Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **41** (2021) no. 2, 19–29. doi:10.1285/i15900932v41n2p19

L^r inequalities for the derivative of a polynomial

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Received: 17.2.2021; accepted: 5.8.2021.

Abstract. Let $p(z)$ be a polynomial of degree n having no zero in $|z| < k$, $k \leq 1$, then Govil [Proc. Nat. Acad. Sci., **50**, (1980), 50-52] proved

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k^n} \max_{|z|=1} |p(z)|,$$

provided $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on the circle $|z| = 1$, where

$$q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}.$$

In this paper, we not only obtain an integral mean inequality for the above inequality but also extend an improved version of it into L^r norm.

Keywords: Inequalities, Polynomials, Zeros, Maximum modulus, L^r norm

MSC 2020 classification: primary 30C15, secondary 30C10

1 Introduction

Let $p(z)$ be a polynomial of degree n. We define

$$\|p\|_r = \left\{ \int_0^{2\pi} \left| p(e^{i\theta}) \right|^r d\theta \right\}^{\frac{1}{r}}, \quad 0 < r < \infty. \tag{1.1}$$

If we let $r \to \infty$ in (1.1) and make use of the well-known fact from analysis (see [18],[19]) that

$$\lim_{r\to\infty} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| p(e^{i\theta}) \right|^r d\theta \right)^{\frac{1}{r}} = \max_{|z|=1} |p(z)|, \tag{1.2}$$

we can suitably denote

$$\|p\|_\infty = \max_{|z|=1} |p(z)|. \tag{1.3}$$

Similarly, we can define

$$\|p\|_0 = \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \log |p(e^{i\theta})| d\theta \right\},$$

and show that $\lim\limits_{r\to 0^+} \|p\|_r = \|p\|_0$. It would be of further interest that by taking limit as $r \to 0^+$ that the stated results on L^r norm inequalities holding for $r > 0$, hold for $r = 0$ as well.

The famous result of Bernstein [3] states that if $p(z)$ is a polynomial of degree n, then

$$\|p'\|_\infty \leq n\|p\|_\infty. \tag{1.4}$$

Inequality (1.4) can be obtained by letting $r \to \infty$ in the inequality

$$\|p'\|_r \leq n\|p\|_r, \quad r > 0. \tag{1.5}$$

Inequality (1.5) was proved by Zygmund [20] for $r \geq 1$ and by Arestov [1] for $0 < r < 1$.

If we restrict to the class of polynomials having no zero in $|z| < 1$, then inequalities (1.4) and (1.5) can be respectively improved as

$$\|p'\|_\infty \leq \frac{n}{2}\|p\|_\infty, \tag{1.6}$$

$$\|p'\|_r \leq \frac{n}{\|1+z\|_r}\|p\|_r, \quad r > 0. \tag{1.7}$$

Inequality (1.6) was conjectured by Erdös and later verified by Lax [13] whereas inequality (1.7) was proved by de-Bruijn [5] for $r \geq 1$ and by Rahman and Schmeisser [16] for $0 < r < 1$.

As a generalization of (1.6), Malik [14] proved that if $p(z)$ is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$, then

$$\|p'\|_\infty \leq \frac{n}{1+k}\|p\|_\infty, \tag{1.8}$$

whereas, under the same hypotheses of the polynomial $p(z)$, Govil and Rahman [11] extended inequality (1.8) to L^r norm by showing that

$$\|p^{'}\|_r \leq \frac{n}{\|z+k\|_r}\|p\|_r, \quad r \geq 1. \tag{1.9}$$

Gardner and Weems [9] and independently by Rather [17] showed that inequality (1.9) holds true for $0 < r < 1$ as well.

For the class of polynomials $p(z)$ of degree n having no zero in $|z| < k$, $k \leq 1$, the precise upper bound estimate for maximum of $|p'(z)|$ on $|z| = 1$, in general, does not seem to be easily obtainable. For quite sometime, it was believed that if $p(z)$ has no zero in $|z| < k$, $k \leq 1$, then the inequality analogous to (1.8) should be

$$\|p'\|_\infty \leq \frac{n}{1+k^n}\|p\|_\infty, \tag{1.10}$$

untill E.B. Saff gave the example $p(z) = \left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)$ to counter this belief.

There are many extensions of inequality (1.9) (see Chan and Malik [6], Dewan and Bidkham [7], and Dewan and Mir [8]). However, for the class of polynomials having no zero in $|z| < k$, $k \leq 1$, Govil [10] proved inequality (1.10) with extra condition.

Theorem 1. *If $p(z)$ is a polynomial of degree n having no zero in $|z| < k$, $k \leq 1$, then*

$$\|p'\|_\infty \leq \frac{n}{1+k^n}\|p\|_\infty, \tag{1.11}$$

provided $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on the circle $|z| = 1$, where

$$q(z) = z^n\overline{p\left(\frac{1}{\bar{z}}\right)}. \tag{1.12}$$

In this paper, we shall prove the following more general result which as a special case gives inequality (1.11). In fact, we prove

Theorem 2. *If $p(z)$ is a polynomial of degree n having no zero in $|z| < k$, $k \leq 1$, then for every $r > 0$,*

$$k^n n\|p\|_r \leq \|z + k^n\|_r \left\{n\|p\|_\infty - \|p'\|_\infty\right\}, \tag{1.13}$$

provided $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on the circle $|z| = 1$, where

$$q(z) = z^n\overline{p\left(\frac{1}{\bar{z}}\right)}. \tag{1.14}$$

Further, we prove the following improved result which sharpens Theorem 2. More precisely, we obtain

Theorem 3. *If $p(z)$ is a polynomial of degree n having no zero in $|z| < k$, $k \leq 1$, then for every real or complex number α with $|\alpha| < 1$ and for every $r > 0$,*

$$k^n n \left\| z^n \overline{p(z)} + \alpha \frac{m}{k^n} \right\|_r \leq \|z + k^n\|_r \left\{ n \|p\|_\infty - \|p'\|_\infty \right\}, \tag{1.15}$$

provided $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on the circle $|z| = 1$, where

$$q(z) = z^n \overline{p\left(\frac{1}{\overline{z}}\right)} \tag{1.16}$$

and $m = \min_{|z|=k} |p(z)|$.

Letting $r \to \infty$ on both sides of (1.13), we readily get inequality (1.11) of Theorem 1.

Remark 1. Further, taking limit as $r \to \infty$ on both sides of (1.15), we get

$$k^n n \max_{|z|=1} \left| z^n \overline{p(z)} + \alpha \frac{m}{k^n} \right| \leq (1 + k^n) \left\{ n \max_{|z|=1} |p(z)| - \max_{|z|=1} |p'(z)| \right\}. \tag{1.17}$$

Suppose z_0 on $|z| = 1$ be such that $\max_{|z|=1} |p(z)| = |p(z_0)|$. Then, in particular,

$$\left| z_0^n \overline{p(z_0)} + \alpha \frac{m}{k^n} \right| \leq \max_{|z|=1} \left| z^n \overline{p(z)} + \alpha \frac{m}{k^n} \right|. \tag{1.18}$$

Now we can choose the argument of α suitably such that

$$\left| z_0^n \overline{p(z_0)} + \alpha \frac{m}{k^n} \right| = |p(z_0)| + |\alpha| \frac{m}{k^n}. \tag{1.19}$$

Using (1.19) to (1.18), we have

$$|p(z_0)| + |\alpha| \frac{m}{k^n} \leq \max_{|z|=1} \left| z^n \overline{p(z)} + \alpha \frac{m}{k^n} \right|. \tag{1.20}$$

On combining (1.17) and (1.20), we have

$$k^n n \left\{ \max_{|z|=1} |p(z)| + |\alpha| \frac{m}{k^n} \right\} \leq (1 + k^n) \left\{ n \max_{|z|=1} |p(z)| - \max_{|z|=1} |p'(z)| \right\}, \tag{1.21}$$

which implies

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1 + k^n} \left\{ \max_{|z|=1} |p(z)| - |\alpha| m \right\}. \tag{1.22}$$

If we take limit as $|\alpha| \to 1$ in (1.22), we get the following inequality proved by Aziz and Ahmad [2, Theorem 3].

Corollary 1. *If $p(z)$ is a polynomial of degree n having no zero in $|z| < k$, $k \leq 1$, then*

$$\|p'\|_\infty \leq \frac{n}{1+k^n}\left\{\|p\|_\infty - m\right\}, \tag{1.23}$$

provided $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on the circle $|z| = 1$, where

$$q(z) = z^n\overline{p\left(\frac{1}{\overline{z}}\right)} \tag{1.24}$$

and $m = \min\limits_{|z|=k}|p(z)|$.

Remark 2. For $\alpha = 0$, inequality (1.22) reduces to inequality (1.11).

2 Lemmas.

For the proofs of the theorems, we require the following lemmas. The first lemma is a special case of a result due to Govil and Rahman [11].

Lemma 1. *If $p(z)$ is a polynomial of degree n, then on $|z| = 1$*

$$|p'(z)| + |q'(z)| \leq n\max_{|z|=1}|p(z)|, \tag{2.1}$$

where

$$q(z) = z^n\overline{p\left(\frac{1}{\overline{z}}\right)}.$$

Lemma 2. *If $p(z)$ is a polynomial of degree n having no zero in $|z| < 1$, then for every $R \geq 1$ and every $r > 0$,*

$$\int_0^{2\pi}\left|p\left(Re^{i\theta}\right)\right|^r d\theta \leq (C_r)^r\int_0^{2\pi}\left|p\left(e^{i\theta}\right)\right|^r d\theta, \tag{2.2}$$

where

$$C_r = \frac{\left\{\int_0^{2\pi}\left|1+R^ne^{i\theta}\right|^r d\theta\right\}^{\frac{1}{r}}}{\left\{\int_0^{2\pi}\left|1+e^{i\theta}\right|^r d\theta\right\}^{\frac{1}{r}}}. \tag{2.3}$$

Lemma 2 was proved by Boas and Rahman [4] for $r \geq 1$ and by Rahman and Schmeisser [16] for $0 < r < 1$.

Lemma 3. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \geq 1$, then for every $r > 0$,*

$$n\|p\|_r \leq \|1 + k^n z\|_r \|p'\|_\infty, \tag{2.4}$$

where,

$$q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}.$$

Proof of Lemma 3. Since $p(z)$ has all its zeros in $|z| \leq k$, $k \geq 1$, the polynomial $E(z) = p(kz)$ has all its zeros in $|z| \leq 1$ and hence the polynomial $F(z) = z^n \overline{E(\frac{1}{\bar{z}})}$ has all its zeros in $|z| \geq 1$. If z_ν, $\nu = 1, 2, 3, ..., n$ are the zeros of $F(z)$, then obviously $|z_\nu| \geq 1$, $1 \leq \nu \leq n$ and

$$\frac{zF'(z)}{F(z)} = \sum_{v=1}^{n} \frac{z}{z - z_v}, \tag{2.5}$$

so that for points $e^{i\theta}$, $0 \leq \theta < 2\pi$, for which $F(e^{i\theta}) \neq 0$, we have

$$Re\left(\frac{e^{i\theta}F'(e^{i\theta})}{F(e^{i\theta})}\right) = \sum_{\nu=1}^{n} Re\left(\frac{e^{i\theta}}{e^{i\theta} - z_\nu}\right) \leq \frac{n}{2}, \tag{2.6}$$

which gives

$$\left|\frac{e^{i\theta}F'\left(e^{i\theta}\right)}{nF\left(e^{i\theta}\right)}\right| \leq \left|1 - \frac{e^{i\theta}F'\left(e^{i\theta}\right)}{nF\left(e^{i\theta}\right)}\right|, \tag{2.7}$$

for points $e^{i\theta}$, $0 \leq \theta < 2\pi$, for which $F\left(e^{i\theta}\right) \neq 0$.

Inequality (2.7) is equivalent to

$$\left|F'\left(e^{i\theta}\right)\right| \leq \left|nF\left(e^{i\theta}\right) - e^{i\theta}F'\left(e^{i\theta}\right)\right|, \tag{2.8}$$

for points $e^{i\theta}$, $0 \leq \theta < 2\pi$, for which $F\left(e^{i\theta}\right) \neq 0$. Also inequality (2.8) trivially holds for the points $e^{i\theta}$, $0 \leq \theta < 2\pi$, for which $F\left(e^{i\theta}\right) = 0$. Hence it follows that for $|z| = 1$

$$|F'(z)| \leq \left|nF(z) - zF'(z)\right|. \tag{2.9}$$

Since $E(z)$ has all its zeros in $|z| \leq 1$, by Gauss Lucas Theorem $E'(z)$ has all its zeros in $|z| \leq 1$ and hence the polynomial

$$z^{n-1}\overline{E'\left(\frac{1}{\bar{z}}\right)} = nF(z) - zF'(z) \tag{2.10}$$

has all its zeros in $|z| \geq 1$.

From (2.10) it follows that the function

$$W(z)=\frac{zF'(z)}{nF(z)-zF'(z)} \tag{2.11}$$

is analytic in $|z|\leq 1$ with $|W(z)|\leq 1$ for $|z|\leq 1$ and $W(0)=0$, hence the function $1+W(z)$ is subordinate to the function $1+z$ for $|z|\leq 1$. Hence, by a well- known property of subordination [12], we have for every $r>0$,

$$\int_0^{2\pi}\left|1+W(e^{i\theta})\right|^r d\theta\leq\int_0^{2\pi}\left|1+e^{i\theta}\right|^r d\theta. \tag{2.12}$$

Now,

$$1+W(z)=\frac{nF(z)}{nF(z)-zF'(z)}. \tag{2.13}$$

For $|z|=1$, we have from (2.10)

$$|E'(z)|=\left|z^{n-1}\overline{E'\left(\frac{1}{\overline{z}}\right)}\right|=\left|nF(z)-zF'(z)\right|. \tag{2.14}$$

For $|z|=1$, using equation (2.14), relation (2.13) gives

$$n|F(z)|=|1+W(z)||nF(z)-zF'(z)|=|1+W(z)||E'(z)|. \tag{2.15}$$

Combining (2.12) and (2.15), we have for every $r>0$

$$n^r\int_0^{2\pi}\left|F\left(e^{i\theta}\right)\right|^r d\theta\leq\left(\int_0^{2\pi}\left|1+e^{i\theta}\right|^r d\theta\right)\left\{\max_{|z|=1}\left|E'(z)\right|\right\}^r. \tag{2.16}$$

Using inequality (2.2) of Lemma 2 to $F(z)$, we get for every $k\geq 1$ and every $r>0$

$$\int_0^{2\pi}|F(ke^{i\theta})|^r d\theta\leq (C_r)^r\int_0^{2\pi}|F(e^{i\theta})|^r d\theta, \tag{2.17}$$

where

$$C_r=\frac{\left(\int_0^{2\pi}\left|1+k^n e^{i\theta}\right|^r d\theta\right)^{\frac{1}{r}}}{\left(\int_0^{2\pi}|1+e^{i\theta}|^r d\theta\right)^{\frac{1}{r}}}.$$

Since $F(z) = z^n \overline{E\left(\frac{1}{\bar{z}}\right)} = z^n \overline{p\left(\frac{k}{\bar{z}}\right)}$, we have for $0 \le \theta < 2\pi$

$$\left|F\left(ke^{i\theta}\right)\right| = \left|k^n e^{in\theta}\overline{p(e^{i\theta})}\right| = k^n \left|p(e^{i\theta})\right|. \tag{2.18}$$

From (2.16), (2.17) and (2.18), it follows that for every $r > 0$

$$\begin{aligned} k^{nr} n^r \int_0^{2\pi} \left|p\left(e^{i\theta}\right)\right|^r d\theta \le & n^r \left(C_r\right)^r \int_0^{2\pi} \left|F\left(e^{i\theta}\right)\right|^r d\theta \\ & \le \left(\int_0^{2\pi} \left|1 + k^n e^{i\theta}\right|^r d\theta\right) \left\{\max_{|z|=1} |E'(z)|\right\}^r. \end{aligned} \tag{2.19}$$

Since $E'(z) = kp'(kz)$, we have

$$\max_{|z|=1} |E'(z)| = k \max_{|z|=1} \left|p'(kz)\right| = k \max_{|z|=k} |p'(z)|. \tag{2.20}$$

If $h(z)$ is a polynomial of degree n, then it is a simple deduction from the maximum modulus principle [15] that

$$\max_{|z|=R\ge 1} |h(z)| \le R^n \max_{|z|=1} |h(z)|. \tag{2.21}$$

Applying (2.21) to $p'(z)$ for $R = k \ge 1$ and using the result to (2.20), we have

$$\max_{|z|=1} |E'(z)| \le k^n \max_{|z|=1} |p'(z)|. \tag{2.22}$$

Using (2.22) to (2.19), we get

$$n^r \int_0^{2\pi} \left|p(e^{i\theta})\right|^r d\theta \le \left(\int_0^{2\pi} \left|1 + k^n e^{i\theta}\right|^r d\theta\right) \left\{\max_{|z|=1} |p'(z)|\right\}^r, \tag{2.23}$$

which is equivalent to

$$n \left(\int_0^{2\pi} \left|p(e^{i\theta})\right|^r d\theta\right)^{\frac{1}{r}} \le \left(\int_0^{2\pi} \left|1 + k^n e^{i\theta}\right|^r d\theta\right)^{\frac{1}{r}} \max_{|z|=1} |p'(z)|, \tag{2.24}$$

which completes the proof of Lemma 3. QED

3 Proofs of the Theorems

We first prove Theorem 3.

Proof of Theorem 3. Let $p(z)$ be a polynomial of degree n having no zero in $|z| < k$, $k \leq 1$. Then $q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$ has all its zeros in $|z| \leq 1/k$, $1/k \geq 1$. If m' denotes $\min\limits_{|z|=\frac{1}{k}} |q(z)|$, then

$$m' = \min_{|z|=\frac{1}{k}} |q(z)| = \frac{1}{k^n} \min_{|z|=k} |p(z)| = \frac{m}{k^n},$$

where $m = \min\limits_{|z|=k} |p(z)|$. Now, for every real or complex number α with $|\alpha| < 1$, it follows by Rouche's theorem that the polynomial

$$Q(z) = q(z) + \alpha m' = q(z) + \alpha \frac{m}{k^n}, \tag{3.1}$$

has all its zeros in $|z| \leq \frac{1}{k}$, $\frac{1}{k} \geq 1$. Applying Lemma 3 to the polynomial $Q(z)$, we have for every $r > 0$

$$n \left(\int_0^{2\pi} \left| Q\left(e^{i\theta}\right) \right|^r d\theta \right)^{\frac{1}{r}} \leq \left(\int_0^{2\pi} \left| 1 + \frac{e^{i\theta}}{k^n} \right|^r d\theta \right)^{\frac{1}{r}} \max_{|z|=1} |Q'(z)|,$$

which is equivalent to

$$k^n n \left(\int_0^{2\pi} \left| q(e^{i\theta}) + \alpha \frac{m}{k^n} \right|^r d\theta \right)^{\frac{1}{r}} \leq \left(\int_0^{2\pi} \left| k^n + e^{i\theta} \right|^r d\theta \right)^{\frac{1}{r}} \max_{|z|=1} |q'(z)|. \tag{3.2}$$

By Lemma 1, we have for $|z| = 1$

$$|p'(z)| + |q'(z)| \leq n \max_{|z|=1} |p(z)|. \tag{3.3}$$

Since $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on $|z| = 1$, let z_0 on $|z| = 1$ be such that $\max\limits_{|z|=1} |q'(z)| = |q'(z_0)|$, then

$$\max_{|z|=1} |p'(z)| = |p'(z_0)|.$$

Now, in particular (3.3) gives

$$|q'(z_0)| + |p'(z_0)| \leq n \max_{|z|=1} |p(z)|, \tag{3.4}$$

which implies

$$\max_{|z|=1}|q'(z)| \le n\max_{|z|=1}|p(z)| - \max_{|z|=1}|p'(z)|. \tag{3.5}$$

Using (3.5) to (3.2), we have

$$k^n n\left(\int_0^{2\pi}\left|q(e^{i\theta})+\alpha\frac{m}{k^n}\right|^r d\theta\right)^{\frac{1}{r}} \le \left(\int_0^{2\pi}\left|k^n+e^{i\theta}\right|^r d\theta\right)^{\frac{1}{r}} \times\left\{n\max_{|z|=1}|p(z)|-\max_{|z|=1}|p'(z)|\right\}. \tag{3.6}$$

From $q(z) = z^n\overline{p\left(\frac{1}{\overline{z}}\right)}$, we have

$$q\left(e^{i\theta}\right) = e^{ni\theta}\overline{p\left(e^{i\theta}\right)}. \tag{3.7}$$

Using (3.7) to (3.6), we have

$$k^n n\left(\int_0^{2\pi}\left|e^{ni\theta}\overline{p\left(e^{i\theta}\right)}+\alpha\frac{m}{k^n}\right|^r d\theta\right)^{\frac{1}{r}} \le \left(\int_0^{2\pi}\left|k^n+e^{i\theta}\right|^r d\theta\right)^{\frac{1}{r}} \times\left\{n\max_{|z|=1}|p(z)|-\max_{|z|=1}|p'(z)|\right\}, \tag{3.8}$$

which completes the proof of Theorem 3. QED

Proof of Theorem 2. The proof of this theorem follows on the same lines as that of Theorem 3 but instead of applying Lemma 3 to $Q(z)$ given by (3.1), we simply apply the same lemma to $q(z) = z^n\overline{p(\frac{1}{\overline{z}})}$ and we omit it. QED

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