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Existence and approximation of solutions for a class of degenerate elliptic equations with Neumann boundary condition

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Abstract. In this work we study the equation $Lu = f$, where L is a degenerate elliptic operator, with Neumann boundary condition in a bounded open set Ω. We prove the existence and uniqueness of weak solutions in the weighted Sobolev space $\mathrm{W}^{1,2}(\Omega,\omega)$ for the Neumann problem. The main result establishes that a weak solution of degenerate elliptic equations can be approximated by a sequence of solutions for non-degenerate elliptic equations

Keywords: Neumann problem, weighted Sobolev spaces

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Introduction

In this paper, we prove the existence and uniqueness of (weak) solutions in the weighted Sobolev space $W^{1,2}(\Omega,\omega)$ (see Definition 1) for the Neumann problem

$$(P)\begin{cases} Lu(x) = f(x) \ \text{ in } \ \Omega, \\ \langle A(x)\nabla u, \vec{\eta}(x)\rangle = 0 \ \text{ on } \ \partial\Omega, \end{cases}$$

where $\vec{\eta}(x) = (\eta_1(x), ..., \eta_n(x))$ is the outward unit normal to $\partial\Omega$ at x, $\langle .,. \rangle$ denotes the usual inner product in $\mathbb{R}^n$, the symbol ∇ indicates the gradient and L is a degenerate elliptic operator

$$Lu = -\sum_{i,j=1}^{n} D_j\big(a_{ij}\, D_i u\big) + \sum_{i=1}^{n} b_i\, D_i u + g\, u \, + \theta\, u\, \omega, \tag{0.1}$$

with $D_j = \dfrac{\partial}{\partial x_j}$, $(j = 1, ..., n)$, θ is positive a constant, the coefficients a_{ij}, b_i and g are measurable, real-valued functions, the coefficient matrix $A(x) = (a_{ij}(x))$ is symmetric and satisfies the *degenerate ellipticity condition*

$$\lambda|\xi|^2\omega(x) \le \langle A(x)\xi, \xi\rangle \le \Lambda|\xi|^2\omega(x) \tag{0.2}$$

for all $\xi\in\mathbb{R}^n$ and almost every $x\in\Omega\subset\mathbb{R}^n$ a bounded open set with piecewise smooth boundary (i.e., $\partial\Omega\in C^{0,1}$), ω is a weight function (that is, locally integrable and nonnegative function on $\mathbb{R}^n$), λ and Λ are positive constants.

Remark 1. In the case $A(x) = Id$ (where Id denotes the Identity matrix in $\mathbb{R}^n$) then the second equation of (P) is $\dfrac{\partial u}{\partial\vec{\eta}} = 0$ on $\partial\Omega$ namely the normal derivative of u vanishes on $\partial\Omega$. In the general case and with the summation convention the second equation of (P) can be written $a_{ij}(x)\dfrac{\partial u}{\partial x_j}\,\eta_i = 0$ on $\partial\Omega$. This expression is called *conormal derivative of u*.

The main purpose of this paper (see Theorem 2) is to establish that a weak solution $u\in W^{1,2}(\Omega,\omega)$ for the Neumann problem (P) can be approximated by a sequence of solutions of non-degenerate elliptic equations.

By a *weight*, we shall mean a locally integrable function ω on $\mathbb{R}^n$ such that $0<\omega(x)<\infty$ for a.e. $x\in\mathbb{R}^n$. Every weight ω gives rise to a measure on the measurable subsets on $\mathbb{R}^n$ through integration. This measure will be denoted by μ. Thus, $\mu(E)=\int_E\omega(x)\,dx$ for measurable sets $E\subset\mathbb{R}^n$.

In general, the Sobolev spaces $W^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [3], [6], [7] and [9]). In various applications, we can meet boundary value problems for elliptic equations whose ellipticity is disturbed in the sense that some degeneration or singularity appears. There are several very concrete problems from practice which lead to such differential equations, e.g from glaceology, non-Newtonian fluid mechanics, flows through porous media, differential geometry, celestial mechanics, climatology, petroleum extraction and reaction-diffusion problems (see some examples of applications of degenerate elliptic equations in [1] and [5]).

A class of weights, which is particularly well understood, is the class of A_p weights (or Muckenhoupt class) that was introduced by B. Muckenhoupt (see [13]). These weights have found many useful applications in harmonic analysis (see [14]). Another reason for studying A_p-weights is the fact that powers of distance to submanifolds of $\mathbb{R}^n$ often belong to A_p (see [12]). There are, in fact, many interesting examples of weights (see [11] for p-admissible weights).

1 Definitions and basic results

Let ω be a locally integrable nonnegative function in $\mathbb{R}^n$ and assume that

$0 < \omega < \infty$ almost everywhere. We say that ω belongs to the Muckenhoupt class A_p, $1 < p < \infty$, or that ω is an A_p-weight, if there is a constant C such that

$$\left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B \omega^{1/(1-p)}(x)\,dx\right)^{p-1} \leq C$$

for all balls $B \subset \mathbb{R}^n$, where $|.|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$. The infimum over all such constants C is called the A_p constant of ω and this constant will be denoted by $C_{p,\omega}$. If $1 < q \leq p$, then $A_q \subset A_p$ (see [10], [11] or [14] for more information about A_p-weights). The weight ω satisfies the doubling condition if $\mu(2B) \leq C\mu(B)$, for all balls $B \subset \mathbb{R}^n$, where $\mu(B) = \displaystyle\int_B \omega(x)\,dx$ and $2B$ denotes the ball with the same center as B which is twice as large (i.e., $2B(x;r) = B(x;2r)$). If $\omega \in A_p$, then ω is doubling (see Corollary 15.7 in [11]).

As an example of A_p-weight, the function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^n$, is in A_p if and only if $-n < \alpha < n(p-1)$ (see Corollary 4.4, Chapter IX in [14]).

Remark 2. (a) If $\omega \in A_p$, $1 < p < \infty$, then $\left(\dfrac{|E|}{|B|}\right)^p \leq C_{p,\omega}\,\dfrac{\mu(E)}{\mu(B)}$ for all measurable subset E of B (see 15.5 strong doubling property in [11]). Therefore $\mu(E) = 0$ if and only if $|E| = 0$; so there is no need to specify the measure when using the ubiquitous expression *almost everywhere* and *almost every*, both abbreviated a.e..
(b) If $\omega \in A_p$, $1 < p < \infty$, then there exist $\delta > 0$ and $C > 0$ depending only on n, p and $C_{p,\omega}$ such that, every time we have a measurable set E contained in a cube K_0, the following inequality holds: $\dfrac{\mu(E)}{\mu(K_0)} \leq C\left(\dfrac{|E|}{|K_0|}\right)^\delta$ (see Theorem 2.9, Chapter IV in [10] or Lemma 15.8 in [11]).

Let ω be a weight and $\Omega \subset \mathbb{R}^n$ be open. For $1 < p < \infty$ we define by $L^p(\Omega,\omega)$ the set of measurable function f defined on Ω for which

$$\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f(x)|^p\omega(x)dx\right)^{1/p} < \infty.$$

If $\omega \in A_p$, $1 < p < \infty$, then $\omega^{-1/(p-1)}$ is locally integrable and we have $L^p(\Omega,\omega) \subset L^1_{\rm loc}(\Omega)$ for every open set Ω (see Remark 1.2.4 in [15]). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$.

Definition 1. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $1 < p < \infty$, $k \in \mathbb{N}$ and $\omega \in A_p$. We define the weighted Sobolev space $W^{k,p}(\Omega,\omega)$ as the set of functions $u \in L^p(\Omega,\omega)$ with weak derivatives $D^\alpha u \in L^p(\Omega,\omega)$, $1 \leq |\alpha| \leq k$. The norm of u in

$W^{k,p}(\Omega,\omega)$ is defined by

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \left(\int_\Omega |u|^p \omega\, dx + \sum_{1\le|\alpha|\le k} \int_\Omega |D^\alpha u|^p \omega\, dx \right)^{1/p}. \tag{1.1}$$

If $\omega \in A_p$, then $W^{k,p}(\Omega,\omega)$ is the closure of $C^\infty(\Omega)$ with respect to the norm (1.1) (see Proposition 3.5 in [4] or Theorem 2.1.4 in [15]). The space $W_0^{k,p}(\Omega,\omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W_0^{k,p}(\Omega,\omega)} = \left(\sum_{1\le|\alpha|\le k} \int_\Omega |D^\alpha u|^p \omega\, dx \right)^{1/p}.$$

The spaces $W^{k,p}(\Omega,\omega)$ and $W_0^{k,p}(\Omega,\omega)$ are Banach spaces and for $k = 1$ and $p = 2$ the spaces $W^{1,2}(\Omega,\omega)$ and $W_0^{1,2}(\Omega,\omega)$ are Hilbert spaces.

It is evident that the weight function ω which satisfy $0 < c_1 \le \omega(x) \le c_2$ for $x \in \Omega$, give nothing new (the space $W^{k,p}(\Omega,\omega)$ is then identical with the classical Sobolev space $W^{k,p}(\Omega)$). Consequently, we shall be interested above all in such weight functions ω which either vanish somewhere in $\bar{\Omega}$ or increase to infinity (or both).

Remark 3. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with boundary $\partial\Omega \in C^{0,1}$. Using integration by parts, with $u, \varphi \in W^{1,2}(\Omega,\omega)$, if u satisfies the boundary condition in problem (P), we have (by Remark 1)

$$\begin{aligned} \int_\Omega \varphi\, Lu\, dx &= \int_\Omega a_{ij}\, D_i u\, D_j \varphi\, dx + \int_\Omega b_i\, \varphi\, D_i u\, dx + \int_\Omega g\, u\, \varphi\, dx \\ &+ \theta \int_\Omega u\, \varphi\, \omega\, dx + \underbrace{\int_{\partial\Omega} a_{ij} \frac{\partial u}{\partial x_j}\, \eta_i\, \varphi\, dx}_{=0} \\ &= B(u,\varphi) + \theta \int_\Omega u\, \varphi\, \omega\, dx. \end{aligned}$$

where $B(u,\varphi) = \displaystyle\int_\Omega a_{ij}\, D_i u\, D_j \varphi\, dx + \int_\Omega b_i\, \varphi\, D_i u\, dx + \int_\Omega g\, u\, \varphi\, dx$ is a bilinear form.

We introduce the following definition of weak solution of the Neumann problem (P).

Definition 2. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $\partial\Omega \in C^{0,1}$ and $f/\omega \in L^2(\Omega,\omega)$. A function $u \in W^{1,2}(\Omega,\omega)$ is a weak solution of the Neumann problem (P) if

$$\sum_{i,j=1}^{n}\int_{\Omega} a_{ij}D_iuD_j\varphi\,dx+\sum_{i=1}^{n}\int_{\Omega}\varphi b_i\,D_iu\,dx+\int_{\Omega} g\,u\,\varphi\,dx+\theta\int_{\Omega} u\,\varphi\,\omega\,dx$$
$$=\int_{\Omega} f\varphi\,dx \tag{1.2}$$

for all $\varphi\in W^{1,2}(\Omega,\omega)$.

Before we prove the main result of this section, Theorem 1, we need the following lemma.

Lemma 1. *Let $\Omega\subset\mathbb{R}^n$ be a bounded open set with boundary $\partial\Omega\in C^{0,1}$. Suppose that*
(**H1**) $\omega\in A_2$*;*
(**H2**) $f/\omega\in L^2(\Omega,\omega)$*;*
(**H3**) $b_i/\omega\in L^\infty(\Omega)$ *(i=1,...,n) and* $g/\omega\in L^\infty(\Omega)$*.*
Then, there exists a constant $\mathbf{C}>0$ *such that*

$$B(u,u)+\mathbf{C}\|u\|^2_{L^2(\Omega,\omega)}\geq\frac{\lambda}{2}\|u\|^2_{W^{1,2}(\Omega,\omega)}$$

for all $u\in W^{1,2}(\Omega,\omega)$*.*

Proof. Using (0.2) and (H3), for all $u\in W^{1,2}(\Omega,\omega)$, we have

$$\begin{aligned}
B(u,u)&=\int_{\Omega} a_{ij}D_iuD_ju\,dx+\int_{\Omega} b_iuD_iu\,dx+\int_{\Omega} u^2\,g\,dx\\
&\geq\lambda\int_{\Omega}|\nabla u|^2\omega\,dx+\int_{\Omega}\frac{b_i}{\omega}\,u\,D_iu\,\omega dx+\int_{\Omega}\frac{g}{\omega}\,u^2\,\omega\,dx\\
&\geq\lambda\int_{\Omega}|\nabla u|^2\omega\,dx-\left(\max\left\|\frac{b_i}{\omega}\right\|_{L^\infty(\Omega)}\right)\int_{\Omega}|u||D_iu|\omega\,dx\\
&\quad-\left\|\frac{g}{\omega}\right\|_{L^\infty(\Omega)}\int_{\Omega}u^2\omega\,dx\\
&\geq\lambda\int_{\Omega}|\nabla u|^2\omega\,dx-C_1\Big(\int_{\Omega}u^2\omega\,dx\Big)^{1/2}\Big(\int_{\Omega}|D_iu|^2\omega\,dx\Big)^{1/2}-C_2\int_{\Omega}u^2\omega\,dx\\
&\geq\lambda\int_{\Omega}|\nabla u|^2\omega\,dx-C_1\|u\|_{L^2(\Omega,\omega)}\|u\|_{\mathrm{W}^{1,2}(\Omega,\omega)}-C_2\|u\|^2_{L^2(\Omega,\omega)}
\end{aligned}\tag{1.3}$$

where $C_1=\max\left\|\frac{b_i}{\omega}\right\|_{L^\infty(\Omega)}$ $(i=1,...,n)$ and $C_2=\left\|\frac{g}{\omega}\right\|_{L^\infty(\Omega)}$. Using the elementary inequality $ab \leq \varepsilon a^2 + \dfrac{1}{4\varepsilon} b^2$, for all $\varepsilon > 0$, we obtain in (1.3),

$$\begin{aligned}
&B(u,u)\\
&\geq \lambda \int_\Omega |\nabla u|^2 \omega\, dx - C_1\left(\varepsilon \|u\|^2_{L^2(\Omega,\omega)} + \frac{1}{4\varepsilon}\|u\|^2_{W^{1,2}(\Omega,\omega)}\right) - C_2\|u\|^2_{L^2(\Omega,\omega)}\\
&= \lambda \int_\Omega |\nabla u|^2 \omega\, dx - (C_1\varepsilon + C_2)\|u\|^2_{L^2(\Omega,\omega)} - \frac{C_1}{4\varepsilon}\|u\|^2_{\mathrm{W}^{1,2}(\Omega,\omega)}\\
&= \lambda \int_\Omega |\nabla u|^2 \omega\, dx + \lambda \int_\Omega u^2 \omega\, dx - \lambda \int_\Omega u^2\omega\, dx - (C_1\varepsilon + C_2)\|u\|^2_{L^2(\Omega,\omega)}\\
&- \frac{C_1}{4\varepsilon}\|u\|^2_{\mathrm{W}^{1,2}(\Omega,\omega)}\\
&= \lambda\|u\|^2_{W^{1,2}(\Omega,\omega)} - (C_1\varepsilon + C_2 + \lambda)\|u\|^2_{L^2(\Omega,\omega)} - \frac{C_1}{4\varepsilon}\|u\|^2_{W^{1,2}(\Omega,\omega)}\\
&= \left(\lambda - \frac{C_1}{4\varepsilon}\right)\|u\|^2_{W^{1,2}(\Omega,\omega)} - (C_1\varepsilon + C_2 + \lambda)\|u\|^2_{L^2(\Omega,\omega)}. \qquad (1.4)
\end{aligned}$$

If $C_1 > 0$, we can choose $\varepsilon > 0$ such that

$$\lambda - \frac{C_1}{4\varepsilon} = \frac{\lambda}{2}, \text{that is, } \varepsilon = \frac{C_1}{2\lambda}.$$

Thus, in (1.4) we obtain

$$B(u,u) \geq \frac{\lambda}{2}\|u\|^2_{W^{1,2}(\Omega,\omega)} - \mathbf{C}\|u\|^2_{L^2(\Omega,\omega)},$$

where $\mathbf{C} = C_1\varepsilon + C_2 + \lambda = \dfrac{C_1^2}{2\lambda} + C_2 + \lambda > 0$. Therefore,

$$B(u,u) + \mathbf{C}\|u\|^2_{L^2(\Omega,\omega)} \geq \frac{\lambda}{2}\|u\|^2_{W^{1,2}(\Omega,\omega)}.$$

If $C_1 = 0$ (that is, $b_i(x) \equiv 0$, $i = 1, ..., n$) then (1.3) reduces to

$$\begin{aligned}
B(u,u) &\geq \lambda \int_\Omega |\nabla u|^2 \omega\, dx - C_2\|u\|^2_{L^2(\Omega,\omega)}\\
&= \lambda\left(\int_\Omega |u|^2\omega\, dx + \int_\Omega |\nabla u|^2 \omega\, dx\right)\\
&- (C_2 + \lambda)\|u\|^2_{L^2(\Omega,\omega)}\\
&\geq \frac{\lambda}{2}\|u\|^2_{\mathrm{W}^{1,2}(\Omega,\omega)} - \mathbf{C}\|u\|^2_{L^2(\Omega,\omega)}.
\end{aligned}$$

Therefore $B(u,u) + \mathbf{C}\|u\|^2_{L^2(\Omega,\omega)} \geq \dfrac{\lambda}{2}\|u\|^2_{W^{1,2}(\Omega,\omega)}$. $\boxed{QED}$

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with boundary $\partial\Omega \in C^{0,1}$. Suppose that (H1) - (H3) holds. Then, there exists a constant $\mathbf{C} > 0$ such that for all $\theta \geq \mathbf{C}$ the Neumann problem (P) has a unique solution $u \in W^{1,2}(\Omega, \omega)$. Moreover, we have that*

$$\|u\|_{W^{1,2}(\Omega,\omega)} \leq \frac{2}{\lambda}\left\|\frac{f}{\omega}\right\|_{L^2(\Omega,\omega)}.$$

Proof. We define bilinear form

$$\begin{aligned}\tilde{B} &: W^{1,2}(\Omega,\omega) \times W^{1,2}(\Omega,\omega) \to \mathbb{R}\\ \tilde{B}(u,\varphi) &= B(u,\varphi) + \theta \int_\Omega u\,\varphi\,\omega\,dx\end{aligned}$$

and a linear mapping

$$\begin{aligned}T &: W^{1,2}(\Omega,\omega) \to \mathbb{R}\\ T(\varphi) &= \int_\Omega f\,\varphi\,dx.\end{aligned}$$

Then $u \in W^{1,2}(\Omega,\omega)$ is a weak solution of the Neumann problem (P) if

$$\tilde{B}(u,\varphi) = T(\varphi), \quad \text{for all } \varphi \in W^{1,2}(\Omega,\omega).$$

Step 1. If $\theta \geq \mathbf{C}$ then $\tilde{B}$ is coercive. In fact, by Lemma 1 there exists a constant $\mathbf{C} > 0$ such that

$$B(u,u) + \mathbf{C}\|u\|^2_{L^2(\Omega,\omega)} \geq \frac{\lambda}{2}\|u\|_{\mathrm{W}^{1,2}(\Omega,\omega)}.$$

Hence, if $\theta \geq \mathbf{C}$, we have

$$\begin{aligned}\tilde{B}(u,u) &= B(u,u) + \theta \int_\Omega u^2 \omega\,dx\\ &= B(u,u) + \theta\|u\|^2_{L^2(\Omega,\omega)}\\ &\geq B(u,u) + \mathbf{C}\|u\|^2_{L^2(\Omega,\omega)}\\ &\geq \frac{\lambda}{2}\|u\|^2_{W^{1,2}(\Omega,\omega)}.\end{aligned}$$

Therefore, for $\theta \geq \mathbf{C}$, we have that

$$\tilde{B}(u,u) \geq \frac{\lambda}{2}\|u\|^2_{W^{1,2}(\Omega,\omega)}, \tag{1.5}$$

for all $u \in W^{1,2}(\Omega,\omega)$.

Step 2. $\tilde{B}$ is bounded. In fact, using that the coefficient matrix $A = (a_{ij})$ is symmetric, (0.2) and (H3), we obtain

$$
\begin{aligned}
&|\tilde{B}(u,\varphi)| \\
&\leq |B(u,\varphi)| + \theta\left|\int_\Omega u\varphi\,\omega\,dx\right| \\
&\leq \int_\Omega |\langle A\nabla u, \nabla\varphi\rangle|\,dx + \int_\Omega |b_i||\varphi||D_i u|\,dx + \int_\Omega |g||\varphi||u|\,dx \\
&+ \theta\int_\Omega |u||\varphi|\,\omega\,dx \\
&\leq \int_\Omega \langle A\nabla u, \nabla u\rangle^{1/2}\langle A\nabla\varphi, \nabla\varphi\rangle^{1/2}\,dx \\
&+ \int_\Omega \frac{|b_i|}{\omega}|\varphi||D_i u|\,\omega\,dx + \int_\Omega \frac{|g|}{\omega}|\varphi||u|\,\omega\,dx + \theta\int_\Omega |u||\varphi|\,\omega\,dx \\
&\leq \left(\int_\Omega \langle A\nabla u, \nabla u\rangle\,dx\right)^{1/2}\left(\int_\Omega \langle A\nabla\varphi, \nabla\varphi\rangle\,dx\right)^{1/2} \\
&+ \left(\max\left\|\frac{b_i}{\omega}\right\|_{L^\infty(\Omega)}\right)\left(\int_\Omega |\varphi|^2\omega dx\right)^{1/2}\left(\int_\Omega |D_i u|^2\omega\,dx\right)^{1/2} \\
&+ \left[\left\|\frac{g}{\omega}\right\|_{L^\infty(\Omega)}\left(\int_\Omega |u|^2\omega dx\right)^{1/2} + \theta\left(\int_\Omega |u|^2\omega dx\right)^{1/2}\right]\left(\int_\Omega |\varphi|^2\omega\,dx\right)^{1/2} \\
&\leq \Lambda\left(\int_\Omega |\nabla u|^2\omega\,dx\right)^{1/2}\left(\int_\Omega |\nabla\varphi|^2\omega\,dx\right)^{1/2} \\
&+ \left(\max\left\|\frac{b_i}{\omega}\right\|_{L^\infty(\Omega)} + \left\|\frac{g}{\omega}\right\|_{L^\infty(\Omega)} + \theta\right)\|u\|_{W^{1,2}(\Omega,\omega)}\|\varphi\|_{W^{1,2}(\Omega,\omega)} \\
&\leq \left(\Lambda + \max\left\|\frac{b_i}{\omega}\right\|_{L^\infty(\Omega)} + \left\|\frac{g}{\omega}\right\|_{L^\infty(\Omega)} + \theta\right)\|u\|_{W^{1,2}(\Omega,\omega)}\|\varphi\|_{W^{1,2}(\Omega,\omega)} \\
&= \tilde{C}\,\|u\|_{W^{1,2}(\Omega,\omega)}\|\varphi\|_{W^{1,2}(\Omega,\omega)},
\end{aligned}
$$

where $\tilde{C} = \left(\Lambda + \max\left\|\frac{b_i}{\omega}\right\|_{L^\infty(\Omega)} + \left\|\frac{g}{\omega}\right\|_{L^\infty(\Omega)} + \theta\right)$, for all $u, \varphi \in W^{1,2}(\Omega,\omega)$.

Step 3. The linear mapping T is bounded (that is, $T \in [W^{1,2}(\Omega,\omega)]^*$). In fact, using (H2), we have

$$
\begin{aligned}
|T(\varphi)| &\leq \int_\Omega |f||\varphi|dx \\
&= \int_\Omega \frac{|f|}{\omega}|\varphi|\,\omega\,dx
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left[\int_\Omega \left(\frac{|f|}{\omega}\right)^2 \omega\,dx\right]^{1/2} \left[\int_\Omega |\varphi|^2 \omega\,dx\right]^{1/2} \\
&\leq \left\|\frac{f}{\omega}\right\|_{L^2(\Omega,\omega)} \|\varphi\|_{W^{1,2}(\Omega,\omega)},
\end{aligned}
$$

for all $\varphi \in W^{1,2}(\Omega,\omega)$.

Therefore the bilinear form $\tilde{B}$ and the linear functional T satisfy the hypotheses of the Lax-Milgram Theorem. Thus, for every f, with $f/\omega \in L^2(\Omega,\omega)$, there is a unique solution $u \in W^{1,2}(\Omega,\omega)$ such that $\tilde{B}(u,\varphi) = T(\varphi)$ for all $\varphi \in W^{1,2}(\Omega,\omega)$, that is, u is a unique solution of the Neumann problem (P). In particular, by setting $\varphi = u$, we have $\tilde{B}(u,u) = \displaystyle\int_\Omega f\,u\,dx$. Using the definition of $\tilde{B}$, we obtain

$$
\begin{aligned}
\tilde{B}(u,u) &= B(u,u) + \theta \int_\Omega u^2 \omega\,dx \\
&= \int_\Omega \frac{f}{\omega}\, u\,\omega\,dx \\
&\leq \|u\|_{L^2(\Omega,\omega)} \|f/\omega\|_{L^2(\Omega,\omega)} \\
&\leq \|u\|_{W^{1,2}(\Omega,\omega)} \|f/\omega\|_{L^2(\Omega,\omega)}.
\end{aligned}
$$

Using (1.5), we obtain

$$
\begin{aligned}
\frac{\lambda}{2}\|u\|^2_{W^{1,2}(\Omega,\omega)} &\leq \tilde{B}(u,u) \\
&\leq \|u\|_{W^{1,2}(\Omega,\omega)} \|f/\omega\|_{L^2(\Omega,\omega)}.
\end{aligned}
$$

Therefore,

$$
\|u\|_{W^{1,2}(\Omega,\omega)} \leq \frac{2}{\lambda}\left\|\frac{f}{\omega}\right\|_{L^2(\Omega,\omega)}. \tag{1.6}
$$

$\boxed{QED}$

2 Approximation of solution

In this section we present our main result: the weak solution to the problem (P) can be approximated by a sequence of solutions for non-degenerate elliptic equations.

The following lemma can be proved in exactly the same way as Lemma 2.1 in [8] (see also, Lemma 3.1 and Lemma 4.13 in [2]). Our lemma provides a general approximation theorem for A_p weights $(1 < p < \infty)$ by means of weights which are bounded away from 0 and infinity and whose A_p-constants depend only on the A_p-constant of ω. Lemma 2 is the key point for Theorem 2, and the crucial point consists of showing that a weak limit of a sequence of solutions of approximate problems is in fact a solution of the original problem.

Lemma 2. *Let $\alpha, \beta > 1$ be given and let $\omega \in A_p$ $(1 < p < \infty)$, with A_p-constant $C(\omega, p)$ and let $a_{ij} = a_{ji}$ be measurable, real-valued functions satisfying*

$$\lambda\, \omega(x)|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda\, \omega(x)\, |\xi|^2, \tag{2.1}$$

for all $\xi \in \mathbb{R}^n$ and a.e. $x \in \Omega$. Then there exist weights $\omega_{\alpha\beta} \geq 0$ a.e. and measurable real-valued functions $a_{ij}^{\alpha\beta}$ such that the following conditions are met.
(i) $c_1(1/\beta) \leq \omega_{\alpha\beta} \leq c_2\, \alpha$ in Ω, where c_1 and c_2 depend only on ω and Ω.
(ii) There exist weights $\tilde{\omega}_1$ and $\tilde{\omega}_2$ such that $\tilde{\omega}_1 \leq \omega_{\alpha\beta} \leq \tilde{\omega}_2$, where $\tilde{\omega}_i \in A_p$ and $C(\tilde{\omega}_i, p)$ depends only on $C(\omega, p)$ $(i = 1, 2)$.
(iii) $\omega_{\alpha\beta} \in A_p$, with constant $C(\omega_{\alpha\beta}, p)$ depending only on $C(\omega, p)$ uniformly on α and β.
(iv) There exists a closed set $F_{\alpha\beta}$ such that $\omega_{\alpha\beta} \equiv \omega$ in $F_{\alpha\beta}$ and $\omega_{\alpha\beta} \sim \tilde{\omega}_1 \sim \tilde{\omega}_2$ in $F_{\alpha\beta}$ with equivalence constants depending on α and β (i.e., there are positive constants $c_{\alpha\beta}$ and $C_{\alpha\beta}$ such that $c_{\alpha\beta}\, \tilde{\omega}_i \leq \omega_{\alpha\beta} \leq C_{\alpha\beta}\, \tilde{\omega}_i$, $i = 1, 2$). Moreover, $F_{\alpha\beta} \subset F_{\alpha'\beta'}$ if $\alpha \leq \alpha'$, $\beta \leq \beta'$, and the complement of $\bigcup_{\alpha,\beta \geq 1} F_{\alpha\beta}$ has zero measure.
(v) $\omega_{\alpha\beta} \to \omega$ a.e. in $\mathbb{R}^n$ as $\alpha, \beta \to \infty$.
(vi) $\lambda\, \omega_{\alpha\beta}(x)\, |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}^{\alpha\beta}(x)\, \xi_i\xi_j \leq \Lambda\omega_{\alpha\beta}(x)\, |\xi|^2$, $\forall\, \xi \in \mathbb{R}^n$ and a.e. $x \in \Omega$, and $a_{ij}^{\alpha\beta}(x) = a_{ji}^{\alpha\beta}(x)$.
(vii) $a_{ij}^{\alpha\beta}(x) = a_{ij}(x)$ in $F_{\alpha\beta}$.

Proof. See [2], Lemma 3.1 or Lemma 4.13. $\boxed{QED}$

The main results of this paper are the following.

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with boundary $\partial\Omega \in C^{0,1}$. Suppose that*
(**H1**) *$\omega \in A_2$;*
(**H2***) *$f/\omega \in L^2(\Omega, \omega) \cap L^2(\Omega, \omega^3)$;*
(**H3**) *$b_i/\omega \in L^\infty(\Omega)$ (i=1,...,n) and $g/\omega \in L^\infty(\Omega)$.*
Then the unique solution $u \in W^{1,2}(\Omega, \omega)$ of problem (P) is the weak limit in $W^{1,2}(\Omega, \tilde{\omega}_1)$ of a sequence of solutions $u_m \in W^{1,2}(\Omega, \omega_m)$ of the problems

$$(P_m) \begin{cases} L_m u_m(x) = f_m(x), & \text{in } \Omega, \\ \langle A^m(x)\nabla u_m, \vec{\eta}(x)\rangle = 0, & \text{on } \partial\Omega, \end{cases}$$

with

$$L_m u_m = -\sum_{i,j=1}^{n} D_j(a_{ij}^{mm}\, D_i u_m) + \sum_{i=1}^{n} b_{mi}\, D_i u_m + g_m\, u_m + \theta\, u_m\, \omega_m,$$

$f_m = f\,(\omega_m/\omega)^{1/2}$, $g_m = g\,\omega_m/\omega$, $b_{mi} = b_i\,\omega_m/\omega$ *and* $\omega_m = \omega_{mm}$ *(where* ω_{mm}*,* a_{ij}^{mm} *and* $\tilde{\omega}_1$ *are as Lemma 2 and* $A^m(x) = \big(a_{ij}^{mm}(x)\big)$*).*

Proof. **Step 1.** First, if $f_m = f(\omega/\omega_m)^{-1/2}$, $g_m = g\,\omega_m/\omega$ and $b_{mi} = b_i\,\omega_m/\omega$, we note that

$$\begin{aligned}
&\left\|\frac{f_m}{\omega_m}\right\|_{L^2(\Omega,\omega_m)} = \left\|\frac{f}{\omega}\right\|_{L^2(\Omega,\omega)}, \quad \left\|\frac{g_m}{\omega_m}\right\|_{L^\infty(\Omega)} = \left\|\frac{g}{\omega}\right\|_{L^\infty(\Omega)} \\
&\left\|\frac{b_{mi}}{\omega_m}\right\|_{L^\infty(\Omega)} = \left\|\frac{b_i}{\omega}\right\|_{L^\infty(\Omega)}.
\end{aligned} \tag{2.2}$$

Then, if $u_m \in W^{1,2}(\Omega,\omega_m)$ is a solution of problem (P_m) we have (by (1.6))

$$\begin{aligned}
\|u_m\|_{W^{1,2}(\Omega,\omega_m)} &\leq \frac{2}{\lambda}\left\|\frac{f_m}{\omega_m}\right\|_{L^2(\Omega,\omega_m)} \\
&= \frac{2}{\lambda}\left\|\frac{f}{\omega}\right\|_{L^2(\Omega,\omega)} = C_3.
\end{aligned}$$

Using Lemma 2, $\tilde{\omega}_1 \leq \omega_m$, we obtain

$$\|u_m\|_{W^{1,2}(\Omega,\tilde{\omega}_1)} \leq \|u_m\|_{W^{1,2}(\Omega,\omega_m)} \leq C_3. \tag{2.3}$$

Consequently, $\{u_m\}$ is a bounded sequence in $W^{1,2}(\Omega,\tilde{\omega}_1)$. Therefore, there is a subsequence, again denoted by $\{u_m\}$, and $\tilde{u} \in W^{1,2}(\Omega,\tilde{\omega}_1)$ such that

$$u_m \rightharpoonup \tilde{u} \text{ in } L^2(\Omega,\tilde{\omega}_1), \tag{2.4}$$

$$\frac{\partial u_m}{\partial x_j} \rightharpoonup \frac{\partial \tilde{u}}{\partial x_j} \text{ in } L^2(\Omega,\tilde{\omega}_1), \tag{2.5}$$

$$u_m \to \tilde{u} \text{ a.e. in } \Omega, \tag{2.6}$$

where the symbol "$\rightharpoonup$" denotes weak convergence (see Theorem 1.31 in [11]).

Step 2. We have that $\tilde{u} \in W^{1,2}(\Omega,\omega)$. In fact, for $F_k = F_{kk}$ fixed (see Lemma 2), we have by (2.4) and (2.5), for all $\varphi \in W^{1,2}(\Omega,\tilde{\omega}_1)$, we obtain

$$\begin{aligned}
&\int_\Omega u_m \varphi\, \tilde{\omega}_1\, dx \to \int_\Omega \tilde{u}\, \varphi\, \tilde{\omega}_1\, dx, \\
&\int_\Omega D_i u_m D_i \varphi\, \tilde{\omega}_1\, dx \to \int_\Omega D_i \tilde{u}\, D_i \varphi\, \tilde{\omega}_1\, dx.
\end{aligned}$$

If $\psi \in L^2(\Omega,\omega)$, then $\psi\,\chi_{F_k} \in L^2(\Omega,\tilde{\omega_1})$ (since $\omega \sim \tilde{\omega_1}$ in F_k, i.e., there is a constant $c>0$ such that $\tilde{\omega}_1 \leq c\,\omega$ in F_k, and χ_E denotes the characteristic function of a measurable set $E\subset \mathbb{R}^n$). Consequently,

$$\int_\Omega u_m \varphi \chi_{F_k}\,\tilde{\omega}_1\,dx \to \int_\Omega \tilde{u}\,\varphi\,\chi_{F_k}\,\tilde{\omega}_1\,dx,$$
$$\int_\Omega D_i u_m\,\varphi\,\chi_{F_k}\,\tilde{\omega}_1\,dx \to \int_\Omega D_i\tilde{u}\,\varphi\,\chi_{F_k}\,\tilde{\omega}_1\,dx,$$

for all $\varphi \in L^2(\Omega,\omega)$, that is, the sequence $\{\dfrac{\partial u_m}{\partial x_i}\,\chi_{F_k}\}$ is weakly convergent to a function in $L^2(\Omega,\omega)$, again since $\omega \sim \tilde{\omega}_1$ on F_k. Therefore, we have

$$\begin{aligned}\|\,|\nabla\tilde{u}|\,\|^2_{L^2(F_k,\omega)} &= \int_{F_k} |\nabla\tilde{u}|^2\omega\,dx \\ &\leq \limsup_{m\to\infty}\int_{F_k} |\nabla u_m|^2\,\omega\,dx,\end{aligned}$$

and for $m\geq k$ we have $\omega=\omega_m$ in F_k. Hence, by (2.3), we obtain

$$\begin{aligned}\|\,|\nabla\tilde{u}|\,\|^2_{L^2(F_k,\omega)} &\leq \limsup_{m\to\infty}\int_{F_k} |\nabla u_m|^2\,\omega\,dx \\ &= \limsup_{m\to\infty}\int_{F_k} |\nabla u_m|^2\,\omega_m\,dx \\ &\leq \limsup_{m\to\infty}\int_{\Omega} |\nabla u_m|^2\omega_m\,dx \\ &\leq C_3^2.\end{aligned}$$

By the Monotone Convergence Theorem we obtain $\|\,|\nabla\tilde{u}|\,\|_{L^2(\Omega,\omega)} \leq C_3$. Analogously, $\|\tilde{u}\|_{L^2(\Omega,\omega)} \leq C_3$. Therefore, we have $\tilde{u}\in W^{1,2}(\Omega,\omega)$.

Step 3. We need to show that $\tilde{u}$ is a solution of problem (P), i.e, for every $\varphi\in W^{1,2}(\Omega,\omega)$ we have

$$\sum_{i,j=1}^n \int_\Omega a_{ij}\,D_i\tilde{u}\,D_j\varphi\ dx + \sum_{i=1}^n\int_\Omega b_i\,\varphi\,D_i\tilde{u}\,dx + \int_\Omega g\,\tilde{u}\,\varphi\,dx + \theta\int_\Omega \tilde{u}\,\varphi\,\omega\,dx$$
$$= \int_\Omega f\,\varphi\,dx.$$

Using the fact that u_m is a solution of (P_m), we have

$$\sum_{i,j=1}^n \int_\Omega a_{ij}^{mm}\,D_i u_m\,D_j\varphi\,dx + \sum_{i=1}^n\int_\Omega b_{mi}\,\varphi\,D_i u_m\,dx + \int_\Omega g_m\,u_m\,\varphi\,dx$$
$$+\,\theta\int_\Omega u_m\,\varphi\,\omega_m\,dx = \int_\Omega f_m\,\varphi\,dx,$$

for every $\varphi \in W^{1,2}(\Omega, \omega_m)$. Moreover, by Lemma 2 and (2.2), over $F_k = F_{kk}$ (for $m \geq k$) we have the following properties:
(i) $\omega = \omega_m$;
(ii) $f_m = f$, $g_m = g$ and $b_{mi} = b_i$;
(iii) $a_{ij}^{mm}(x) = a_{ij}(x)$.

For $\varphi \in W^{1,2}(\Omega, \omega)$ and $k > 0$(fixed), we define $G_1, G_2 : W^{1,2}(\Omega, \tilde{\omega}_1) \to \mathbb{R}$ by

$$G_1(u) = \sum_{i,j=1}^{n} \int_\Omega a_{ij}\, D_i u\; D_j\varphi\, \chi_{F_k}\, dx,$$
$$G_2(u) = \sum_{i=1}^{n} \int_\Omega \varphi\, b_i\, D_i u\, \chi_{F_k}\, dx + \int_\Omega g\, u\, \varphi\, \chi_{F_k}\, dx + \theta \int_\Omega u\, \varphi\, \omega\, \chi_{F_k}\, dx.$$

(a) We have that G_1 is linear and continuous functional. In fact, we have (by Lemma 2(iv)) $\omega \sim \tilde{\omega}_1$ in F_k (there is a constant $c > 0$ such that $\omega \leq c\,\tilde{\omega}_1$ in F_k). And by (0.2) we obtain

$$\begin{aligned}
|G_1(u)| &\leq \int_{F_k} |\langle A\nabla u, \nabla\varphi\rangle|\, dx \\
&\leq \int_{F_k} (\langle A\nabla u, \nabla u\rangle)^{1/2}\, (\langle A\nabla\varphi, \nabla\varphi\rangle)^{1/2}\, dx \\
&\leq \left(\int_{F_k} \langle A\nabla u, \nabla u\rangle\, dx\right)^{1/2} \left(\int_{F_k} \langle A\nabla\varphi, \nabla\varphi\rangle\, dx\right)^{1/2} \\
&\leq \Lambda\left(\int_{F_k} |\nabla u|^2\, \omega\, dx\right)^{1/2} \left(\int_{F_k} |\nabla\varphi|^2\, \omega\, dx\right)^{1/2} \\
&\leq \Lambda\left(\int_{F_k} c\, |\nabla u|^2 \tilde{\omega}_1\, dx\right)^{1/2} \left(\int_{\Omega} |\nabla\varphi|^2\, \omega\, dx\right)^{1/2} \\
&\leq \Lambda\, c^{1/2} \|\varphi\|_{W^{1,2}(\Omega,\omega)}\, \|u\|_{W^{1,2}(\Omega,\tilde{\omega}_1)}.
\end{aligned}$$

(b) We have that G_2 is linear and continuous functional. In fact,

$$\begin{aligned}
&|G_2(u)| \\
&\leq \sum_{i=1}^{n} \int_{F_k} |\varphi|\, |b_i|\, |D_i u|\, dx + \int_{F_k} |g|\, |u|\, |\varphi|\, dx + \theta \int_{F_k} |u|\, |\varphi|\, \omega\, dx \\
&\leq \sum_{i=1}^{n} \left\|\frac{b_i}{\omega}\right\|_{L^\infty(F_k)} \left(\int_{F_k} |D_i u|^2\, \omega\, dx\right)^{1/2} \left(\int_{F_k} |\varphi|^2\, \omega\, dx\right)^{1/2} \\
&+ \left\|\frac{g}{\omega}\right\|_{L^\infty(F_k)} \left(\int_{F_k} |u|^2\, \omega\, dx\right)^{1/2} \left(\int_{F_k} |\varphi|^2\, \omega\, dx\right)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&+\theta\Big(\int_{F_k}|u|^2\,\omega\,dx\Big)^{1/2}\Big(\int_{F_k}|\varphi|^2\,\omega\,dx\Big)^{1/2}\\
&\leq\Big(\max\Big\|\frac{b_i}{\omega}\Big\|_{L^\infty(\Omega)}\Big)\Big(c\int_{F_k}|D_iu|^2\,\tilde{\omega_1}\,dx\Big)^{1/2}\Big(\int_\Omega|\varphi|^2\,\omega\,dx\Big)^{1/2}\\
&+\Big\|\frac{g}{\omega}\Big\|_{L^\infty(\Omega)}\Big(c\int_{F_k}|u|^2\,\tilde{\omega_1}\,dx\Big)^{1/2}\Big(\int_\Omega|\varphi|^2\,\omega\,dx\Big)^{1/2}\\
&+\theta\Big(c\int_{F_k}|u|^2\,\tilde{\omega_1}\,dx\Big)^{1/2}\Big(\int_\Omega|\varphi|^2\,\omega\,dx\Big)^{1/2}\\
&\leq\Big(\max\Big\|\frac{b_i}{\omega}\Big\|_{L^\infty(\Omega)}+\Big\|\frac{g}{\omega}\Big\|_{L^\infty(\Omega)}+\theta\Big)\,c^{1/2}\ \|u\|_{W^{1,2}(\Omega,\tilde{\omega}_1)}\|\varphi\|_{W^{1,2}(\Omega,\omega)}.
\end{aligned}
$$

Using (a), (b), properties (i),(ii) and (iii), and that u_m is solution of (P_m), we obtain

$$
\begin{aligned}
&\sum_{i,j=1}^n\int_{F_k}a_{ij}\,D_i\tilde{u}\,D_j\varphi\,dx+\sum_{i=1}^n\int_{F_k}\varphi\,b_i\,D_i\tilde{u}\,dx+\int_{F_k}g\,\tilde{u}\,\varphi\,dx+\theta\int_{F_k}\tilde{u}\,\varphi\,\omega\,dx\\
&=\lim_{m\to\infty}\big(G_1(u_m)+G_2(u_m)\big)\\
&=\lim_{m\to\infty}\Big(\sum_{i,j=1}^n\int_{F_k}a_{ij}^{mm}\,D_iu_m\,D_j\varphi\,dx+\sum_{i=1}^n\int_{F_k}\varphi\,b_{mi}\,D_iu_m\,dx\\
&+\int_{F_k}g_m\,u_m\,\varphi\,dx+\ \theta\int_{F_k}u_m\,\varphi\,\omega_m\,dx\Big)\\
&=\lim_{m\to\infty}\Big(\sum_{i,j=1}^n\int_{\Omega}a_{ij}^{mm}\,D_iu_m\,D_j\varphi\,dx+\sum_{i=1}^n\int_{\Omega}\varphi\,b_i\,D_iu_m\,dx+\int_\Omega g_m\,u_m\,\varphi\,dx\\
&+\theta\int_\Omega u_m\,\varphi\,\omega_m\,dx\\
&-\sum_{i,j=1}^n\int_{\Omega\cap F_k^c}a_{ij}^{mm}\,D_iu_m\,D_j\varphi\,dx-\sum_{i=1}^n\int_{\Omega\cap F_k^c}\varphi\,b_i\,D_iu_m\,dx\\
&-\int_{\Omega\cap F_k^c}g_m\,u_m\,\varphi\,dx-\ \theta\int_{\Omega\cap F_k^c}u_m\,\varphi\,\omega_m\,dx\Big)\\
&=\lim_{m\to\infty}\Big(\int_\Omega f_m\,\varphi\,dx-\sum_{i,j=1}^n\int_{\Omega\cap F_k^c}a_{ij}^{mm}\,D_iu_m\,D_j\varphi\,dx\\
&-\sum_{i=1}^n\int_{\Omega\cap F_k^c}\varphi\,b_i\,D_iu_m\,dx-\int_{\Omega\cap F_k^c}g_m\,u_m\,\varphi\,dx-\theta\int_{\Omega\cap F_k^c}u_m\,\varphi\,\omega_m\,dx,\qquad(2.7)
\end{aligned}
$$

where E^c denotes the complement of a set $E \subset \mathbb{R}^n$.

(I) By Lemma 2(v) we have $f_m = \dfrac{f\omega_m^{1/2}}{\omega^{1/2}} \to f$ a.e. in Ω. Since $\omega_m = \omega$ in F_k $(m \geq k)$ we also have

$$\begin{aligned}
\int_\Omega f_m^2\, \omega\, dx &= \int_\Omega f^2\, \omega_m\, dx \\
&= \int_{F_k} f^2\, \omega_m\, dx + \int_{\Omega \cap F_k^c} f^2\, \omega_m\, dx \\
&= \int_{F_k} f^2\, \omega\, dx + \int_{\Omega \cap F_k^c} f^2\, \omega_m\, dx \\
&\leq \int_\Omega f^2\, \omega\, dx + \int_{\Omega \cap F_k^c} f^2\, \omega_m\, dx \\
&= \int_\Omega \left(\frac{f}{\omega}\right)^2 \omega^3\, dx + \int_{\Omega \cap F_k^c} f^2\, \omega_m\, dx.
\end{aligned}$$

By Lemma 2(iv), we we know that $|\Omega \cap F_k^c| \to 0$ when $k \to \infty$. Then, for sufficiently large k we have

$$\int_{\Omega \cap F_k^c} f^2\, \omega_m\, dx \leq 1.$$

Therefore, for sufficiently large m and (H2*), we obtain

$$\int_\Omega f_m^2\, \omega\, dx \leq \int_\Omega \left(\frac{f}{\omega}\right)^2 \omega^3\, dx + 1 < \infty.$$

Hence the sequence $\{f_m\}$ is bounded in $L^2(\Omega, \omega)$. Then there is a subsequence, still denoted by $\{f_m\}$, and a function $\tilde{f}$ such that

$$\begin{aligned}
&f_m \rightharpoonup \tilde{f} \text{ in } L^2(\Omega, \omega), \\
&f_m \to \tilde{f} \text{ a.e. in } \Omega.
\end{aligned}$$

Since $f_m \to f$ a.e. in Ω, then $\tilde{f} = f$ a.e. in Ω. Therefore, for all $\varphi \in W^{1,2}(\Omega, \omega)$, we have

$$\int_\Omega f_m\, \varphi\, dx \to \int_\Omega f\, \varphi\, dx.$$

(II) Since the matrix $A^m(x) = (a_{ij}^{mm})(x)$ is symmetric, we have

$$|\langle A^m \nabla u_m, \nabla\varphi\rangle| \leq \langle A^m \nabla u_m, \nabla u_m\rangle^{1/2}\, \langle A^m \nabla\varphi, \nabla\varphi\rangle^{1/2}.$$

Then, by Lemma 2(vi) and (2.3), we obtain

$$\begin{aligned}
&\Big|\sum_{i,j=1}^{n}\int_{\Omega\cap F_k^c} a_{ij}^{mm} D_i u_m D_j\varphi\,dx\Big| \\
&\leq \int_{\Omega\cap F_k^c} |\langle A^m \nabla u_m, \nabla\varphi\rangle|\,dx \\
&\leq \Lambda\Big(\int_{\Omega\cap F_k^c} |\nabla u_m|^2\omega_m\,dx\Big)^{1/2}\Big(\int_{\Omega\cap F_k^c} |\nabla\varphi|^2\omega_m\,dx\Big)^{1/2} \\
&\leq \Lambda\|u_m\|_{W^{1,2}(\Omega,\omega_m)}\Big(\int_{\Omega\cap F_k^c} |\nabla\varphi|^2\,w_m\,dx\Big)^{1/2} \\
&\leq \Lambda\,C_3\,\Big(\int_{\Omega\cap F_k^c} |\nabla\varphi|^2\,w_m\,dx\Big)^{1/2}.
\end{aligned} \tag{2.8}$$

(III) By (H3), (2.2) and (2.3) we have

$$\begin{aligned}
&\Big|\int_{\Omega\cap F_k^c} \varphi\,b_{mi}\,D_i u_m\,dx\Big| \\
&\leq \int_{\Omega\cap F_k^c} |\varphi|\,|b_{mi}|\,|D_i u_m|\,dx \\
&\leq \Big\|\frac{b_{mi}}{\omega_m}\Big\|_{L^\infty(\Omega)}\Big(\int_{\Omega\cap F_k^c} |D_i u_m|^2\omega_m\,dx\Big)^{1/2}\Big(\int_{\Omega\cap F_k^c} |\varphi|^2\,\omega_m\,dx\Big)^{1/2} \\
&\leq \Big\|\frac{b_i}{\omega}\Big\|_{L^\infty(\Omega)}\|u_m\|_{W^{1,2}(\Omega,\omega_m)}\Big(\int_{\Omega\cap F_k^c} |\varphi|^2\omega_m\,dx\Big)^{1/2} \\
&\leq C_3\,\Big\|\frac{b_i}{\omega}\Big\|_{L^\infty(\Omega)}\Big(\int_{\Omega\cap F_k^c} |\varphi|^2\omega_m\,dx\Big)^{1/2},
\end{aligned} \tag{2.9}$$

and analogously

$$\Big|\int_{\Omega\cap F_k^c} g_m\,u_m\,\varphi\,dx\Big| \leq C_3\Big\|\frac{g}{\omega}\Big\|_{L^\infty(\Omega)}\Big(\int_{\Omega\cap F_k^c} |\varphi|^2\omega_m\,dx\Big)^{1/2}, \tag{2.10}$$

and

$$\Big|\int_{\Omega\cap F_k^c} u_m\,\varphi\,\omega_m\,dx\Big| \leq C_3\Big(\int_{\Omega\cap F_k^c} |\varphi|^2\omega_m\,dx\Big)^{1/2}. \tag{2.11}$$

Note now that $\omega_m \leq \tilde{\omega}_2$ and $\tilde{\omega}_2 \in A_2$ (by Lemma 2). Hence, by Remark 2(b), there exist $\delta > 0$ and $C > 0$ such that, if K_0 is a cube containing $\bar{\Omega}$ then

$$\begin{aligned} \mu_m(\Omega \cap F_k^c) &= \int_{\Omega \cap F_k^c} \omega_m(x)\, dx \\ &\leq \int_{\Omega \cap F_k^c} \tilde{\omega}_2(x)\, dx \\ &= \tilde{\mu}_2(\Omega \cap F_k^c) \\ &\leq C\, \tilde{\mu}_2(K_0) \left(\frac{|F_k^c|}{|K_0|} \right)^{\delta}, \end{aligned}$$

which is independent of m and tends to zero as $k \to \infty$ by Lemma 2(iv). Then

$$\lim_{k\to\infty} \left(\int_{\Omega\cap F_k^c} |\varphi|^2 \omega_m\, dx \right)^{1/2} = \lim_{k\to\infty} \left(\int_{\Omega\cap F_k^c} |\nabla\varphi|^2\, \omega_m\, dx \right)^{1/2} = 0,$$

and we obtain in (2.8), (2.9), (2.10) and (2.11)

$$\lim_{k\to\infty} \int_{\Omega\cap F_k^c} a_{ij}^{mm}(x) D_i u(x)\, D_j\varphi(x)\, dx = 0, \tag{2.12}$$

$$\lim_{k\to\infty} \int_{\Omega\cap F_k^c} \varphi\, b_{mi}\, D_i u_m\, dx\ = 0, \tag{2.13}$$

$$\lim_{k\to\infty} \int_{\Omega\cap F_k^c} g_m\, u_m\, \varphi\, dx = 0, \tag{2.14}$$

$$\lim_{k\to\infty} \int_{\Omega\cap F_k^c} u_m\, \varphi\, \omega_m\, dx = 0. \tag{2.15}$$

Therefore, by (2.7), (2.12), (2.13), (2.14) and (2.15) we conclude, when $k \to \infty$ (and $m \geq k$),

$$\sum_{i,j=1}^{n} \int_\Omega a_{ij} D_i \tilde{u}\, D_j\varphi\, dx + \sum_{i=1}^{n} \int_\Omega b_i\, \varphi\, D_i\tilde{u}\, dx + \int_\Omega g\, \tilde{u}\, \varphi\, dx + \theta \int_\Omega \tilde{u}\, \varphi\, \omega\, dx$$
$$= \int_\Omega f\, \varphi\, dx,$$

for all $\varphi \in W^{1,2}(\Omega, \omega)$, that is, $\tilde{u}$ is a solution of problem (P). Therefore, $u = \tilde{u}$ (by the uniqueness). $\boxed{QED}$

Example. Consider the domain $\Omega = \{(x, y)\in\mathbb{R}^2 :\ x^2 + y^2 < 1\}$, the weight function $\omega(x, y) = (x^2 + y^2)^{-1/2}$ and the coefficient matrix

$$A(x, y) = \begin{pmatrix} 2(x^2 + y^2)^{-1/2} & 0 \\ 0 & 4(x^2 + y^2)^{-1/2} \end{pmatrix}$$

We have for all $\xi\in\mathbb{R}^2$ and almost every $(x,y)\in\Omega$,

$$\frac{2}{(x^2+y^2)^{1/2}}|\xi|^2 \leq \langle A(x,y)\xi,\xi\rangle \leq \frac{4}{(x^2+y^2)^{1/2}}|\xi|^2.$$

If $(x,y)\in\partial\Omega = \{(x,y)\in\mathbb{R}^2 \ : x^2+y^2=1\}$, then $\vec{\eta}(x,y)=(x,y)$ is the unit outward normal to $\partial\Omega$. By Theorem 1 the Neumann problem

$$\begin{cases} Lu(x,y) = (x^2+y^2)^{-1/5}\cos(xy) \quad \text{on} \quad \Omega, \\ \langle A(x,y)\nabla u, \vec{\eta}\rangle = 0, \quad \text{on} \quad \partial\Omega, \end{cases}$$

where

$$\begin{aligned} Lu(x,y) \quad &= \quad -\left[\frac{\partial}{\partial x}\left(\frac{2}{(x^2+y^2)^{1/2}}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{4}{(x^2+y^2)^{1/2}}\frac{\partial u}{\partial y}\right)\right] \\ &+ \quad \frac{\cos(xy)}{(x^2+y^2)^{1/3}}\frac{\partial u}{\partial x} + \frac{\sin(xy)}{(x^2+y^2)^{1/4}}\frac{\partial u}{\partial y} \\ &+ \quad \frac{u(x,y)\sin(xy)}{(x^2+y^2)^{1/3}} + \theta\frac{u(x,y)}{(x^2+y^2)^{1/2}} \end{aligned}$$

has a unique solution $u\in W^{1,2}(\Omega,\omega)$ (if $\theta \geq 13/4$), and by Theorem 2 the solution u can be approximated by a sequence of solutions of non-degenerate elliptic equations.

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