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Bilateral Riemann-Liouville Fractional Sobolev spaces

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Abstract. We establish some notation and properties of the bilateral Riemann-Liouville fractional derivative D^s. We introduce the associated Sobolev spaces of fractional order s, denoted by $W^{s,1}(a,b)$, and the Bounded Variation spaces of fractional order s, denoted by $BV^s(a,b)$: these spaces are studied with the aim of providing a suitable functional framework for fractional variational models in image analysis.

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1 Introduction

There are many different definitions of fractional derivatives and related functional spaces: this paper focuses the analysis on some classical pointwise defined notions of fractional derivatives connected to integral or convolution operators.
First, we introduce refined bilateral definitions of such fractional derivatives and describe their basic properties; then we provide a definition of related Sobolev and BV spaces.
The basic idea connecting various pointwise classical definitions is the remark that, if u is a smooth function, then its classical derivative $u^{(n)}$ of integer order $n \geq 1$ is the n-th iteration of the derivative, while the integral or primitive $\int_0^x u(t)\,dt$, evaluated at $x > 0$, is the "antiderivative", or derivative of order -1; hence the "derivative of order $-n$" can be defined as the n-th iteration of the antiderivative:

$$u^{(-n)}(x) \;=\; \int_0^x dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \cdots \int_0^{t_{n-1}} u(t_n)\,dt_n\,.$$

The iterated integral above actually is an n-dimensional integral over the n dimensional simplex $\Sigma_x^n = \big\{ (t_1, t_2, \dots, t_n) : \ 0 \le t_1 \le t_2 \le \dots \le t_n \le x \big\}$

$$u^{(-n)}(x) \ = \ \int_{\Sigma_x^n} u(t_n) \ dt_1 \, dt_2 \dots dt_n$$

by changing the order of integration and denoting Σ_{x-t}^{n-1} the $(n-1)$-dimensional simplex provided by the intersection of Σ_x^n and the hyperplane $t_n = t$, one gets

$$u^{(-n)}(x) \ = \ \int_0^x |\Sigma_{x-t}^{n-1}| \, u(t) \, dt \ = \ \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} \, u(t) \, dt \ . \tag{1.1}$$

By applying (1.1) with $u^{(1)} := u'$ in place of u, one gets for integers $n \ge 1$

$$u^{(-n+1)}(x) \ = \ \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} \, u'(t) \, dt \ . \tag{1.2}$$

By exploiting the identity $\Gamma(n) = (n-1)!$ and the fact that Euler Gamma function Γ is defined for every $z \in \mathbb{C}$ with the exception of 0 and negative integers, we can perform the substitution $s = -n+1$ and allow $s \in \mathbb{R}$ in order to extend relationship (1.2) to primitives of noninteger order $s \in [-1, 0]$ and derivatives of noninteger order $s \in (0,1)$ for a function $u \in C^1$: e.g. the fractional derivatives of order s of u is expressed as follows

$$u^{(s)}(x) \ = \ \int_0^x \frac{(x-t)^{-s}}{\Gamma(1-s)} \, u'(t) \, dt \qquad s \in \mathbb{R}, \ s < 1 \, , \ s \text{ not integer.} \tag{1.3}$$

We emphasize that condition $s < 1$ entails convergence of the integral, which is actually the convolution of two $L^1_{loc}(\mathbb{R})$ functions with support in $[0, +\infty)$.
Indeed (1.3) provides a hint for the extension of pointwise noninteger or fractional derivative of order s with $0 < s < 1$: actually (1.3) is known in the literature as the (left) Caputo fractional derivative of order s ([5],[19]).
Another classical definition is the (left) Riemann-Liouville fractional derivative of order s, whose definition, for functions in $L^1(a,b)$, is as follows

$${}_{RL}D^s_{a+}[u](x) \ = \ \frac{1}{\Gamma(1-s)} \, \frac{d}{dx} \int_a^x \frac{u(t)}{(x-t)^s} dt \qquad a < x < b. \tag{1.4}$$

Riemann-Liouville and Caputo derivatives are closely related, indeed they fulfil

$${}_{RL}D^s_{0+}[u](x) \ = \ u^{(s)}(x) \ + \ \frac{u(0)}{\Gamma(1-s)} \, x^{-s} \qquad 0 < s < 1 \, . \tag{1.5}$$

In the opposite direction, by the same tools, we can extend (1.1) to every real $s > 0$, in place of strictly positive integers n:

$$u^{(-s)}(x) \;=\; \int_0^x \frac{(x-t)^{s-1}}{\Gamma(s)}\, u(t)\, dt \quad s > 0\,. \tag{1.6}$$

Indeed (1.6) is known as the (left) Riemann-Liouville fractional integral of order s of u ([3],[19]). Moreover the right-hand side of (1.6) is the convolution of the trivial extension of u in $\mathbb{R}$ with $t^{1-s}H(t)/\Gamma(s)$, where H denotes the Heaviside function.

Actually formulae (1.3), (1.4) and (1.6) can be set for every x in (a, b) without any differentiability assumption on u, provided the right-hand side Lebesgue integrals exists finite, namely: $u \in L^1(a, b)$ is a sufficient condition to achieve (1.6) defined a.e. on (a, b); $u' \in L^1$ is a sufficient condition to achieve (1.3) defined a.e. on (a, b); $\int_a^x u(t)(x-t)^{-s} dt \in L^1$ is a sufficient condition to achieve (1.4) defined a.e. on (a, b).

In this paper we study some properties of classical Riemann-Liouville left and right fractional derivatives D^s_+, D^s_-, which are inspired by (1.4) and defined for non integer order $s \in (0, 1)$ (see Definition 2.2), moreover we introduce "bilateral" notions D^s_e, D^s_o, respectively "even" and "odd" (see Definition 2.3) mainly focusing on the distributional interpretation of their limit as s tends either to 0 or to 1 (see Lemmas 2.3, 2.4, and Remark 2.1).

In addition to these bilateral fractional derivatives suitable "bilateral" fractional integrals I^s_e, I^s_o are introduced by Definitions 3.1 and 3.2 in such a way that, up to a normalization, they represent respectively the inverse operators of D^s_e and D^s_o (see Lemma 2.5).

Eventually, we introduce the notions of fractional Sobolev spaces $W^{s,1}$ and fractional Bounded Variation spaces BV^s, associated to these bilateral derivatives (see Definitions 4.3 and 5.2). The space $W^{s,1}$ turns out to be the natural space for data of Abel integral equations in order to make such equations well posed problems: the forthcoming paper [15] is focused on the basic properties of these functional spaces and comparison with their non-bilateral counterpart (see [6], [12], [13]). The spaces $W^{s,1}$ and BV^s are introduced with the aim of providing a suitable functional framework for fractional variational models in image analysis (see [2], [4], [7], [8], [9], [10], [11],[20]), which are the object of another forthcoming paper [16].

2 Bilateral fractional integral and derivatives

In the sequel $(a, b) \subset \mathbb{R}$ is a non empty (possibly unbounded) open interval, u is a real function of one variable and $0 < s < 1$. The support of

a function u is denoted by $\operatorname{spt} u$. The notation d/dx stands for the classical pointwise derivative, D denotes the distributional derivative with respect to the variable x, for every open set $A \subset \mathbb{R}$ we denote by $AC(A)$ the space of absolutely continuous functions in A ([18]), which coincides with the Gagliardo-Sobolev space $W_G^{1,1}(A) = \{u \in L^1(A) \mid Du \in L^1(A)\}$. Moreover, we set $AC_{loc}(A) = W_{G,loc}^{1,1}(A) = \{u \in L^1_{loc}(A) \mid Du \in L^1_{loc}(A)\}$ and $BV(A) = \{u \in L^1(A) \mid Du \in \mathcal{M}(A)\}$, where $\mathcal{M}(A)$ are the bounded measures on A.
For reader's convenience we recall the definition of Gagliardo's fractional Sobolev Spaces $W_G^{s,1}$ ([1], [14]): for any $s \in (0,1)$ we set

$$W_G^{s,1} = \left\{ u \in L^1(a,b) \;:\; \frac{|u(x)-u(y)|}{|x-y|^{1+s}} \in L^1([a,b]\times[a,b]) \right\}, \qquad (2.1)$$

which is a Banach space endowed with the norm

$$\|u\|_{W_G^{s,1}} = \left[\int_{[a,b]} |u(x)| dx \; + \int_{[a,b]} \int_{[a,b]} \frac{|u(x)-u(y)|}{|x-y|^{1+s}} dx\, dy \right],$$

and we recall the definition of the Riemann-Liouville fractional integral and derivative of order s for L^1-functions, whose standard references can be found in the book by Samko and al. [19].

Definition 2.1. (**Riemann-Liouville fractional integral**)
Assume $u \in L^1(a,b)$ and $s > 0$.
The left-side and right-side Riemann-Liouville fractional integrals are defined by setting respectively

$${}_{RL}I^s_{a+}[u](x) \;=\; \frac{1}{\Gamma(s)} \int_a^x \frac{u(t)}{(x-t)^{1-s}} dt\,, \qquad x \in [a,b]\,, \qquad (2.2)$$

$${}_{RL}I^s_{b-}[u](x) \;=\; \frac{1}{\Gamma(s)} \int_x^b \frac{u(t)}{(t-x)^{1-s}} dt\,, \qquad x \in [a,b]\,, \qquad (2.3)$$

Here Γ stands for the classical Gamma function [17].
Notice that ${}_{RL}I^1_{a+}[u](x) = \int_a^x u(t)\, dt$, and in general, for strictly positive integer values of $s = n \in \mathbb{N}$, we recover n-th order primitive vanishing together with all derivatives up to order $n-1$ at $x = a$.
Obviously, both ${}_{RL}I^s_{a+}[u]$ and ${}_{RL}I^s_{b-}[u]$ are absolutely continuous functions if $s \geq 1$; whereas they are only L^1 functions if $0 < s < 1$. Indeed they may have jump discontinuities, as shown by next example.

Example 2.1. Set $(a,b) = (-1,1)$, $u(x) = H(x)/\sqrt{x}$, $s = 1/2$. Then

$$ {}_{RL}I^{1/2}_{-1+}[u](x) \;=\; \frac{1}{\sqrt{\pi}}\int_{-1}^{x} \frac{H(t)}{\sqrt{t}\,\sqrt{x-t}}\,dt \;=\; \begin{cases} 0 & \text{if } -1<x\le 0,\\ \dfrac{1}{\sqrt{\pi}}\displaystyle\int_0^x \frac{H(t)}{\sqrt{t}\,\sqrt{x-t}}\,dt & \text{if } 0<x<1\,. \end{cases} $$

$$ x\in(-1,0) \quad\Rightarrow\quad {}_{RL}I^{1/2}_{-1+}[u](x) \;=0\,, $$

$$ x\in(0,1) \quad\Rightarrow\quad {}_{RL}I^{1/2}_{-1+}[u](x) \;=\; \frac{1}{\sqrt{\pi}}\int_0^1 \frac{1}{\sqrt{t(1-t)}}dt \;=\; \sqrt{\pi}\,. $$

Next we may define the Riemann-Liouville fractional derivative as in [3],[19].

Definition 2.2. (**Riemann-Liouville fractional derivative**)
Assume $u \in L^1(a,b)$ and $0 < s < 1$.
The left Riemann-Liouville derivative of u at $x \in [a,b]$ is defined by

$$ {}_{RL}D^{s}_{a+}[u](x) \;=\; \frac{d}{dx}\,{}_{RL}I^{1-s}_{a+}[u](x) \;=\; \frac{1}{\Gamma(1-s)}\frac{d}{dx}\int_a^x \frac{u(t)}{(x-t)^s}dt \tag{2.4} $$

at values x such that it exists.
Similarly, we may define the right Riemann-Liouville derivative of u at $x \in [a,b]$ as

$$ {}_{RL}D^{s}_{b-}[u](x) \;=\; -\frac{d}{dx}\,{}_{RL}I^{1-s}_{b-}[u](x) \;=\; \frac{-1}{\Gamma(1-s)}\frac{d}{dx}\int_x^b \frac{u(t)}{(t-x)^s}dt \tag{2.5} $$

at values x such that it exists.

In Lemma 2.6 we examine the case when above pointwise defined derivative exists a.e. and defines an L^1 function coincident with the distributional derivative, respectively of ${}_{RL}I^{1-s}_{a+}[u]$ and ${}_{RL}I^{1-s}_{b-}[u]$.
In the sequel we omit the RL index and the endpoint of the interval without loss of information, since in the sequel we do not consider other fractional derivatives (as the Caputo or the Weyl ones) since the forthcoming results can be proved in the same way for such derivatives. So we shortly write $I^s_+[u]$, $I^s_-[u]$, $D^s_+[u]$ and $D^s_-[u]$ respectively in place of ${}_{RL}I^s_{a+}[u]$, ${}_{RL}I^s_{b-}[u]$, ${}_{RL}D^s_{a+}[u]$ and ${}_{RL}D^s_{b-}[u]$.

One of the disadvantages of the one-side Riemann-Liouville derivative (and integral) is the fact that only one endpoint of the interval plays a role. If we aim to plug such a (point-wise) "anisotropic" definition in a variational framework we have to deal with boundary conditions. In other terms the interval bounds (or the boundary in several dimensions) should play the same role. Therefore, we introduce the bilateral fractional integral and derivative, though keeping separate "even" and "odd" parts:

Definition 2.3. For every $u \in L^1(a,b)$ we set the even and odd version of bilateral fractional integrals and derivatives:

$$\begin{aligned} I_e^s[u](x) &:= \frac{1}{2}\big(I_+^s[u](x) + I_-^s[u](x)\big) = \\ &= \frac{1}{2\,\Gamma(s)}\int_a^b \frac{u(t)}{|x-t|^{1-s}}dt = \frac{(u * 1/|t|^{1-s})(x)}{2\,\Gamma(s)}, \end{aligned} \tag{2.6}$$

$$\begin{aligned} D_e^s[u](x) &:= \frac{d}{dx} I_e^{1-s}[u](x) = \\ &= \frac{1}{2}\left(D_+^s[u](x) - D_-^s[u](x)\right) = \frac{d}{dx}\frac{(u * 1/|t|^s)(x)}{2\,\Gamma(1-s)}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} I_o^s[u](x) &:= \frac{1}{2}\left(I_+^s[u](x) - I_-^s[u](x)\right) = \\ &= \frac{1}{2\,\Gamma(s)}\int_a^b u(t)\,\frac{\mathrm{sign}(x-t)}{|x-t|^{1-s}}\,dt = \frac{(u * \frac{\mathrm{sign}(t)}{|t|^{1-s}})(x)}{2\,\Gamma(s)}, \end{aligned} \tag{2.8}$$

$$\begin{aligned} D_o^s[u](x) &:= \frac{d}{dx} I_o^{1-s}[u](x) = \\ &= \frac{1}{2}\left(D_+^s[u](x) + D_-^s[u](x)\right) = \frac{d}{dx}\frac{(u * \mathrm{sign}(t)/|t|^s)(x)}{2\,\Gamma(1-s)}. \end{aligned} \tag{2.9}$$

So that

$$I_+^s[u] = I_e^s[u] + I_o^s[u], \qquad D_+^s[u] = D_e^s[u] + D_o^s[u], \tag{2.10}$$

$$I_-^s[u] = I_e^s[u] - I_o^s[u], \qquad D_-^s[u] = D_o^s[u] - D_e^s[u]. \tag{2.11}$$

Whenever $(a,b) \neq \mathbb{R}$ the convolution in (2.6), (2.7), (2.8), (2.9) has to be understood, without relabeling, as the convolution of the trivial extension of u (still an $L^1(\mathbb{R})$ function) with either $1/|t|^s$ or $\mathrm{sign}(t)/|t|^s$ (both belonging to $L^1_{loc}(\mathbb{R})$). Also $I_\pm^s[u](x)$, $I_e^s[u](x)$, $I_o^s[u](x)$ have to be understood, without relabeling, as the natural extension for $x \in \mathbb{R}\backslash[a,b]$, provided by the convolution of the trivial extension of u with the corresponding kernels (here H denotes the Heaviside function):

$$I_+^s[u] = u * \frac{H(x)}{\Gamma(s)|x|^{1-s}}, \qquad I_-^s[u] = u * \frac{H(-x)}{\Gamma(s)|x|^{1-s}}, \tag{2.12}$$

$$I_e^s[u] = u * \frac{1}{2\Gamma(s)|x|^{1-s}}, \qquad I_o^s[u] = u * \frac{\mathrm{sign}(x)}{2\Gamma(s)|x|^{1-s}}, \tag{2.13}$$

namely

$$I_+^s[u](x) = \frac{1}{\Gamma(s)} \int_a^b \frac{u(t)\,H(x-t)}{|x-t|^{1-s}}\,dt \qquad \text{for every } x \in \mathbb{R}\,, \quad (2.14)$$

$$I_-^s[u](x) = \frac{1}{\Gamma(s)} \int_a^b \frac{u(t)\,H(t-x)}{|x-t|^{1-s}}\,dt \qquad \text{for every } x \in \mathbb{R}\,, \quad (2.15)$$

$$I_e^s[u](x) = \frac{1}{2\,\Gamma(s)} \int_a^b \frac{u(t)}{|x-t|^{1-s}}\,dt \qquad \text{for every } x \in \mathbb{R}\,, \quad (2.16)$$

$$I_o^s[u](x) = \frac{1}{2\,\Gamma(s)} \int_a^b \frac{u(t)\ \mathrm{sign}(x-t)}{|x-t|^{1-s}}\,dt \qquad \text{for every } x \in \mathbb{R}\,. \quad (2.17)$$

In this way $I_+^s[u]$, $I_-^s[u]$, $I_e^s[u]$, $I_o^s[u]$ turn out to be $L^p_{loc}(\mathbb{R})$ (hence $L^p(I)$ for any bounded interval I) for any $1 \le p < 1/(1-s)$, since it is a convolution of $u \in L^1(\mathbb{R})$ with an $L^p_{loc}(\mathbb{R})$ kernel. Moreover we have the next result.

Lemma 2.1. If $-\infty < a < b < +\infty$, $u \in L^\infty(\mathbb{R})$, $\mathrm{spt}(u) \subset [a,b]$ and $0 < s < 1$ then $I_+^s[u]$, $I_-^s[u]$, $I_e^s[u]$, $I_o^s[u]$ belong to $L^\infty(\mathbb{R})$.

Proof. Due to previous remarks the functions $I_\pm^s[u]$ are almost everywhere pointwise defined L^1_{loc} functions. Moreover we get

$$\left| I_\pm^s[u](x) \right| \le \frac{\|u\|_{L^\infty}}{\Gamma(s)} \int_a^b \frac{dt}{|x-t|^{1-s}} = \frac{\|u\|_{L^\infty}}{s\,\Gamma(s)} \left| |x-b|^s - |x-a|^s \right|, \quad (2.18)$$

hence

$$\left| I_\pm^s[u](x) \right| \le \frac{\|u\|_{L^\infty}}{\Gamma(s)} \max\left(|x-b|^{s-1},\, |x-a|^{s-1} \right) |b-a|\,, \quad (2.19)$$

Choose a real number M such that $M > \max(|a|,|b|)$; then $I_\pm^s[u] \in L^\infty(-M,+M)$ due to (2.18) and $I_\pm^s[u] \in L^\infty\big(\mathbb{R} \setminus (-M,+M)\big)$ due to (2.19). QED

Up to a conventional constant $I_e^{1-s}[u]$ is called Riesz potential of u. For our purposes it is useful considering suitable normalization of these potentials:

$$\frac{1}{\cos(s\pi/2)}\, I_e^s[u](x) = \frac{1}{2\,\Gamma(s)\,\cos(s\pi/2)} \int_a^b \frac{u(t)}{|x-t|^{1-s}}\,dt = A^s[u]\,, \quad (2.20)$$

$$\frac{1}{\sin(s\pi/2)}\, I_o^s[u](x) = \frac{1}{2\,\Gamma(s)\,\sin(s\pi/2)} \int_a^b u(t)\,\frac{\mathrm{sign}(x-t)}{|x-t|^{1-s}}\,dt = B^s[u], \quad (2.21)$$

where $0 < s < 1$, $a < x < b$, and for comparison we mention notation A^s, B^s as defined in [19], eq. 12.44, eq.12.45.

In order to clarify the meaning of definitions (2.6)-(2.17) and (2.20)-(2.21) one can consider the case when (a,b) is replaced by $\mathbb{R}$, e.g. u is defined on $\mathbb{R}$ and all integrals over (a,b) are replaced by integrals over the whole real line.
Everything can be transferred back to (a,b) up to standard smooth corrections, dependent only on boundary data and whose integrals are computed in (a,b):

$$I_e^s[l(u)],\;\; I_o^s[l(u)],\;\; D_e^s[l(u)],\;\; D_o^s[l(u)], \quad \text{where } l(u) = u(a) + \frac{u(b)-u(a)}{b-a}(x-a).$$

So that if $u \in W_G^{1,2}(\mathbb{R}) := \{u \in L^2(\mathbb{R}) : Du \in L^2(\mathbb{R})\}$, where Du denote the distributional derivative of u, then by setting $\widetilde{u}(x) = u(x) - l(u)(x)$, for $a < x < b$, and denoting its trivial extension still by $\widetilde{u}$ without relabeling, we get $\widetilde{u} \in W_G^{1,2}(\mathbb{R})$, $\operatorname{spt}\widetilde{u} \subset [a,b]$.
For functions u with domain in $\mathbb{R}$ we introduce the symmetric part (or "even") u_e^x and the antisymmetric (or "odd") u_o^x of u with respect to the point x, as follows:

$$u_e^x(t) \;:=\; (1/2)\Big(u(t) + u(2x-t)\Big), \qquad u_o^x(t) \;:=\; (1/2)\Big(u(t) - u(2x-t)\Big)$$

so that

$$u(t) \;=\; u_e^x(t) \,+\, u_o^x(t) \qquad \forall t,\, x \in \mathbb{R}$$

By exploiting cancellation under integration and exchanges under derivation of even and odd terms, we get the next statement.

Lemma 2.2. For every $u \in C^1_{loc}(\mathbb{R})$ and $x \in \mathbb{R}$:

$$I_e^s[u_o^x](x) \;=\; 0\,, \quad I_o^s[u_e^x](x) \;=\; 0 \quad D_e^s[u_e^x](x) \;=\; 0\,, \quad D_o^s[u_o^x](x) \;=\; 0\,, \tag{2.22}$$

which together with (2.10),(2.11) provides:

$$I_+^s[u](x) \;=\; I_e^s[u_e^x](x) + I_o^s[u_o^x](x)\,, \tag{2.23}$$
$$D_+^s[u](x) \;=\; D_e^s[u_o^x](x) + D_o^s[u_e^x](x)\,, \tag{2.24}$$
$$I_-^s[u](x) \;=\; I_e^s[u_e^x](x) - I_o^s[u_o^x](x)\,, \tag{2.25}$$
$$D_-^s[u](x) \;=\; D_o^s[u_e^x](x) - D_e^s[u_o^x](x)\,. \tag{2.26}$$

In the present paper we make extensive use of the Fourier transform defined by

$$\widehat{u}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} u(x)dx \qquad u \in L^1(\mathbb{R}) \tag{2.27}$$

and the corresponding extension (namely, with the same choice of the constants) when u is a tempered distribution.

The meaning of all the above operators is clarified by subsequent Lemmas, which rely on two identities concerning the Fourier transform, that are recalled here:

$$\mathcal{F}\left\{ \frac{1}{|x|^s} \right\}(\xi) \;=\; 2\,\sin(s\,\pi/2)\;\Gamma(1-s)\,\frac{1}{|\xi|^{1-s}} \qquad 0 < s < 1\,, \quad (2.28)$$

$$\mathcal{F}\left\{ \frac{\operatorname{sign}(x)}{|x|^s} \right\}(\xi) \;=\; -\,2\,i\,\cos(s\,\pi/2)\,\Gamma(1-s)\,\frac{\operatorname{sign}(\xi)}{|\xi|^{1-s}} \qquad 0 < s < 1\,, \quad (2.29)$$

and on the notions of two distributions (respectively of order 1 and 2): *principal value of* $1/x$ (notation p.v. $\frac{1}{x}$) and *finite part of* $1/x^2$ (notation f.p. $\frac{1}{x^2}$), whose definitions are provided by duality: for any test function $\varphi \in C_0^\infty(\mathbb{R})$ one has

$$\langle \text{p.v.}\,\frac{1}{x}, \varphi\rangle = \lim_{\varepsilon\to 0_+} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x}\,dx\,, \quad (2.30)$$

$$\langle\,\text{f.p.}\,\frac{1}{x^2}, \varphi\,\rangle = \lim_{\varepsilon\to 0_+} \left(\int_{|x|>\varepsilon} \frac{\varphi(x)}{x^2} dx - \frac{2}{\varepsilon}\varphi(0) \right)\,. \quad (2.31)$$

Notice that both $1/|x|^s$ and $\operatorname{sign}(x)/|x|^s$ belong to $L^1_{loc}(\mathbb{R})$, for $0 < s < 1$, hence the convolution with any L^1 function is well defined and belongs to $L^1_{loc}(\mathbb{R})$; moreover $\operatorname{sign}(x)/|x|^s \to$ p.v. $\frac{1}{x}$ in $\mathcal{S}'$ as $s \to 1^-$, while $1/|x|^s$ has no limit in $\mathcal{S}'$ as $s \to 1^-$, where $\mathcal{S}'$ denotes the space of tempered distributions.
Fractional derivatives degenerate developing singularities as $s \to 1^-$; nevertheless they can be made convergent to meaningful limits by suitable normalization.

Lemma 2.3. Assume $0 < s < 1$ and $u \in W^{1,2}_G(\mathbb{R})$. Then

$$\frac{D^s_e[u]}{\sin(s\,\pi/2)} \;\longrightarrow\; \mathcal{F}^{-1}\,\{\,i\,\xi\,\widehat{u}(\xi)\,\} \;= Du \quad \text{in } L^2(\mathbb{R}) \quad \text{as } s \to 1^-\,, \quad (2.32)$$

$$\frac{D^s_o[u]}{\cos(s\,\pi/2)} \;\longrightarrow\; \mathcal{F}^{-1}\,\{\,|\xi|\,\widehat{u}(\xi)\,\} \qquad \text{in } L^2(\mathbb{R}) \quad \text{as } s \to 1^-\,, \quad (2.33)$$

$$D^s_+[u] \;\longrightarrow\; Du \qquad \text{in } L^2(\mathbb{R}) \qquad \text{as } s \to 1^-\,, \quad (2.34)$$

$$D^s_-[u] \;\longrightarrow\; -\,Du \qquad \text{in } L^2(\mathbb{R}) \quad \text{as } s \to 1^-\,. \quad (2.35)$$

Proof. Since $u \in W^{1,2}_G(\mathbb{R})$, we get $(1+|\xi|^2)^{1/2}\,\widehat{u}(\xi) \in L^2(\mathbb{R})$. Hence both $\widehat{u}(\xi)$, $|\xi|\,\widehat{u}(\xi)$ belong to $L^2(\mathbb{R})$, then $|\xi|^s\widehat{u}(\xi) \in L^2(\mathbb{R})$ for every $0<s<1$, and by (2.6),(2.28):

$$\begin{aligned}\mathcal{F}\left\{\frac{D^s_e[u](x)}{\sin(s\,\pi/2)}\right\}(\xi) \;&=\; \frac{1}{2\,\Gamma(1-s)\,\sin(s\,\pi/2)}\; i\,\xi\,\widehat{u}(\xi)\;\mathcal{F}\left\{\frac{1}{|x|^s}\right\}(\xi) \;=\\ &=\; \frac{i\,\xi\,\widehat{u}(\xi)}{2\,\Gamma(1-s)\,\sin(s\,\pi/2)}\Big(2\,|\xi|^{s-1}\,\Gamma(1-s)\,\sin(s\,\pi/2)\Big) =\\ &=\; i\,\xi\,|\xi|^{s-1}\,\widehat{u}(\xi) \;=\; i\,|\xi|^s\,\operatorname{sign}(\xi)\,\widehat{u}(\xi) \;\stackrel{s\to1^-}{\longrightarrow}\; i\,\xi\,\widehat{u}(\xi) \;=\; \mathcal{F}\{Du\}\;,\end{aligned}$$

hence (2.32) follows by continuity of Fourier transform in L^2.

Analogously by (2.8),(2.29),

$$\begin{aligned}\mathcal{F}\left\{\frac{D^s_o[u](x)}{\cos(s\,\pi/2)}\right\}(\xi) \;&=\; \frac{1}{2\,\Gamma(1-s)\,\cos(s\,\pi/2)}\; i\,\xi\,\widehat{u}(\xi)\;\mathcal{F}\left\{\frac{\operatorname{sign}(x)}{|x|^s}\right\}(\xi) \;=\\ &=\; \frac{i\,\xi\,\widehat{u}(\xi)}{2\,\Gamma(1-s)\,\cos(s\,\pi/2)}\Big(-\,2i\,|\xi|^{s-1}\,\Gamma(1-s)\,\cos(s\pi/2)\,\operatorname{sign}(\xi)\Big) =\\ &=\; |\xi|^s\,\widehat{u}(\xi) \;\stackrel{s\to1^-}{\longrightarrow}\; |\xi|\,\widehat{u}(\xi)\;.\end{aligned}$$

Then (2.33) follows by continuity of Fourier transform in L^2.
Eventually, since $\sin(s\pi/2)\to1$, $\cos(s\pi/2)\to0$ as $s\to1^-$, (2.34) follows by (2.32),(2.33) and $D^s_+[u](x) \;=\; D^s_e[u](x)+D^s_o[u](x)$, while (2.35) follows by (2.32),(2.33) and $D^s_-[u](x) \;=\; D^s_o[u](x)-D^s_e[u](x)$. QED

Remark 2.1. Notice that relations (2.32),(2.34) and (2.35) tell that, as $s\to1^-$, both $D^s_+[u]$ (left Riemann-Liouville fractional derivative of order s of u) and $D^s_e[u]$ (even Riemann-Liouville fractional derivative of order s of u) converge in L^2 to the distributional derivative Du, while $D^s_-[u]$ converges in L^2 to $-Du$.
On the other hand relation (2.33) means that $D^s_o[u]$ (odd Riemann-Liouville fractional derivative of order s of u) fades as $s\to1^-$ but, when suitably normalized as $D^s_o[u]/\cos(s\,\pi/2)$, it converges in L^2 to the Gagliardo fractional derivative of order 1 of u, say $(-\Delta)^{1/2}u \;:=\; \mathcal{F}^{-1}\,\{\,|\xi|\,\widehat{u}(\xi)\,\}$.
If in addition $u\in C^2(\mathbb{R})$, this last convergence can be made even more explicit

$$\frac{D^s_o[u]}{\cos(s\,\pi/2)} \;\to\; \mathcal{F}^{-1}\left\{|\xi|\,\widehat{u}(\xi)\right\} = \frac{1}{\pi}\left(\text{p.v.}\,\frac{1}{x}\right)*Du = -\frac{1}{\pi}\left(\text{f.p.}\,\frac{1}{x^2}\right)*u \quad (2.36)$$

in $L^2(\mathbb{R})$ as $s\to1^-$.

Indeed, if $u \in C^2(\mathbb{R})$, then

$$|\xi|\, \widehat{u}(\xi) \;=\; \big(i\,\xi\,\widehat{u}(\xi)\big)\,\big(-i\,\mathrm{sign}(\xi)\big) \;=\; \mathcal{F}\{Du\}\;\mathcal{F}\left\{\frac{1}{\pi}\left(\mathrm{p.v.}\,\frac{1}{x}\right)\right\} \;=\; \tag{2.37}$$
$$=\frac{1}{\pi}\,\mathcal{F}\left\{Du * \left(\mathrm{p.v.}\,\frac{1}{x}\right)\right\} \;=\; -\frac{1}{\pi}\,\mathcal{F}\left\{u * \left(\mathrm{f.p.}\,\frac{1}{x^2}\right)\right\}\,,$$

Notice that, by density of $\mathcal{S}(\mathbb{R})$ in $W^{1,2}_G(\mathbb{R})$, we can exploit (2.36),(2.37) to set a (unique extensions of the) definition of both convolutions

$$\frac{1}{\pi}\left(\mathrm{p.v.}\,\frac{1}{x}\right) * Du \;=\; -\frac{1}{\pi}\left(\mathrm{f.p.}\,\frac{1}{x^2}\right) * u \;:=\; \mathcal{F}^{-1}\,\{\,|\xi|\,\widehat{u}(\xi)\,\} \qquad \forall u \in W^{1,2}_G(\mathbb{R}).$$

Fractional integrals degenerate producing singularities as $s \to 0^+$; indeed the convolution term fulfils $|x|^{s-1}/(2\Gamma(s)\cos(s\pi/2)) \to \delta$ in $\mathcal{S}'$ as $s \to 0^+$; nevertheless fractional integrals are convergent to meaningful limits by suitable normalization.

Lemma 2.4. Assume $0<s<1$, $u \in L^1(\mathbb{R})$ with $\widehat{u} \in L^1(\mathbb{R})$ and set the constants in the Fourier transform such that $\widehat{u}(\xi) = \int_{\mathbb{R}} \exp(-i\xi x)\,u(x)\,dx$. Then

$$\frac{1}{\cos(s\,\pi/2)}\; I^s_e[u](x) \;\longrightarrow\; u(x) \qquad \text{uniformly in } \mathbb{R} \quad \text{as } s \to 0^+\,, \tag{2.38}$$

$$\frac{\pi}{\sin(s\,\pi/2)}\; I^s_o[u](x) \;\longrightarrow\; (\mathrm{p.v.}\,1/x) * u \quad \text{in } \mathcal{S}'(\mathbb{R}) \quad \text{as } s \to 0^+\,. \tag{2.39}$$

Proof. Since $u,\, \widehat{u} \in L^1(\mathbb{R})$, we get $u,\, \widehat{u} \in L^\infty(\mathbb{R})$, $|\xi|^{-s}\,\widehat{u}(\xi) \in L^1(\mathbb{R})$ for $0<s<1$, then by (2.6),(2.28) and dominated convergence Theorem:

$$\mathcal{F}\left\{\frac{1}{\cos(s\,\pi/2)}\; I^s_e[u](x)\right\}(\xi) \;=\; \frac{1}{\cos(s\,\pi/2)}\,\frac{\widehat{u}(\xi)}{2\Gamma(s)}\;\mathcal{F}\left\{\frac{1}{|x|^{1-s}}\right\}(\xi) \;=$$
$$=\; \frac{\widehat{u}(\xi)}{2\;\Gamma(s)\;\cos(s\,\pi/2)}\Big(2\,|\xi|^{-s}\,\Gamma(s)\;\cos(s\,\pi/2)\Big) \;=$$
$$=\; |\xi|^{-s}\,\widehat{u}(\xi) \;\overset{s\to 0^+}{\longrightarrow}\; \widehat{u}(\xi) \;=\; \mathcal{F}\,\{u\} \qquad \text{in } L^1(\mathbb{R})\,,$$

hence (2.38) follows by continuity of inverse Fourier transform from L^1 to L^∞. Since $u,\, \widehat{u} \in L^1(\mathbb{R})$, we get $u,\, \widehat{u} \in L^\infty(\mathbb{R})$, $|\xi|^{-s}\,\widehat{u}(\xi) \in L^1(\mathbb{R})$ for $0<s<1$, then by (2.8),(2.29)

$$\mathcal{F}\left\{ \frac{1}{\sin(s\,\pi/2)}\; I_o^s[u](x) \right\}(\xi) \;=\; \frac{1}{\sin(s\,\pi/2)}\,\frac{\widehat{u}(\xi)}{2\Gamma(s)}\;\mathcal{F}\left\{\frac{\mathrm{sign}(x)}{|x|^{1-s}}\right\}(\xi) \;=$$

$$= \;\frac{\widehat{u}(\xi)}{2\,\Gamma(s)\,\sin(s\,\pi/2)}\Big(-\,2\,i\,\Gamma(s)\,\sin(s\,\pi/2)\,\mathrm{sign}(\xi)\,|\xi|^{-s}\Big) \;=$$

$$= \;-\,i\;\mathrm{sign}(\xi)\,|\xi|^{-s}\,\widehat{u}(\xi) \;\overset{s\to 0^+}{\longrightarrow}\; \widehat{u}(\xi) \;=\; -\,i\;\mathrm{sign}(\xi)\;\widehat{u}(\xi) \;=$$

$$= \;\frac{1}{\pi}\,\mathcal{F}\{(\mathrm{p.v.}\,1/x)\,*\,u\} \qquad \text{in } \mathcal{S}'(\mathbb{R})\,,$$

hence (2.39) follows by continuity of inverse Fourier transform in $\mathcal{S}'(\mathbb{R})$. *QED*

It is well known that, under suitable assumptions, the one sided operators $D^s_\pm$ are the right and left inverse respectively of operators $I^s_\pm$ for functions defined in the bounded interval (a,b).
Concerning the issue "whether I^s is the right and/or left inverse of D^s, at least up to a suitable constant, even in the case of bilateral definitions", the next lemma provides an answer by focusing the analysis separately on the even and odd bilateral definitions.

Lemma 2.5. Assume $0 < s < 1$, $u \in L^1(\mathbb{R})$.
If $I_o^{1-s}[u] \in AC_{loc}(\mathbb{R})$, then

$$\frac{1}{(\cos(s\,\pi/2))^2}\; I_e^s\,[\,D_o^s[u]\,] \;=\; u\,. \tag{2.40}$$

If $I_e^{1-s}[u] \in AC_{loc}(\mathbb{R})$, then

$$\frac{1}{(\sin(s\,\pi/2))^2}\; I_o^s\,[\,D_e^s[u]\,] \;=\; u\,. \tag{2.41}$$

If $I_o^{1-s}\,[I_e^s[u]] \in AC_{loc}(\mathbb{R})$, then

$$\frac{1}{(\cos(s\,\pi/2))^2}\; D_o^s\,[\,I_e^s[u]\,] \;=\; u\,. \tag{2.42}$$

If $I_e^{1-s}\,[I_o^s[u]] \in AC_{loc}(\mathbb{R})$, then

$$\frac{1}{(\sin(s\,\pi/2))^2}\; D_e^s\,[\,I_o^s[u]\,] \;=\; u\,. \tag{2.43}$$

Proof. By (2.28), (2.29), exploiting the associativity of convolution and the standard properties of the Fourier transform, we can perform the subsequent computations.

Assumption $I_o^{1-s}[u] \in AC_{loc}(\mathbb{R})$ entails the derivative d/dx appearing in D_o^s actually exists almost everywhere and coincides with the distributional derivative D, since $D\, I_o^{1-s}[u]$ is absolutely continuous with respect to the Lebesgue measure.

$$\begin{aligned}
&\mathcal{F}\left\{ \frac{1}{(\cos(s\,\pi/2))^2}\; I_e^s\,[\,D_o^s[u]\,]\right\}(\xi) \;= \\
&\quad= \frac{1}{(\cos(s\,\pi/2))^2}\;\mathcal{F}\left\{ \frac{1}{4\,\Gamma(s)\,\Gamma(1-s)}\;\frac{d}{dx}\left(u * \frac{\operatorname{sign}(x)}{|x|^s}\right) * \frac{1}{|x|^{1-s}}\right\}(\xi) \\
&\quad= \frac{i\,\xi\,\widehat{u}(\xi)}{4\;(\cos(s\,\pi/2))^2\;\Gamma(s)\,\Gamma(1-s)}\;\mathcal{F}\left\{\frac{\operatorname{sign}(x)}{|x|^s}\right\}(\xi)\;\mathcal{F}\left\{\frac{1}{|x|^{1-s}}\right\}(\xi) \\
&\quad= i\,\xi\,\widehat{u}(\xi)\;\frac{-i\;\operatorname{sign}(\xi)}{|\xi|^{1-s}}\;\frac{1}{|\xi|^s}\;=\;\widehat{u}(\xi)
\end{aligned}$$

and (2.40) follows by inverse Fourier transform.
Assumption $I_e^{1-s}[u] \in AC_{loc}(\mathbb{R})$ entails the derivative d/dx appearing in D_e^s actually exists almost everywhere and coincides with the distributional derivative D, since $D\, I_e^{1-s}[u]$ is absolutely continuous with respect to the Lebesgue measure.

$$\begin{aligned}
&\mathcal{F}\left\{ \frac{1}{(\sin(s\,\pi/2))^2}\; I_o^s\,[\,D_e^s[u]\,]\right\}(\xi) \;= \\
&\quad= \frac{1}{(\sin(s\,\pi/2))^2}\;\mathcal{F}\left\{ \frac{1}{4\,\Gamma(s)\,\Gamma(1-s)}\;\frac{d}{dx}\left(u * \frac{1}{|x|^s}\right) * \frac{\operatorname{sign}(x)}{|x|^{1-s}}\right\}(\xi) \\
&\quad= \frac{i\,\xi\,\widehat{u}(\xi)}{4\;(\sin(s\,\pi/2))^2\;\Gamma(s)\,\Gamma(1-s)}\;\mathcal{F}\left\{\frac{1}{|x|^s}\right\}(\xi)\;\mathcal{F}\left\{\frac{\operatorname{sign}(x)}{|x|^{1-s}}\right\}(\xi) \\
&\quad= i\,\xi\,\widehat{u}(\xi)\;\frac{1}{|\xi|^{1-s}}\;\frac{-i\;\operatorname{sign}(\xi)}{|\xi|^s}\;=\;\widehat{u}(\xi)
\end{aligned}$$

and (2.41) follows by inverse Fourier transform.
Assumption $I_o^{1-s}\,[I_e^s[u]] \in AC_{loc}(\mathbb{R})$ entails the derivative d/dx appearing in D_o^s actually exists almost everywhere and coincides with the distributional derivative D, since $D\, I_o^{1-s}\,[I_e^s[u]]$ is absolutely continuous with respect to the Lebesgue measure.

$$\begin{aligned}
&\mathcal{F}\left\{ \frac{1}{(\cos(s\,\pi/2))^2}\; D_o^s\,[\,I_e^s[u]\,]\right\}(\xi) \;= \\
&\quad= \frac{1}{(\cos(s\,\pi/2))^2}\;\mathcal{F}\left\{ \frac{1}{4\,\Gamma(s)\,\Gamma(1-s)}\;\frac{d}{dx}\left(u * \frac{1}{|x|^{1-s}}\right) * \frac{\operatorname{sign}(x)}{|x|^s}\right\}(\xi) \\
&\quad= \frac{i\,\xi\,\widehat{u}(\xi)}{4\;(\cos(s\,\pi/2))^2\;\Gamma(s)\,\Gamma(1-s)}\;\mathcal{F}\left\{\frac{1}{|x|^{1-s}}\right\}(\xi)\;\mathcal{F}\left\{\frac{\operatorname{sign}(x)}{|x|^s}\right\}(\xi)
\end{aligned}$$

$$= \; i\,\xi\,\widehat{u}(\xi)\;\frac{1}{|\xi|^s}\;\frac{-i\;\mathrm{sign}(\xi)}{|\xi|^{1-s}} \;=\; \widehat{u}(\xi)$$

and (2.42) follows by inverse Fourier transform.
Assumption $I_o^{1-s}\,[I_o^s[u]] \in AC_{loc}(\mathbb{R})$ entails the derivative d/dx appearing in D_e^s actually exists almost everywhere and coincides with the distributional derivative D, since $D\,I_o^{1-s}\,[I_o^s[u]]$ is absolutely continuous with respect to the Lebesgue measure.

$$\begin{aligned}
&\mathcal{F}\left\{\frac{1}{(\sin(s\,\pi/2))^2}\;D_e^s\,[\,I_o^s[u]\,]\right\}(\xi) \;= \\
&\quad= \frac{1}{(\sin(s\,\pi/2))^2}\;\mathcal{F}\left\{\frac{1}{4\,\Gamma(s)\,\Gamma(1-s)}\;\frac{d}{dx}\left(u*\frac{\mathrm{sign}(x)}{|x|^{1-s}}\right)*\frac{1}{|x|^s}\right\}(\xi) \\
&\quad= \frac{i\,\xi\,\widehat{u}(\xi)}{4\,(\sin(s\,\pi/2))^2\;\Gamma(s)\,\Gamma(1-s)}\;\mathcal{F}\left\{\frac{\mathrm{sign}(x)}{|x|^{1-s}}\right\}(\xi)\;\mathcal{F}\left\{\frac{1}{|x|^s}\right\}(\xi) \\
&\quad= i\,\xi\,\widehat{u}(\xi)\;\frac{-i\;\mathrm{sign}(\xi)}{|\xi|^s}\;\frac{1}{|\xi|^{1-s}} \;=\; \widehat{u}(\xi)
\end{aligned}$$

and (2.43) follows by inverse Fourier transform. QED

For the sake of completeness here we provide also a direct proof that the one sided fractional derivative is the inverse of the corresponding fractional integral, in the framework of definitions extended to the whole real line: say (2.14),(2.15) and

$$D_+^s[u](x) = \frac{1}{\Gamma(1-s)}\,\frac{d}{dx}\left(u\,*\,\frac{H(x)}{|x|^s}\right), \tag{2.44}$$

$$D_-^s[u](x) = \frac{-1}{\Gamma(1-s)}\,\frac{d}{dx}\left(u\,*\,\frac{H(-x)}{|x|^s}\right). \tag{2.45}$$

Lemma 2.6. Assume $0 < s < 1$, $u \in L^1(\mathbb{R})$. Then

$$D_+^s\left[\,I_+^s[u]\,\right] \;=\; u \tag{2.46}$$

$$D_-^s\left[\,I_-^s[u]\,\right] \;=\; u\,. \tag{2.47}$$

If in addition $I_+^{1-s}[u] \in AC_{loc}(\mathbb{R})$, then

$$I_+^s\left[\,D_+^s[u]\,\right] \;=\; u\,. \tag{2.48}$$

If in addition $I_-^{1-s}[u] \in AC_{loc}(\mathbb{R})$, then

$$I_-^s\left[\,D_-^s[u]\,\right] \;=\; u\,. \tag{2.49}$$

Proof. By (2.28), (2.29), $H(x) = 1/2 + \mathrm{sign}(x)/2$ and $H(-x) = 1/2 - \mathrm{sign}(x)/2$

$$\mathcal{F}\left\{ \frac{H(x)}{|x|^s} \right\}(\xi) \ = \ \Gamma(1-s)\, \frac{\sin(s\pi/2) - i\, \cos(s\pi/2)\, \mathrm{sign}(\xi)}{|\xi|^{1-s}} \qquad 0 < s < 1, \tag{2.50}$$

$$\mathcal{F}\left\{ \frac{H(-x)}{|x|^s} \right\}(\xi) \ = \ \Gamma(1-s)\, \frac{\sin(s\pi/2) + i\cos(s\pi/2)\, \mathrm{sign}(\xi)}{|\xi|^{1-s}} \qquad 0 < s < 1. \tag{2.51}$$

By (2.50), (2.51), exploiting the associativity of convolution and the standard properties of the Fourier transform, we can perform the subsequent computations.
By the semigroup property of fractional integrals ((2.21) and Thm 2.5) in [19]) $I_+^{1-s}\left[I_+^s[u]\right] \ = \ I_+^1[u] \ = \ \int_a^x u(t)dt \ \in \ AC_{loc}(\mathbb{R})$. Hence the derivative d/dx appearing in representation (2.45) of D_+^s actually exists almost everywhere and coincides with the distributional derivative D, since $D\, I_+^{1-s}[u]$ is absolutely continuous with respect to the Lebesgue measure.

$$\begin{aligned}
&\mathcal{F}\big\{D_+^s\,\left[\,I_+^s[u]\,\right]\big\}\,(\xi) \ = \\
&\quad = \ \mathcal{F}\left\{\frac{1}{\Gamma(s)\,\Gamma(1-s)}\ \frac{d}{dx}\left(u * \frac{H(x)}{|x|^{1-s}}\right) * \frac{H(x)}{|x|^s}\right\}(\xi) \\
&\quad = \ \frac{i\,\xi\,\widehat{u}(\xi)}{\Gamma(s)\,\Gamma(1-s)}\ \mathcal{F}\left\{\frac{H(x)}{|x|^{1-s}}\right\}(\xi)\ \mathcal{F}\left\{\frac{H(x)}{|x|^s}\right\}(\xi) \\
&\quad = \ \frac{i\,\xi\,\widehat{u}(\xi)}{\Gamma(s)\,\Gamma(1-s)}\ \frac{\Gamma(s)\,\big(\cos(s\pi/2) - i\sin(s\pi/2)\,\mathrm{sign}(\xi)\,\big)}{|\xi|^s} \ \times \\
&\qquad\qquad \times\ \frac{\Gamma(1-s)\,\big(\sin(s\pi/2) - i\cos(s\pi/2)\,\mathrm{sign}(\xi)\,\big)}{|\xi|^{1-s}} \ = \\
&\quad = \ \frac{i\,\xi\,\widehat{u}(\xi)}{|\xi|}\ \Big(\sin(s\pi/2)\cos(s\pi/2)\big(1 - (\mathrm{sign}(\xi))^2\big) - i\ \mathrm{sign}(\xi)\Big) \ = \\
&\quad = \ \widehat{u}(\xi)
\end{aligned}$$

and (2.46) follows by inverse Fourier transform.
By the semigroup property of fractional integrals ((2.21) and Thm 2.5 in [19]) $I_-^{1-s}\left[I_-^s[u]\right] \ = \ I_-^1[u] \ = \ \int_x^b u(t)dt \ \in \ AC_{loc}(\mathbb{R})$. Hence the derivative d/dx appearing in representation (2.45) of D_+^s actually exists almost everywhere and coincides with the distributional derivative D, since $D\, I_-^{1-s}[u]$ is absolutely con-

tinuous with respect to the Lebesgue measure.

$$
\begin{aligned}
&\mathcal{F}\big\{D^s_-\,\big[\,I^s_-[u]\,\big]\big\}\,(\xi) \;= \\
&\quad= \mathcal{F}\left\{\frac{-1}{\Gamma(s)\,\Gamma(1-s)}\;\frac{d}{dx}\left(u * \frac{H(-x)}{|x|^{1-s}}\right) * \frac{H(-x)}{|x|^{s}}\right\}(\xi)\\
&\quad= \frac{-i\,\xi\,\widehat{u}(\xi)}{\Gamma(s)\,\Gamma(1-s)}\;\mathcal{F}\left\{\frac{H(x)}{|x|^{1-s}}\right\}(\xi)\;\mathcal{F}\left\{\frac{H(x)}{|x|^{s}}\right\}(\xi)\\
&\quad= \frac{-i\,\xi\,\widehat{u}(\xi)}{\Gamma(s)\,\Gamma(1-s)}\;\frac{\Gamma(s)\,\big(\cos(s\pi/2)+i\sin(s\pi/2)\,\mathrm{sign}(\xi)\,\big)}{|\xi|^{s}}\;\times\\
&\qquad\qquad\times\;\frac{\Gamma(1-s)\,\big(\sin(s\pi/2)+i\cos(s\pi/2)\,\mathrm{sign}(\xi)\,\big)}{|\xi|^{1-s}}\;=\\
&\quad= \frac{-i\,\xi\,\widehat{u}(\xi)}{|\xi|}\;\Big(\sin(s\pi/2)\cos(s\pi/2)\big(1-(\mathrm{sign}(\xi))^2\big)+\,i\;\mathrm{sign}(\xi)\Big)\;=\\
&\quad= \;\widehat{u}(\xi)
\end{aligned}
$$

and (2.47) follows by inverse Fourier transform.
Assumption $I^{1-s}_+[u] \in AC_{loc}(\mathbb{R})$ entails the derivative d/dx appearing in representation (2.44) of D^s_+ actually exists almost everywhere and coincides with the distributional derivative D, since $D\,I^{1-s}_+[u]$ is absolutely continuous with respect to the Lebesgue measure.

$$
\begin{aligned}
&\mathcal{F}\big\{I^s_+\,\big[\,D^s_+[u]\,\big]\big\}\,(\xi) \;= \\
&\quad= \mathcal{F}\left\{\frac{1}{\Gamma(s)\,\Gamma(1-s)}\;\frac{d}{dx}\left(u * \frac{H(x)}{|x|^{s}}\right) * \frac{H(x)}{|x|^{1-s}}\right\}(\xi)\\
&\quad= \frac{i\,\xi\,\widehat{u}(\xi)}{\Gamma(s)\,\Gamma(1-s)}\;\mathcal{F}\left\{\frac{H(x)}{|x|^{s}}\right\}(\xi)\;\mathcal{F}\left\{\frac{H(x)}{|x|^{1-s}}\right\}(\xi)\\
&\quad= \frac{i\,\xi\,\widehat{u}(\xi)}{\Gamma(s)\,\Gamma(1-s)}\;\frac{\Gamma(1-s)\,\big(\sin(s\pi/2)-i\cos(s\pi/2)\,\mathrm{sign}(\xi)\,\big)}{|\xi|^{1-s}}\;\times\\
&\qquad\qquad\times\;\frac{\Gamma(s)\,\big(\cos(s\pi/2)-i\sin(s\pi/2)\,\mathrm{sign}(\xi)\,\big)}{|\xi|^{s}}\;=\\
&\quad= \frac{i\,\xi\,\widehat{u}(\xi)}{|\xi|}\;\Big(\sin(s\pi/2)\cos(s\pi/2)\big(1-(\mathrm{sign}(\xi))^2\big)-\,i\;\mathrm{sign}(\xi)\Big)\;=\\
&\quad= \;\widehat{u}(\xi)
\end{aligned}
$$

and (2.48) follows by inverse Fourier transform.
Assumption $I^{1-s}_-[u] \in AC_{loc}(\mathbb{R})$ entails the derivative d/dx appearing in representation (2.45) of D^s_- actually exists almost everywhere and coincides with the distributional derivative D, since $D\,I^{1-s}_-[u]$ is absolutely continuous with

respect to the Lebesgue measure.

$$
\begin{aligned}
&\mathcal{F}\left\{ I_-^s \left[D_-^s[u] \right] \right\}(\xi) \;= \\
&\quad = \mathcal{F}\left\{ \frac{-1}{\Gamma(s)\,\Gamma(1-s)} \, \frac{d}{dx} \left(u * \frac{H(-x)}{|x|^s} \right) * \frac{H(-x)}{|x|^{1-s}} \right\}(\xi) \\
&\quad = \frac{-i\,\xi\,\widehat{u}(\xi)}{\Gamma(s)\,\Gamma(1-s)} \, \mathcal{F}\left\{ \frac{H(-x)}{|x|^s} \right\}(\xi) \; \mathcal{F}\left\{ \frac{H(-x)}{|x|^{1-s}} \right\}(\xi) \\
&\quad = \frac{-i\,\xi\,\widehat{u}(\xi)}{\Gamma(s)\,\Gamma(1-s)} \, \frac{\Gamma(1-s)\left(\sin(s\pi/2) + i\cos(s\pi/2)\,\mathrm{sign}(\xi) \right)}{|\xi|^{1-s}} \times \\
&\qquad\qquad \times \frac{\Gamma(s)\left(\cos(s\pi/2) + i\sin(s\pi/2)\,\mathrm{sign}(\xi) \right)}{|\xi|^{s}} = \\
&\quad = \frac{-i\,\xi\,\widehat{u}(\xi)}{|\xi|} \left(\sin(s\pi/2)\cos(s\pi/2)\left(1-(\mathrm{sign}(\xi))^2\right) + i\ \mathrm{sign}(\xi) \right) = \\
&\quad = \widehat{u}(\xi)
\end{aligned}
$$

and (2.49) follows by inverse Fourier transform. QED

Remark 2.2. We emphasize that, in contrast to Lemma 2.5, in the last Lemma both conditions $I_+^{1-s}\left[I_+^s[u]\right] = I_+^1[u] = \int_a^x u \in AC_{loc}(\mathbb{R})$ and $I_-^{1-s}\left[I_-^s[u]\right] = I_-^1[u] \in AC_{loc}(\mathbb{R})$, useful to achieve respectively (2.48) and (2.49), are automatically fulfilled here, due to semigroup property of $\sigma \to I_\pm^\sigma$. Whereas, $I_o^{1-s}\left[I_e^s[u]\right] \in AC_{loc}(\mathbb{R})$ and $I_e^{1-s}\left[I_o^s[u]\right] \in AC_{loc}(\mathbb{R})$, had to be assumed there, in order to achieve respectively (2.42), (2.43),

3 Basic properties of bilateral fractional derivatives

Results of the previous Section (mainly Lemmas 2.4 and 2.5) provide some hints to suitably define the operators representing the bilateral versions of Riemann-Liouville fractional derivatives and integrals.

Definition 3.1. (**bilateral Riemann-Liouville fractional integral of order** s)

$$
I^s[u] \;=\; \frac{1}{\cos(s\,\pi/2)}\, I_e^s[u] \;=\; \frac{1}{2\,\Gamma(s)\,\cos(s\pi/2)} \left(I_+^s[u] + I_-^s[u] \right)
$$

Definition 3.2.(**bilateral Riemann-Liouville fractional derivative of order** s)

$$
D^s[u] \;=\; \frac{1}{\cos(s\,\pi/2)}\, D_o^s[u] \;=\; \frac{1}{2\,\Gamma(s)\,\cos(s\pi/2)} \left(D_+^s[u] - D_-^s[u] \right)
$$

In Definitions 3.1 and 3.2 we made one conventional choice of possible coefficients.
Obviously, the alternative choice provided by subsequent definition (namely $\mathcal{I}^s$, $\mathcal{D}^s$ in place of I^s, D^s) would work as well.

Definition 3.3.

$$\mathcal{I}^s[u] \;=\; \frac{1}{\sin(s\,\pi/2)}\, I^s_o[u] \qquad \text{together with} \qquad \mathcal{D}^s[u] \;=\; \frac{1}{\sin(s\,\pi/2)}\, D^s_e[u]\,.$$

Theorem 3.1. Assume $0 < s < 1$, $u \in L^1(\mathbb{R})$.
If $\mathcal{I}^{1-s}_o[u] \in AC_{loc}(\mathbb{R})$, then

$$I^s\,[\,D^s[u]\,] \;=\; u\,. \tag{3.1}$$

If $I^{1-s}[u] \in AC_{loc}(\mathbb{R})$, then

$$\mathcal{I}^s\,[\,\mathcal{D}^s[u]\,] \;=\; u\,. \tag{3.2}$$

If $\mathcal{I}^{1-s}\,[I^s[u]] \in AC_{loc}(\mathbb{R})$, then

$$D^s\,[\,I^s[u]\,] \;=\; u\,. \tag{3.3}$$

If $I^{1-s}\,[\mathcal{I}^s[u]] \in AC_{loc}(\mathbb{R})$, then

$$\mathcal{D}^s_e\,[\,\mathcal{I}^s_o[u]\,] \;=\; u\,. \tag{3.4}$$

Proof. It is a straightforward consequence of Lemma 2.5, with notations introduced by Definitions 3.1, 3.2, 3.3. *QED*

Using classical results on the left-hand and the right-hand side RL integral [19] it is straightforward to get the following:

Proposition 3.1. For any $s \in (0,1)$, the following properties hold true.

(i) The fractional integral I^s is a continuous operator from $L^1(a,b)$ into $L^q(a,b)$ with $q \in [1, 1/(1-s)]$;

(ii) The fractional integral I^s is a continuous operator from $L^p(a,b)$ to $L^p(a,b)$:

$$\begin{aligned} &I^s : L^p(a,b) \to L^p(a,b) \quad \forall\, p \geq 1 \\ &\|I^s u\|_{L^p(a,b)} \;\leq\; C(a,b,s,p)\|u\|_{L^p(a,b)}\,. \end{aligned} \tag{3.5}$$

(iii) I^s is a continuous operator from $L^p(a,b)$ into $L^r(a,b)$ for every $p \in [1, 1/s)$ and $r \in [1, p/(1-sp))$;

(iv) For every $u \in L^p(a,b)$, with $p \geq 1$, we have

$$\lim_{s \to 0^+} \|I^s u - u\|_{L^p(a,b)} = 0\,. \tag{3.6}$$

Next theorem concern the mapping properties of fractional integral on Lebesgue and Hölder spaces.

Proposition 3.2. For any $s \in (0,1)$, we get

(i) For every $p > 1/s$ the fractional integral I^s is a continuous operator from $L^p(a,b)$ into $\mathcal{C}^{0,s-\frac{1}{p}}(a,b)$;

(ii) For $p = 1/s$ the fractional integral I^s is a continuous operator from $L^p(a,b)$ into $L^r(a,b)$ with $r \in [1,\infty)$;

(iii) the fractional integral I^s is a continuous operator from $L^\infty(a,b)$ into $\mathcal{C}^{0,s}(a,b)$.

Here $\mathcal{C}^{0,s}(a,b)$ denotes the space of Hölder continuous functions of order s:

$$\mathcal{C}^{0,s}(a,b) := \{u \mid \exists C > 0 \text{ s.t. } \forall x, y \in [a,b]\, |u(y) - u(x)| \leq C\, |y-x|^s\, \}.$$

The proofs of the two previous propositions are clear since the claims hold for I^s_+ and I^s_- respectively (see [19]: Corollary 2 p.56, Theorem 3.5 p.66, Theorem 3.6 p.67, paragraph 3.3 p.91 and Thorem 14.2 with $p = 1$). The previous proposition shows that the fractional integration improves the function regularity.

We get more precise results providing the function has an Hölder regularity using Theorem 3.1 of [19].

Proposition 3.3. Let $s,\ \alpha \in (0,1)$ and $u \in \mathcal{C}^{0,\alpha}(a,b)$ then the fractional integral has the form

$$I^s_+[u] = \frac{u(a)}{\Gamma(1+s)}(x-a)^s + \psi(x)\ ,$$

where $\psi \in \mathcal{C}^{0,\alpha+s}(a,b)$ if $\alpha + s < 1$ and $\psi \in \mathcal{C}^{1,\alpha+s-1}(a,b) \subset \mathcal{C}^1(a,b)$ if $\alpha + s > 1$.

Proposition 3.4. Let $s,\ \alpha \in (0,1)$ be such that $s < \alpha \leq 1$ and $u \in \mathcal{C}^{0,\alpha}(a,b)$; then there exists $\psi_+,\ \psi_- \in \mathcal{C}^{0,\alpha-s}(a,b)$such that $\psi_+(a) = 0,\ \psi_-(b) = 0$ and

$$D^s[u](x) = \frac{1}{2\Gamma(1-s)}\left(\frac{u(a)}{(x-a)^s} + \frac{u(b)}{(b-x)^s}\right) + \psi_+(x) + \psi_-(x)\ .$$

Therefore $D^s u$ exists for every $s \in [0,\alpha)$ and $D^s u \in \mathcal{C}^{0,\alpha-s}(a,b)$ for every u such that $u(a) = u(b) = 0$.

Proof. The result comes directly from [19], Lemma 13.1 p.239. QED

4 The bilateral fractional Sobolev space

To develop a satisfactory theory of Riemann-Liouville fractional Sobolev spaces we introduced suitable function spaces in [3], by defining the Fractional Sobolev spaces related to one-sided fractional derivatives as follows.

Definition 4.1. We recall the definitions of *Riemann-Liouville Fractional Sobolev spaces related to one-sided fractional derivatives*, as introduced in [3], here we confine to the case $p = 1$:

$$W_{+}^{s,1}(a,b) := \{u \in L^1(a,b) \mid I_{+}^{1-s}[u] \in W_G^{1,1}(a,b)\ \}\,, \tag{4.1}$$

$$W_{-}^{s,1}(a,b) := \{u \in L^1(a,b) \mid I_{-}^{1-s}[u] \in W_G^{1,1}(a,b)\ \}\,. \tag{4.2}$$

Explicitly, the properties $u \in W_{\pm}^{s,1}(a,b)$ entail respectively that the distributional derivatives $D\left[I_{\pm}^{1-s}[u]\right]$ belong to $L^1(a,b)$.
Here we introduce the "even" and "odd" fractional Sobolev spaces.

Definition 4.2. *The even/odd Riemann-Liouville Fractional Sobolev spaces, respectively denoted by $W_e^{s,1}(a,b)$ and $W_o^{s,1}(a,b)$, are defined as follows*

$$W_{e}^{s,1}(a,b) := \{u \in L^1(a,b) \mid I_{e}^{1-s}[u] \in W_G^{1,1}(a,b)\ \}\,. \tag{4.3}$$

$$W_{o}^{s,1}(a,b) := \{u \in L^1(a,b) \mid I_{o}^{1-s}[u] \in W_G^{1,1}(a,b)\ \}\,, \tag{4.4}$$

Next, we define the bilateral Riemann-Liouville Fractional Sobolev spaces, with the aim to achieve a symmetric framework.

Definition 4.3. Bilateral Riemann-Liouville Fractional Sobolev space. For every $s \in (0,1)$, we set $W^{s,1}(a,b) = W_{+}^{s,1}(a,b) \cap W_{-}^{s,1}(a,b)$, that is:

$$W^{s,1}(a,b) := \{u \in L^1(a,b) \mid I_{+}^{1-s}[u] \in W_G^{1,1}(a,b) \text{ and } I_{-}^{1-s}[u] \in W_G^{1,1}(a,b)\}. \tag{4.5}$$

The even/odd Riemann-Liouville Fractional Sobolev spaces are the object of study in forthcoming paper [15].
It may appear surprising to choose such a definition since a natural one apparently would be, referring to Definition 3.1:

$$\widetilde{W^{s,1}}(a,b) := \{u \in L^1(a,b) \mid I^{1-s}[u] \in W_G^{1,1}(a,b)\}. \tag{4.6}$$

Though, at our knowlwdge, the space $\widetilde{W^{s,1}}$ does not allow to recover suitable representability results, analogous to the ones implied by Definition 2.4 of [3].
However, referring to Definition 4.3, by (2.10) and (2.11) we get

$$W^{s,1}(a,b) \ = \ W_{+}^{s,1}(a,b) \cap W_{-}^{s,1}(a,b) \ = \ W_{o}^{s,1}(a,b) \cap W_{e}^{s,1}(a,b)\,. \tag{4.7}$$

5 A fractional Bounded Variation space

In this section we need refinement of the Riemann-Liouville fractional derivative: simple substitution of the point-wise classical derivative with the distributional derivative

Definition 5.1. (**distributional Riemann-Liouville fractional derivative**) Assume $u \in L^1(a,b)$ and $0 < s < 1$.
The left Riemann-Liouville derivative of u at $x \in [a,b]$ is defined by

$$ {}_{RL}D^s_{a+}[u](x) \;=\; D_x \; {}_{RL}I^{1-s}_{a+}[u](x) \;=\; \frac{1}{\Gamma(1-s)}\, D_x \int_a^x \frac{u(t)}{(x-t)^s} dt \tag{5.1} $$

at values x such that it exists.
Similarly, we may define the right Riemann-Liouville derivative of u at $x \in [a,b]$ as

$$ {}_{RL}D^s_{b-}[u](x) \;=\; -\,D_x \; {}_{RL}I^{1-s}_{b-}[u](x) \;=\; \frac{-1}{\Gamma(1-s)}\, D_x \int_x^b \frac{u(t)}{(t-x)^s} dt \tag{5.2} $$

at values x such that it exists.

Remark 5.1. We emphasize that as long as these derivatives are evaluated on absolutely continuous functions, as it was done in previous section, then Definition 5.1 (based on Definitions 5.1 and 5.2) turns out to be equivalent to Definition 4.1 (which was based on Definition 2.2). For this reason we keep the same notations (${}_{RL}D^s_{a+}$, ${}_{RL}D^s_{b-}$ and the corresponding short forms D^s_+, D^s_-). However, here we have evaluate it on functions of bounded variations, a setting where actually the two Definitions are different.

Next, inspired by [6] where the non symmetric spaces are studied also in the case of higher order derivatives, we introduce the bilateral Riemann-Liouville Bounded Variation space, with the aim to achieve a symmetric framework.

Definition 5.2. *The (bilateral) Riemann-Liouville Fractional Bounded Variation spaces.* For every $s \in (0,1)$, we set

$$ BV^s \;= BV^s_+ \cap BV^s_- \tag{5.3} $$

where

$$ BV^s_+ = \{u \in L^1(a,b) \mid I^{1-s}_+[u] \in BV(a,b)\}, $$
$$ BV^s_- = \{u \in L^1(a,b) \mid I^{1-s}_-[u] \in BV(a,b)\}. $$

The (bilateral) Riemann-Liouville Fractional Bounded Variation spaces are the object of study in the forthcoming paper [15].

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