Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **40** (2020) no. 2, 53–61. doi:10.1285/i15900932v40n2p53

Connections between Fibonacci and Pell numbers

Mircea Merca
Department of Mathematics, University of Craiova, 200585 Craiova, Romania
Academy of Romanian Scientist, Ilfov 3, Sector 5, Bucharest, Romania
mircea.merca@profinfo.edu.ro

Received: 28.12.2019; accepted: 4.5.2020.

Abstract. In this paper, we introduce several convolution identities that combine Fibonacci and Pell numbers. Congruence relations involving Fibonacci and Pell numbers follow as corollaries.

Keywords: Fibonacci numbers, Pell numbers, congruences

MSC 2020 classification: primary 11B39, 11A07, secondary 05A15

1 Introduction

Recall [1, 4] that the Fibonacci polynomials $F_n(x)$ are defined by the recurrence relation

$$F_n(x) = \begin{cases} 0, & \text{for } n = 0, \\ 1, & \text{for } n = 1, \\ x \cdot F_{n-1}(x) + F_{n-2}(x), & \text{for } n \geqslant 2, \end{cases} \tag{1.1}$$

having the generating function

$$\sum_{n=0}^{\infty} F_n(x) \cdot t^n = \frac{t}{1 - x \cdot t - t^2}. \tag{1.2}$$

On the other hand, the nth Fibonacci polynomial is explicitly given by the Binet-type formula

$$F_n(x) = \frac{\alpha(x)^n - \beta(x)^n}{\alpha(x) - \beta(x)}, \tag{1.3}$$

where

$$\alpha(x) = \frac{x + \sqrt{x^2+4}}{2} \qquad \text{and} \qquad \beta(x) = \frac{x - \sqrt{x^2+4}}{2}$$

are the solutions of

$$t^2 - x \cdot t - 1 = 0.$$

Several properties of these polynomials were derived in [2, 3].

The Fibonacci numbers F_n are recovered by evaluating these polynomials at $x = 1$, i.e.,

$$F_n = F_n(1).$$

On the other hand, the Pell numbers P_n are recovered by evaluating Fibonacci polynomials at $x = 2$, i.e.,

$$P_n = F_n(2).$$

For large values of n and nonnegative values of x, the $\alpha(x)^n$ term dominates the expression (1.3). So the Fibonacci numbers are approximately proportional to powers of the golden ratio $(1+\sqrt{5})/2$, analogous to the growth rate of Pell numbers as powers of the silver ratio $1+\sqrt{2}$.

Although the Fibonacci and Pell numbers are known from ancient times, they continue to intrigue the mathematical world with their beauty and applicability. These offer opportunities for exploration, conjecture, and problem-solving techniques, connecting various areas of mathematics. As we can see in [6, Chapter 17], the Fibonacci and Pell numbers coexist in perfect harmony, and share a number of charming properties. In this paper, motivated by these results, we use the method of generating functions to prove new connections between Fibonacci and Pell numbers.

Theorem 1.1. For $n \geqslant 0$,

$$F_{3n} = 2P_n + 2\sum_{k=1}^{n-1} F_{3k}P_{n-k}.$$

Theorem 1.2. For $n \geqslant 0$,

a) $F_{6n} = 4P_{2n} + 6\sum_{k=1}^{n-1} F_{6k}P_{2n-2k}$;

b) $F_{6n+3} = 2P_{2n+1} + 3\sum_{k=1}^{n} F_{6k}P_{2n+1-2k}$.

Theorem 1.3. For $n \geqslant 0$,

a) $F_{12n} = 12P_{4n} + 24\sum_{k=1}^{n-1} F_{12k}P_{4n-4k}$;

b) $6F_{12n+3} + F_{12n} = 12P_{4n+1} + 24\sum_{k=1}^{n} F_{12k}P_{4n+1-4k}$;

c) $4F_{12n+3} + F_{12n} = 4P_{4n+2} + 8\sum_{k=1}^{n} F_{12k}P_{4n+2-4k}$;

d) $30F_{12n+3} + 7F_{12n} = 12P_{4n+3} + 24\sum_{k=1}^{n} F_{12k}P_{4n+3-4k}$.

Seven congruence identities involving Fibonacci and Pell numbers can be easily obtained as consequences of these theorems.

Corollary 1.1. For $n \geqslant 0$,

a) $F_{3n} - 2P_n \equiv 0 \pmod 4$;

b) $F_{6n} - 4P_{2n} \equiv 0 \pmod{96}$;

c) $F_{6n+3} - 2P_{2n+1} \equiv 0 \pmod{24}$;

d) $F_{12n} - 12P_{4n} \equiv 0 \pmod{41472}$;

e) $6F_{12n+3} + F_{12n} - 12P_{4n+1} \equiv 0 \pmod{3456}$;

f) $4F_{12n+3} + F_{12n} - 4P_{4n+2} \equiv 0 \pmod{2304}$;

g) $30F_{12n+3} + 7F_{12n} - 12P_{4n+3} \equiv 0 \pmod{3456}$.

2 Proofs of theorems

Firstly we consider the multisection formula published by Simpson (see [5, Ch. 16], [7], [8, Ch. 4, S. 4.3] and [9]) as early as 1759:

$$\sum_{k\geqslant 0} a_{r+kn}t^{r+kn} = \frac{1}{n}\sum_{k=0}^{n-1} z^{-kr}f(z^k t), \qquad 0 \leqslant r < n, \tag{2.1}$$

where $z = e^{\frac{2\pi i}{n}}$ is the nth root of 1 and

$$f(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n + \cdots$$

is a finite or infinite series. Applying this formula to (1.1) with x replaced by 1, we obtain

$$\sum_{n=0}^{\infty} F_{3n}t^n = \frac{2t}{1-4t-t^2}.$$

By this relation, we deduce the generating function for F_{6n},

$$\sum_{n=0}^{\infty} F_{6n} t^n = \frac{8t}{1-18t+t^2}.$$

Similarly, the generating function for F_{12n}, namely

$$\sum_{n=0}^{\infty} F_{12n} t^n = \frac{144t}{1-322t+t^2},$$

follows as a bisection of the generating function for F_{6n}.

Proof of Theorem 1.1. We consider the sequences $\{a_n\}_{n\geqslant 0}$ and $\{b_n\}_{n\geqslant 0}$ defined by:

$$a_n = \begin{cases} 1, & \text{if } n=0, \\ F_{3n}, & \text{if } n>0, \end{cases}$$

$$b_n = \begin{cases} 1, & \text{if } n=0, \\ (-1)^{n+1}\cdot 2\cdot P_n, & \text{if } n>0. \end{cases}$$

The generating functions of these sequences are given by:

$$\sum_{n=0}^{\infty} a_n t^n = 1 + \frac{2t}{1-4t-t^2}, \tag{2.2}$$

$$\sum_{n=0}^{\infty} b_n t^n = 1 + \frac{2t}{1+2t-t^2}. \tag{2.3}$$

So considering the identity

$$\left(1 - \frac{2t}{1+4t-t^2}\right)\left(1 + \frac{2t}{1+2t-t^2}\right) = 1,$$

we deduce that

$$\left(\sum_{n=0}^{\infty} (-1)^n a_n t^n\right)\left(\sum_{n=0}^{\infty} b_n t^n\right) = 1.$$

Theorem 1.1 follows easily equating coefficients of t^n in this relation. $\boxed{QED}$

Proof of Theorem 1.2. The proof follows from the following identity

$$\left(1 + \frac{12t^2}{1-18t^2+t^4}\right)\left(1 + \frac{2t}{1+2t-t^2}\right) = 1 + \frac{2t}{1-4t-t^2}. \tag{2.4}$$

By (2.2), we deduce that the sequence $\{c_n\}_{n\geqslant 0}$ defined by

$$c_n = \begin{cases} 1, & \text{if } n = 0, \\ \dfrac{3}{2}F_{6n}, & \text{if } n > 0 \end{cases} \tag{2.5}$$

has the generating function given by

$$\sum_{n=0}^{\infty} c_n t^n = 1 + \frac{12t}{1 - 18t + t^2}.$$

Thus we deduce that

$$\left(\sum_{n=0}^{\infty} c_n t^{2n}\right)\left(\sum_{n=0}^{\infty} b_n t^n\right) = \sum_{n=0}^{\infty} a_n t^n.$$

Equating the coefficients of t^n in this relation, we obtain

$$\sum_{k=0}^{\lfloor n/2 \rfloor} c_k b_{n-2k} = a_n$$

and Theorem 1.2 follows easily considering the case n even

$$-2P_{2n} + \frac{3}{2}F_{6n} - 3\sum_{k=1}^{n-1} F_{6k}P_{2n-2k} = F_{6n}$$

and the case n odd

$$2P_{2n+1} + 3\sum_{k=1}^{n} F_{6k}P_{2n+1-2k} = F_{6n+3}.$$

QED

Proof of Theorem 1.3. The proof follows from the following identity

$$\left(1 + \frac{288t^4}{1 - 322t^4 + t^8}\right)\left(1 + \frac{2t}{1 + 2t - t^2}\right) = \left(1 - \frac{12t^2}{1 + 18t^2 + t^4}\right)\left(1 + \frac{2t}{1 - 4t - t^2}\right). \tag{2.6}$$

The sequence $\{d_n\}_{n\geqslant 0}$ defined by

$$d_n = \begin{cases} 1, & \text{if } n = 0, \\ 2F_{12n}, & \text{if } n > 0 \end{cases} \tag{2.7}$$

has the generating function given by

$$\sum_{n=0}^{\infty} d_n t^n = 1 + \frac{288t}{1-322t+t^2}.$$

Thus we deduce that

$$\left(\sum_{n=0}^{\infty} d_n t^{4n}\right)\left(\sum_{n=0}^{\infty} b_n t^n\right) = \left(\sum_{n=0}^{\infty}(-1)^n c_n t^{2n}\right)\left(\sum_{n=0}^{\infty} a_n t^n\right)$$

Equating the coefficients of t^n in this relation, we obtain

$$\sum_{k=0}^{\lfloor n/4\rfloor} d_k b_{n-4k} = \sum_{k=0}^{\lfloor n/2\rfloor} (-1)^k c_k a_{n-2k}. \tag{2.8}$$

By this relation, with n replaced by $4n$, we derive the following identity

$$12P_{4n} + 24\sum_{k=1}^{n-1} F_{12k}P_{4n-4k} = \frac{9}{4}\sum_{k=1}^{2n-1}(-1)^{k-1}F_{6k}F_{12n-6k}.$$

Having the identity

$$\frac{8t}{1+18t+t^2}\cdot\frac{8t}{1-18t+t^2} = \frac{64t^2}{1-322t^2+t^4},$$

we deduce that

$$\left(\sum_{n=0}^{\infty}(-1)^{n-1}F_{6n}t^n\right)\left(\sum_{n=0}^{\infty} F_{6n}t^n\right) = \frac{4}{9}\sum_{n=0}^{\infty} F_{12n}t^{2n}.$$

Equating the coefficients of t^{2n} gives the relation

$$\sum_{k=0}^{2n}(-1)^{k-1}F_{6k}F_{12n-6k} = \frac{4}{9}F_{12n}$$

and the first identity of theorem is proved.

Replacing n by $4n+1$ in (2.8), we obtain

$$2P_{4n+1} + 4\sum_{k=1}^{n} F_{12k}P_{4n+1-4k} = F_{12n+3} + \frac{3}{2}\sum_{k=1}^{2n}(-1)^k F_{6k}F_{12n+3-6k}.$$

Considering the identity

$$\frac{8t}{1+18t+t^2}\cdot\frac{2t}{1-18t+t^2} = \frac{16}{144}\cdot\frac{144t^2}{1-322t^2+t^4},$$

we give

$$\left(\sum_{n=0}^{\infty}(-1)^{n-1}F_{6n}t^n\right)\left(\sum_{n=0}^{\infty}F_{6n+3}t^n\right)=\frac{16}{144}\sum_{n=0}^{\infty}F_{12n}t^{2n}.$$

So we obtain

$$\sum_{k=0}^{2n}(-1)^{k-1}F_{6k}F_{12n+3-6k}=\frac{16}{144}F_{12n}$$

and the second identity follows easily.

The next two identities can be easily derived invoking the first two identities and the recurrence relation (1.1) with x replaced by 2. $\boxed{QED}$

3 Proof of Corollary 1.1

First we remark that the nth Pell number is even if and only if n is even. The first congruence identity of this corollary follows easily from Theorem 1.1, considering that

$$F_{3n} \equiv 0 \pmod 2.$$

This congruence is immediate from the generating function of F_{3n}.

From the generating function of F_{6n}, it is clear that

$$F_{6n} \equiv 0 \pmod 8.$$

So, using Theorem 1.2 we derive the next two congruence identities of Corollary 1.1.

By (1.2), we derive the generating function for P_{2n}, that is

$$\sum_{n=0}^{\infty}P_{2n}t^n=\frac{2t}{1-6t+t^2}.$$

This allows us to obtain the generating function for P_{4n}, namely

$$\sum_{n=0}^{\infty}P_{4n}t^n=\frac{12t}{1-34t+t^2}.$$

Considering the generating functions for F_{12n} and P_{4n}, we obtain

$$F_{12n} \equiv 0 \pmod{144}, \qquad \text{and} \qquad P_{4n} \equiv 0 \pmod{12},$$

respectively. Thus the last four congruences of Corollary 1.1 follows from Theorem 1.3.

4 Concluding remarks

In this paper, we derive a number of non-obvious Fibonacci-Pell congruences by multi-sectioning generating functions. In this context, we discovered the identity (2.6) with which we built the proof of Theorem 1.3. This curious identity can be considered a consequence of the two triple product identities:

$$1-322t^4+t^8=(1+18t^2+t^4)(1+4t-t^2)(1-4t-t^2) \tag{4.1}$$

and

$$1-34t^4+t^8=(1+6t^2+t^4)(1+2t-t^2)(1-2t-t^2). \tag{4.2}$$

They imply that both sides of the identity (2.6) equal

$$\frac{1-2t-t^2}{1-4t-t^2}\cdot\frac{1+6t^2+t^4}{1+18t^2+t^4}.$$

The identities (4.1) and (4.2) are special cases of a more general triple product identity:

$$1-\big((a^2+2)^2-2\big)t^4+t^8=\big(1+(a^2+2)t^2+t^4\big)\big(1+at-t^2\big)\big(1-at-t^2\big). \tag{4.3}$$

In this context, we remark another special case of (4.3):

$$1-L_{8n+4}t^4+t^8=(1+L_{4n+2}t^2+t^4)(1+L_{2n+1}t-t^2)(1-L_{2n+1}t-t^2),$$

where

$$L_n=F_{n-1}+F_{n+1}$$

denotes the nth Lucas number.

Apart from (4.3) there is another triple product identity:

$$1-\big((a^2-2)^2-2\big)t^4+t^8=\big(1+(a^2+2)t^2+t^4\big)\big(1+at+t^2\big)\big(1-at+t^2\big).$$

The following factorization involving Lucas numbers is a particular case of this identity:

$$1-L_{8n}t^4+t^8=(1+L_{4n}t^2+t^4)(1+L_{2n}t+t^2)(1-L_{2n}t+t^2).$$

Acknowledgements. We thank the referees for assisting in clarifying our concluding remarks.

References

[1] M. Bicknell:*A primer for the Fibonacci numbers: Part VII*, Fibonacci Quart., 8 (1970), no. 4, 407–420.

[2] G. S. Doman, J. K. Williams: *Fibonacci and Lucas Polynomials*, Math. Proc. Cambridge Philos. Soc., 90 (1981), 385–387.

[3] F. J. Galvez, J. S. Dehesa: *Novel Properties of Fibonacci and Lucas Polynomials*, Math. Proc. Cambridge Philos. Soc., 97 (1985), 159–164.

[4] V. E. Hoggatt, Jr., M. Bicknell: *Roots of Fibonacci Polynomials*, Fibonacci Quart., 11 (1973), no. 3, 271-–274.

[5] R. Honsberger: Mathematical Gems III, Mathematical Association of America, Washington, DC, 1985.

[6] T. Koshy: Pell and Pell-Lucas Numbers with Applications, Springer, New-York, 2014.

[7] M. Merca: *On Some Power Sums of Sine or Cosine*, Amer. Math. Monthly, 121 (2014) 244–248.

[8] J. Riordan: Combinatorial Identities, John Wiley & Sons, New York, 1968.

[9] T. Simpson: *The Invention of a General Method for Determining the Sum of Every 2d, 3d, 4th, or 5th, &c. Term of a Series, Taken in Order; The Sum of the Whole Series Being Known*, Philosophical Transactions 50 (1757 - 1758) 757–769, available at `http://www.jstor.org/stable/105328`.