $L_0 = V(C)$, where C is a maximal clique in G. If $k = 2p$, then $L_0 = \{v\}$, where v is a vertex of G. Recursively define $L_{i+1} = N(L_i) \backslash (L_0 \cup L_1 \cup \cdots \cup L_i)$ for $i = 0, 1, 2, \ldots, p-1$. Let $D_k = L_0 \cup L_1 \cup \cdots \cup L_p$. The minimum and the maximum colored vertices among the vertices of D_k are in L_α and L_β, respectively.

From the proof of Theorem 5 in [3], it is easy to see that the right hand side of (2.1) is actually counts the number of colors between minimum and maximum colors (both inclusive) among the vertices of D_k and hence we have the following theorem.

Theorem 6. *Let G be a graph, and L_i and D_k be as in Theorem 5. If f is a radio k-coloring of G, and $\lambda_{min} \in L_\alpha$ and $\lambda_{max} \in L_\beta$ are the minimum and the maximum colors respectively, assigned by f to the vertices of D_k, then*

$$\lambda_{max} - \lambda_{min} + 1 \geq |D_k| - 2p + 2\sum_{i=0}^{p-1} |L_i|(p-i) + \alpha + \beta.$$

For a path P_n, if k is odd, we choose L_0 as two adjacent vertices which are at distance at least $\frac{k-1}{2}$ from the pendant vertices of P_n, and if k is even, we choose L_0 as one vertex which is at distance at least $\frac{k}{2}$ from the pendant vertices of P_n. For $k = 2p + 1$, we get $|L_i| = 2$ for all $i = 0, 1, 2, \dots, p$, and for $k = 2p$, we get $|L_0| = 1$ and $|L_i| = 2$ for all $i = 1, 2, 3, \dots, p$. In any case, D_k induces P_{k+1} for which L_0 is the center. Then Theorem 6 gives the theorem below.

Theorem 7. *If f is a radio k-coloring of P_n, then*

$$rc_k(f) \geq \lambda_{max} \geq \begin{cases} \frac{k^2+3}{2} + \alpha + \beta + \lambda_{min} - 1 & if\ k\ is\ odd, \\ \frac{k^2+2}{2} + \alpha + \beta + \lambda_{min} - 1 & if\ k\ is\ even. \end{cases}$$

3 Results

In this section, we determine the radio k-chromatic number of paths P_n where $k + 4 \leq n \leq \frac{5k+4}{2}$ if k is even and $k + 5 \leq n \leq \frac{7k-1}{2}$ if k is odd. We use Theorem 6 and Theorem 7 to get the lower bounds match those with the upper bounds in Theorems 2, 3 and 4. We use the following lemmas in the sequel.

Lemma 1. *If f is a radio k-coloring of a graph G with span λ, then there exists a radio k-coloring g of G with span λ such that the vertices of G receiving 1 and λ by f receive λ and 1, respectively by g.*

Proof. The radio k-coloring g of G defined as $g(v) = \lambda + 1 - f(v)$ for every vertex v of G is one of such colorings. QED

Lemma 2. *If n_1 and n_2 are positive integers such that $n_1 < n_2$, then $rc_k(P_{n_1}) \leq rc_k(P_{n_2})$.*

Theorem 8. *If $k \geq 7$ and $4 \leq s \leq \lfloor \frac{k+1}{2} \rfloor$, then*

$$rc_k(P_{k+s}) = \begin{cases} \frac{k^2+2s+1}{2} & \text{if } k \text{ is odd,} \\ \frac{k^2+2s+2}{2} & \text{if } k \text{ is even.} \end{cases}$$

Proof. Let f be a minimal radio k-coloring of path $P_{k+s} : v_1 v_2 v_3 \dots v_{k+s}$ with span λ. Let i and j be the least positive integers such that $f(v_i) = 1$ and $f(v_j) = \lambda$. Without loss of generality, we assume that $i < j$.

Case I: $k = 2p + 1$

To prove the result, depending on the positions of the maximum and the minimum colored vertices, we choose a P_{k+1} subpath (L_0 is the center of it) of P_n such that $\alpha + \beta \geq s - 1$. If $\alpha + \beta \geq s - 1$, we get the required lower bound and if $\alpha + \beta > s - 1$, we get a contradiction to Theorem 2 (using Theorem 7). If $i \leq s$, then by considering the path $v_i v_{i+1} v_{i+2} \ldots v_{i+p} v_{i+p+1} \ldots v_{i+k}$, we get $\alpha = \frac{k-1}{2}$. Now, by using Theorem 7, we get $rc_k(f) \geq \frac{k^2+k+2}{2}$ which is a contradiction to Theorem 2 if $s \neq \frac{k+1}{2}$. If $s < i < p+1$, then by considering the path $v_s v_{s+1} v_{s+2} \ldots v_{s+p} v_{s+p+1} \ldots v_{s+k}$, we get $\alpha \geq s$. If $j \geq k+1$, then by considering the path $v_{j-k} v_{j-k+1} v_{j-k+2} \ldots v_{j-p-1} v_{j-p} \ldots v_j$, we get $\beta \geq \frac{k-1}{2}$ which is strictly greater than $s-1$ if $s \neq \frac{k+1}{2}$. If $p+s < j < k+1$, then by considering the path $v_1 v_2 v_3 \ldots v_{p+1} v_{p+2} \ldots v_{k+1}$, we get $\beta \geq s-1$. Suppose $p+1 \leq i < j \leq p+s$.

Subcase (i): $s = 2l$

If $i \geq p+l+1$, then by choosing the path $v_1 v_2 v_3 \ldots v_{p+1} v_{p+2} \ldots v_{k+1}$, we get $\alpha \geq l-1$ and $\beta \geq l$. By Theorem 7, we get $rc_k(f) \geq \frac{k^2+3}{2} + l - 1 + l = \frac{k^2+2s+1}{2}$. If $j \leq p+l+1$, then by choosing $v_s v_{s+1} v_{s+2} \ldots v_{s+p} v_{s+p+1} \ldots v_{k+s}$ subpath, we get $\beta \geq l-1$ and $\alpha \geq l$. So, $\alpha + \beta \geq s-1$. Suppose $p+1 \leq i < p+l+1 < j \leq p+s$. Let $i = p+l+1-l_1$ and $j = p+l+1+l_2$ where $1 \leq l_1 \leq l$ and $1 \leq l_2 \leq l-1$. Suppose that $l_1 < l_2$. Then by considering the path $v_1 v_2 v_3 \ldots v_{p+1} v_{p+2} \ldots v_{k+1}$, we get $\alpha = (p+l+1-l_1) - (p+2) = l - l_1 - 1$ and $\beta = (p+l+1+l_2) - (p+2) = l + l_2 - 1$. Now, by Theorem 7, $rc_k(f) \geq \frac{k^2+3}{2} + l - l_1 - 1 + l + l_2 - 1 = \frac{k^2+3}{2} + 2l + (l_2 - l_1) - 2 \geq \frac{k^2+2s+1}{2}$. Suppose that $l_1 > l_2$. Then by considering the path $v_s v_{s+1} v_{s+2} \ldots v_{s+p} v_{s+p+1} \ldots v_{k+s}$, we get $\alpha = (p+2l) - (p+l+1-l_1) = l + l_1 - 1$ and $\beta = (p+2l) - (p+l+1+l_2) = l - l_2 - 1$. So, $\alpha + \beta \geq s-1$. If $l_1 = l_2$, then we choose $L_0 = \{v_p, v_{p+1}\}$ (we get the path $v_1 v_2 v_3 \ldots v_k$). So, we get $|L_p| = 1$ and $|L_t| = 2$, $t = 0, 1, \ldots, p-1$. Also, $\alpha + \beta = p + l + 1 - l_1 - p + 1 + p + l + 1 + l_2 - (p+1) = 2l = s$. Now, by Theorem 6, $rc_k(f) \geq 2p + 1 - 2p + 2 \sum\limits_{t=0}^{p-1} 2(p-t) + 1 = \frac{k^2+2s+1}{2}$.

Subcase (ii): $s = 2l+1$

If $i \geq p+l+1$ or $j \leq p+l+2$, then as in Subcase (i), we get $rc_k(f) \geq \frac{k^2+2s+1}{2}$. So, we assume $p+1 \leq i < p+l+1 < p+l+2 < j \leq p+s$. Let $i = p+l+1-l_1$ and $j = p+l+2+l_2$ where $1 \leq l_1 \leq l$ and $1 \leq l_2 \leq l-1$. Rest of the proof is similar to that of Subcase (i).

Case II: $k = 2p$

Analogous to Case I, depending on the positions of maximum and minimum colored vertices, here also we choose a P_{k+1} subpath such that $\alpha + \beta \geq s$. If $i \leq s$, then we choose the path $v_i v_{i+1} v_{i+2} \ldots v_{i+p} \ldots v_{i+k}$. So, we get $\alpha = \frac{k}{2}$ and by Theorem 7, $rc_k(f) \geq \frac{k^2+k+2}{2}$, which is a contradiction to Theorem 2 if $s \neq \frac{k}{2}$. If $s < i \leq p$, then by choosing $v_s v_{s+1} v_{s+2} \ldots v_{s+p} \ldots v_{s+k}$ subpath, we get $\alpha \geq s$. If $j \geq k+1$, then as in the Case I, we get contradiction only if $s \neq \frac{k}{2}$. Also, if $j > p+s$, then similar to Case I, we get $\beta \geq s$. Suppose that

$p+1 \leq i < j \leq p+s$.

Subcase (i): $s = 2l$

If $i > p+l$, then by choosing the path $v_1 v_2 v_3 \ldots v_{p+1} \ldots v_{k+1}$, we get $\alpha \geq l$ and $\beta \geq l+1$. If $j \leq p+l$, then by considering the subpath $v_s v_{s+1} v_{s+2} \ldots v_{s+p} \ldots v_{s+k}$, we get $\beta \geq l$ and $\alpha \geq l+1$. Suppose $p+1 \leq i \leq p+l < j \leq p+s$. Let $i = p+l+1-l_1$ and $j = p+l+l_2$, where $1 \leq l_1 \leq l$ and $1 \leq l_2 \leq l$. The cases $l_1 < l_2$ and $l_1 > l_2$ are similar to Subcase (i) of Case I. If $l_1 = l_2$, we choose $L_0 = \{v_p\}$. So, we get $|L_0| = |L_p| = 1$ and $|L_t| = 2$, $t = 1, 2, 3, \ldots, p-1$. Also, $\alpha + \beta = p+l+1-l_1-p+p+l+l_2-p = 2l+1 = s+1$. Now by Theorem 6, $rc_k(f) \geq 2p - 2p + 2p + 2\sum_{t=1}^{p-1} 2(p-t) + s + 1 = \frac{k^2+2s+2}{2}$.

Subcase (ii): $s = 2l+1$

If $i > p+l+1$ or $j \leq p+l$, then as in Subcase (i), we get $rc_k(f) \geq \frac{k^2+2s+1}{2}$. So, we assume that $p+1 \leq i < p+l+1 < p+l+2 < j \leq p+s$. Let $i = p+l+1-l_1$ and $j = p+l+1+l_2$ where $1 \leq l_1 \leq l$ and $0 \leq l_2 \leq l-1$. Rest of the proof is similar to that of Subcase (i). QED

Theorem 9. *If $k > 7$ is even and $n = \frac{3k+2}{2}$, then $rc_k(P_n) = \frac{k^2+k+2}{2}$.*

Proof. From Theorem 8, we have $rc_k(P_{\frac{3k}{2}}) = \frac{k^2+k+2}{2}$. By Lemma 2 and Theorem 3, we get the result. QED

Theorem 10. *If $k \geq 7$ is odd and $\frac{3k+1}{2} \leq n \leq \frac{5k-1}{2}$, then $rc_k(P_n) = \frac{k^2+k+2}{2}$.*

Proof. From Theorem 8, we have $rc_k(P_{\frac{3k+1}{2}}) = \frac{k^2+k+2}{2}$. By Lemma 2 and Theorem 4, we get the result. QED

Lemma 3. *Let $k \geq 7$ be odd and f be a minimal radio k-coloring of $P_n : v_1 v_2 \ldots v_n$ where $n = \frac{5k-1}{2}$. If $f(v_i) = 1$ and $f(v_j) = \frac{k^2+k+2}{2}$, then $\{i, j\} = \{k, n-k+1\}$.*

Proof. Let $f(v_i) = 1$ and $f(v_j) = \lambda$ where $\lambda = \frac{k^2+k+2}{2}$. Without loss of generality, we assume that $i < j$. Let $k = 2p+1$. To prove $i = k$ and $j = n-k+1$, we first show that $j - i = p$ or $j - i = p+1$. If $j - i < p$ or $p+1 < j-i \leq k$, then we choose the path $v_{j-k} v_{j-k+1} v_{j-k+2} \ldots v_{j-p-1} v_{j-p} \ldots v_j$ if $j > k$, else we choose the path $v_i v_{i+1} v_{i+2} \ldots v_{i+p} v_{i+p+1} \ldots v_{i+k}$. In any case, we get one of α and β is $\frac{k-1}{2}$ and the other is at least 1. Now, by Theorem 7, $rc_k(f) \geq \frac{k^2+k+4}{2}$, which is a contradiction. Suppose that $j - i > k$. If the color λ is not used in the path $v_i v_{i+1} v_{i+2} \ldots v_{i+p} v_{i+p+1} \ldots v_{i+k}$, using Theorem 7, we get a contradiction. Suppose the color λ is used in the path $v_i v_{i+1} v_{i+2} \ldots v_{i+p} v_{i+p+1} \ldots v_{i+k}$, say $f(v_t) = \lambda$. Since $t - i \leq k$, $t - i = p$ or

$t-i=p+1$. Since $f(v_t)=f(v_j)=\lambda$, $t+k<j\leq n$. If the color 1 is not used in the path $v_tv_{t+1}v_{t+2}\ldots v_{t+p}v_{t+p+1}\ldots v_{t+k}$, using Theorem 7, we get a contradiction. Suppose the color 1 is used in the path $v_tv_{t+1}v_{t+2}\ldots v_{t+p}v_{t+p+1}\ldots v_{t+k}$, say $f(v_l)=1$. Since $l-t\leq k$, $l-t$ is p or $p+1$. Since $f(v_i)=f(v_l)=1$, $l-i\geq k+1$. Therefore $l-i=k+1$. Now, the minimum color used in the path $v_{i+1}v_{i+2}v_{i+3}\ldots v_{l-1}$ (path on k vertices) is not less than $p+2$. So, the colors available to color the path $v_{i+1}v_{i+2}v_{i+3}\ldots v_{l-1}$ is from $p+2=\frac{k+3}{2}$ to $\frac{k^2+k+2}{2}$. Since $rc_k(P_k)=\frac{k^2+3}{2}$ and $\frac{k^2+k+2}{2}-\frac{k+3}{3}+1=\frac{k^2+1}{2}$, the path $v_{i+1}v_{i+2}v_{i+3}\ldots v_{l-1}$ cannot be colored. Therefore in any case, $j-i\not>k$ and hence $j-i=p$ or $p+1$.

Next, we show that $k\leq i<j\leq n-k+1$ and $j-i\neq p$. For that, we first prove that the colors 1 and λ are used only once by f. Suppose $f(v_l)=1$ for some $l\neq i$. Since $f(v_i)=1$, $l\geq i+k+1$ and hence $l>j$. So, $l-j$ is p or $p+1$. Therefore $l-i=l-j+j-i\leq k+1$ and hence $l-i=k+1$. Now, the minimum color used in the path $v_{i+1}v_{i+2}v_{i+3}\ldots v_{l-1}$ (path on k vertices) is not less than $p+2$. So, the colors available to color the path $v_{i+1}v_{i+2}v_{i+3}\ldots v_{l-1}$ is from $p+2=\frac{k+3}{2}$ to $\frac{k^2+k+2}{2}$. Since $rc_k(P_k)=\frac{k^2+3}{2}$ and $\frac{k^2+k+2}{2}-\frac{k+3}{3}+1=\frac{k^2+1}{2}$, the path $v_{i+1}v_{i+2}v_{i+3}\ldots v_{l-1}$ cannot be colored. Hence the color 1 is assigned to only v_i and by Lemma 1, the color λ is assigned only to v_j. If $i<k$, then $v_{i+1},v_{i+2}v_{i+3}\ldots v_n$ is a path of at least $\frac{3k+1}{2}$ vertices. Since $rc_k(P_{\frac{3k+1}{2}})=\frac{k^2+k+2}{2}=\lambda$ and the color 1 is not used in the path $v_{i+1},v_{i+2}v_{i+3}\ldots v_n$, we get a contradiction. Hence $i\geq k$. Suppose that $j>n-k+1$. Then $v_1v_2v_3\ldots v_{j-1}$ is a path of at least $\frac{3k+1}{2}$ vertices and $rc_k(P_{\frac{3k+1}{2}})=\frac{k^2+k+2}{2}=\lambda$. But maximum color used for a vertex of $v_1v_2v_3\ldots v_{j-1}$ is at most $\lambda-1$, which is a contradiction. Therefore $k\leq i<j\leq n-k+1$. If $j-i=p$, then $i=k$, $j=k+p$ or $i=k+1$, $j=k+p+1$. If $i=k$ and $j=k+p$, then by considering the path $v_{k+p}v_{k+p+1}v_{k+p+2}\ldots v_{k+2p}v_{k+2p+1}\ldots v_n$, we get $\beta=\frac{k-1}{2}$ and the color 1 is not used for $v_{k+p}v_{k+p+1}v_{k+p+2}\ldots v_n$. Now, by using Theorem 7, we get $rc_k(f)\geq\frac{k^2+k+4}{2}$, which is a contradiction. If $i=k+1$ and $j=k+p+1$, then for the path $v_1v_2v_3\ldots v_{p+1}v_{p+2}\ldots v_{k+1}$, the color $\frac{k^2+k+2}{2}$ is not used and $\alpha=\frac{k-1}{2}$. Now, by Theorem 7, we get $rc_k(f)\geq\frac{k^2+k+4}{2}$, which is a contradiction. Therefore, $j-i=p+1$, that is, $i=k$ and $j=n-k+1$. *QED*

Theorem 11. *If $k\geq 7$ is odd, then $rc_k(P_n)=\frac{k^2+k+4}{2}$, where $\frac{5k+1}{2}\leq n\leq\frac{7k-1}{2}$.*

Proof. Let $n=\frac{5k+1}{2}$, $P_n:v_1v_2v_3\ldots v_n$ and $\lambda=\frac{k^2+k+2}{2}$. Suppose $rc_k(P_n)=\lambda$. Let f be a minimal radio k-coloring of P_n. Now, f restricted to $v_1v_2v_3\ldots v_{n-1}$ is a minimal radio k-coloring of P_{n-1}. By Lemma 3, we get $\{f(v_k),f(v_{n-k})\}=\{1,\lambda\}$. By restricting f to the path $v_2v_3\ldots v_n$ and using Lemma 3, we get

$\{f(v_{k+1}), f(v_{n-k+1})\} = \{1, \lambda\}$. Therefore, $rc_k(P_n) \geq \frac{k^2+k+4}{2}$ and hence by Theorem 4, $rc_k(P_n) = \frac{k^2+k+4}{2}$. QED

Lemma 4. *Let $k = 2p > 7$ and f be a minimal radio k-coloring of P_n : $v_1v_2 \ldots v_n$ where $n = \frac{3k+2}{2}$. If $f(v_i) = 1$ and $f(v_j) = \frac{k^2+k+2}{2}$, then $\{i, j\} = \{p + 1, n - p\}$.*

Proof. Let $f(v_i) = 1$ and $f(v_j) = \lambda$ where $\lambda = \frac{k^2+k+2}{2}$. Without loss of generality, we assume that $i < j$. To prove $i = p + 1$ and $j = n - p$, we first show that $j - i = p$. Suppose that $j - i < p$. If $j > k$, then by choosing $v_{j-k}v_{j-k+1}v_{j-k+2} \ldots v_{j-p} \ldots v_j$ path and if $i \leq p + 1$, then by choosing $v_iv_{i+1}v_{i+2} \ldots v_{i+p} \ldots v_{i+k}$ path, we get $\alpha + \beta \geq \frac{k}{2} + 1$, a contradiction, by Theorem 7, to the fact that $rc_k(f) = \frac{k^2+k+2}{2}$. If $i \geq \lceil \frac{3p+1}{2} \rceil$, then by considering $L_0 = \{v_p\}$ and using Theorem 6, we get a contradiction as $\alpha + \beta \geq \frac{k}{2} + 2$. If $j \leq \lceil \frac{3p+1}{2} \rceil$, then by considering the path $v_{p+1}v_{p+2}v_{p+3} \ldots v_{2p+1} \ldots v_n$ we get a contradiction. So, $p + 1 < i < \lceil \frac{3p+1}{2} \rceil < j \leq k$. Let $i = \lceil \frac{3p+1}{2} \rceil - l_1$ and $j = \lceil \frac{3p+1}{2} \rceil + l_2$. By applying Theorem 6 with $L_0 = \{v_{2p+2}\}$ if $l_1 \geq l_2$ and with $L_0 = \{v_p\}$ if $l_1 < l_2$, we get a contradiction to the fact that $rc_k(f) = \frac{k^2+k+2}{2}$. Therefore $j - i \not< p$. If $j - i > p$, then by considering an appropriate subpath of $k+1$ vertices (starting with v_i or ending with v_j), again we get a contradiction. Therefore $j - i = p$.

Next, we show that $i = p + 1$ and $j = n - p$. For that, we first show that the colors 1 and λ are not repeated. Suppose $f(v_l) = 1$ for some $l \neq i$. Then $l \geq i + k + 1$ and $l - j = p$. Therefore $l = j + p = i + 2p = i + k$, which is a contradiction. Hence the color 1 is assigned to only v_i and by Lemma 1, the color λ is assigned only to v_j. Suppose that $i \leq p$. Then $v_{i+1}v_{i+2}v_{i+3} \ldots v_{i+p+1} \ldots v_{i+k+1}$ does not contain the color 1. Let λ_{min} be the minimum color used in $v_{i+1}v_{i+2}v_{i+3} \ldots v_{i+p+1} \ldots v_{i+k+1}$, say $f(v_t) = \lambda_{min}$. Since $rc_k(P_{k+1}) = \frac{k^2+6}{2}$ and the maximum color used is $\frac{k^2+k+2}{2}$, $\lambda_{min} \leq p - 1$. Now, $p - 2 \geq \lambda_{min} - 1 \geq 2p + 1 - d(v_i, v_t) = 2p + 1 - (t - i)$, that is $t \geq i + p + 3$. So, $\alpha = t - (i + p + 1) = (t - i) - (p + 1) \geq 2p + 1 - \lambda_{min} + 1 - (p + 1) = p + 1 - \lambda_{min}$ and $\beta = 1$. Now, by Theorem 7, we get $rc_k(f) \geq \frac{k^2+2}{2} + p + 1 - \lambda_{min} + 1 + \lambda_{min} - 1 = \frac{k^2+k+4}{2}$ which is a contradiction. Similarly, by considering the path $v_{j-k-1}v_{j-k}v_{j-k+1} \ldots v_{j-p-1} \ldots v_{j-1}$, we get a contradiction if $j > n - p$. Therefore $j = n - p$ and $i = p + 1$. QED

Theorem 12. *If $k > 7$ is even, then $rc_k(P_n) = \frac{k^2+k+4}{2}$, where $\frac{3k+4}{2} \leq n \leq \frac{5k+4}{2}$.*

Proof. Similar to the proof of Theorem 11, using Lemma 4. QED

4 Conclusion

For any non-trivial class of graphs, the radio k-chromatic number is not known for arbitrary k, in fact, little has been done when $k \leq diam(G) - 2$. One of the possible reasons could be that finding $rc_k(G)$ is difficult for smaller values of k, in general. As far as we know, $rc_k(G)$ has been studied for $k \leq diam(G)-3$ only when $G = P_n$. In this article, we have determined $rc_k(P_n)$ for $k \geq \frac{2n+1}{7}$ if k is odd and for $k \geq \frac{2n-4}{5}$ if k is even. From Theorem 11 and Theorem 12, for the infinite path P_∞, $rc_k(P_\infty) \geq \frac{k^2+k+4}{2}$ which improves the lower bound given by Das et al. [3] by one, a step towards Conjecture 1.

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