Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **40** (2020) no. 1, 73–79. doi:10.1285/i15900932v40n1p73

On rings and Banach algebras with skew derivations

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Received: 21.7.2019; accepted: 21.10.2019.

Abstract. In the present paper, we investigate the commutativity of a prime Banach algebra with skew derivations and prove that if $\mathcal{A}$ is prime Banach algebra and $\mathcal{A}$ has a nonzero continuous linear skew derivation $\mathfrak{F}$ from $\mathcal{A}$ to $\mathcal{A}$ such that $[\mathfrak{F}(\mathfrak{x}^m), \mathfrak{F}(\mathfrak{y}^n)] - [\mathfrak{x}^m, \mathfrak{y}^n] \in \mathcal{Z}(\mathcal{A})$ for an integers $m = m(\mathfrak{x}, \mathfrak{y}) > 1$ and $n = n(\mathfrak{x}, \mathfrak{y}) > 1$ and sufficiently many $\mathfrak{x}, \mathfrak{y}$, then $\mathcal{A}$ is commutative.

Keywords: Prime Banach algebra, skew derivation.

MSC 2000 classification: primary 16W25, secondary 46J10

Introduction

Several theorems in ring theory, mostly due to Herstein, are devoted to showing that certain rings must be commutative as a consequence of conditions which are seemingly too weak to imply commutativity. Consider the following theorem of Herstein [9, p 412] which states that a ring $\mathfrak{R}$ is commutative if for each $\mathfrak{x}$ and $\mathfrak{y} \in \mathfrak{R}$ there is a positive integer $n(\mathfrak{x}, \mathfrak{y}) > 1$ such that $\mathfrak{x}^{n(\mathfrak{x},\mathfrak{y})} - \mathfrak{x}$ permutes with $\mathfrak{y}$.

This research is motivated by the work of Ali and Khan [1] and Yood [16]. Throughout this manuscript $\mathcal{A}$ represents a Banach algebra over the complex field, $\mathcal{Z}(\mathcal{A})$ denote the center of $\mathcal{A}$ and M be a closed linear subspace of $\mathcal{A}$. Recall that an algebra $\mathcal{A}$ is said to be prime if for any $a, b \in \mathcal{A}$, $a\mathcal{A}b = (0)$ implies that $a = 0$ or $b = 0$, and $\mathcal{A}$ is semiprime if for any $a \in \mathcal{A}$, $a\mathcal{A}a = (0)$ implies $a = 0$. We shall use several times the readily fact. Let $p(\mathfrak{t}) = \sum_{r=0}^{n} b_r \mathfrak{t}^r$ be a polynomial in the real variable $\mathfrak{t}$ with coefficients in $\mathcal{A}$. If $p(\mathfrak{t}) \in M$ for all $\mathfrak{t}$ in an infinite subset of the reals, then every b_r lies in M. A linear map $\mathfrak{F}$ of $\mathcal{A}$ into itself is called a linear derivation if $\mathfrak{F}(\mathfrak{x}\mathfrak{y}) = \mathfrak{F}(\mathfrak{x})\mathfrak{y} + \mathfrak{x}\mathfrak{F}(\mathfrak{y})$ for all $\mathfrak{x}, \mathfrak{y} \in \mathcal{A}$. Let σ be an automorphism of $\mathcal{A}$. A linear map $\mathfrak{F} : \mathcal{A} \to \mathcal{A}$ is called a linear skew-derivation if $\mathfrak{F}(\mathfrak{x}\mathfrak{y}) = \mathfrak{F}(\mathfrak{x})\mathfrak{y} + \sigma(\mathfrak{x})\mathfrak{F}(\mathfrak{y})$ for all $\mathfrak{x}, \mathfrak{y} \in \mathcal{A}$. When $\sigma = I_{\mathcal{A}}$ on $\mathcal{A}$, linear skew-derivation is simply an ordinary linear derivation. For $\sigma \neq I_{\mathcal{A}}$, the example of linear skew-derivation is the map $I_{\mathcal{A}} - \sigma$, where $I_{\mathcal{A}}$ denoted the identity automorphism of $\mathcal{A}$. Thus, the results on linear skew-derivations are the generalizations of both linear derivations and automorphisms.

The problems of characterizing maps that preserve certain subsets or relations had been investigated on various rings and algebras. In [2] Bell and Daif initiated the study of a certain kind of commutativity preserving map as follows: "Let S be a subset of a ring $\mathfrak{R}$. A map $f : S \to \mathfrak{R}$ is called strong commutativity preserving (SCP) on S if $[f(\mathfrak{x}), f(\mathfrak{y})] = [\mathfrak{x}, \mathfrak{y}]$ for all $\mathfrak{x}, \mathfrak{y} \in S$." More precisely, they proved that $\mathfrak{R}$ must be commutative if $\mathfrak{R}$ is a prime ring and $\mathfrak{R}$ admits derivation or a non-identity endomorphism which is SCP on right ideal of $\mathfrak{R}$. Later, Brešar and Miers [4] studied additive strong commutativity preserving maps on semiprime rings and characterized an additive map $f : \mathfrak{R} \to \mathfrak{R}$ which is SCP on the entire semiprime ring $\mathfrak{R}$ and showed that f must be of the form $f(\mathfrak{x}) = \lambda\mathfrak{x} + \nu(\mathfrak{x})$, which $\lambda \in \mathfrak{C}$, $\lambda^2 = 1$ and $\nu : \mathfrak{R} \to \mathfrak{C}$ is an additive map, where $\mathfrak{C}$ is the extended centroid of $\mathfrak{R}$. In 2008, Lin and Liu [14] discussed the strong commutativity preserving maps on Lie ideals of prime rings. Many authors have studied the strong commutativity preserving map in the setting of rings and algebras. Our results on commutativity of Banach algebras take a different direction. In [7, 8], Herstein proved that "a ring $\mathfrak{R}$ is commutative if it has no nonzero nilpotent ideal and there is a fixed integer $n > 1$ such that $(\mathfrak{x}\mathfrak{y})^n = \mathfrak{x}^n\mathfrak{y}^n$ for all $\mathfrak{x}, \mathfrak{y} \in \mathfrak{R}$." In the case of Banach algebras, Yood [17] proved these results. More precisely, he proved the following result:

Theorem 1. *Suppose that there are nonempty open subsets $\mathfrak{G}_1$ and $\mathfrak{G}_2$ of $\mathcal{A}$ such that for each $\mathfrak{x} \in \mathfrak{G}_1$ and $\mathfrak{y} \in \mathfrak{G}_2$ there is an integer $n = n(\mathfrak{x}, \mathfrak{y}) > 1$ where either $(\mathfrak{x}\mathfrak{y})^n - \mathfrak{x}^n\mathfrak{y}^n$ or $(\mathfrak{x}\mathfrak{y})^n - \mathfrak{y}^n\mathfrak{x}^n$ lies in M. Then $[\mathfrak{x}, \mathfrak{y}] \in M$ for all $\mathfrak{x}, \mathfrak{y} \in \mathcal{A}$*

Motivated by the work of Ali and Khan [1] and Yood [16, 17, 18], in the present paper our aim is to explore strong commutativity preserving skew-derivation on Banach algebras.

1 Main Results

We begin with the following result due to Lin and Liu [14] which is essential for developing the proof of our main result.

Lemma 1. *[14, Corollary 1.4.] Let $\mathfrak{R}$ be a prime ring, $\mathfrak{L}$ a Lie ideal of $\mathfrak{R}$ and $\mathfrak{F}$ a nonzero σ-derivation of $\mathfrak{R}$. Suppose that $[\mathfrak{F}(\mathfrak{x}), \mathfrak{F}(\mathfrak{y})] - [\mathfrak{x}, \mathfrak{y}] \in \mathcal{Z}(\mathfrak{R})$ for all $\mathfrak{x}, \mathfrak{y} \in \mathfrak{L}$. Then $\mathfrak{R} \subseteq \mathcal{Z}(\mathfrak{R})$ unless char $\mathfrak{R} = 2$ and $\mathfrak{R}$ satisfies the standard identity of degree 4.*

Before proving our main theorem, we prove the following key theorem.

Theorem 2. *Let $\mathfrak{R}$ be a prime ring with characteristic different from two. If $\mathfrak{R}$ admits a skew derivation $\mathfrak{F}$ satisfies $[\mathfrak{F}(\mathfrak{x}^m), \mathfrak{F}(\mathfrak{y}^n)] - [\mathfrak{x}^m, \mathfrak{y}^n] \in \mathcal{Z}(\mathfrak{R})$ for all $\mathfrak{x}, \mathfrak{y} \in \mathfrak{R}$, where m and n are fixed positive integers, then $\mathfrak{R}$ is commutative.*

Proof. We have

$$[\mathfrak{F}(\mathfrak{x}^m), \mathfrak{F}(\mathfrak{y}^n)] - [\mathfrak{x}^m, \mathfrak{y}^n] \in \mathcal{Z}(\mathfrak{R}) \text{ for all } \mathfrak{x}, \mathfrak{y} \in \mathfrak{R}.$$

If $\mathfrak{F} = 0$, then $[\mathfrak{x}^m, \mathfrak{y}^n] \in \mathcal{Z}(\mathfrak{R})$ for all $\mathfrak{x}, \mathfrak{y} \in \mathfrak{R}$. For $m, n = 1$, $\mathfrak{R}$ is obviously commutative. If $m > 1$ and $n = 1$ or $n > 1$ and $m = 1$, then by [15, Lemma 1], $\mathfrak{R}$ has no nonzero nilpotent elements and hence [11, Theorem 1], $\mathfrak{R}$ is commutative and for $m, n > 1$ by [6, Theorem 1.1], $\mathfrak{R}$ is commutative. Now we assume that $\mathfrak{F} \neq 0$.
Let δ_1 be the additive subgroup generated by the subset $\{\mathfrak{r}^m | \mathfrak{r} \in \mathfrak{R}\}$ and δ_2 the additive subgroup generated by the subset $\{\mathfrak{r}^n | \mathfrak{r} \in \mathfrak{R}\}$. It is easy to show that

$$[\mathfrak{F}(\mathfrak{x}), \mathfrak{F}(\mathfrak{y})] - [\mathfrak{x}, \mathfrak{y}] \in \mathcal{Z}(\mathfrak{R}) \text{ for all } \mathfrak{x} \in \delta_1, \mathfrak{y} \in \delta_2. \tag{1.1}$$

In the light of main theorem in [5] either δ_1 have a non-central Lie ideal $\mathfrak{L}_1$ or $\mathfrak{r}^m \in \mathfrak{R}$ for all $\mathfrak{r} \in \mathfrak{R}$. The later case concludes $\mathfrak{R}$ is commutative. Similarly, assume that there exists a non-central Lie ideal $\mathfrak{L}_2$ of $\mathfrak{R}$, which is contained in δ_2. Moreover, by [10, page 4-5], there exist $\mathcal{L}_1$ and $\mathcal{L}_2$ ideals of $\mathfrak{R}$, such that $0 \neq [\mathcal{L}_1, \mathfrak{R}] \subseteq \mathfrak{L}_1$ and $0 \neq [\mathcal{L}_2, \mathfrak{R}] \subseteq \mathfrak{L}_2$. Hence, we have

$$[\mathfrak{F}(\mathfrak{x}), \mathfrak{F}(\mathfrak{y})] - [\mathfrak{x}, \mathfrak{y}] \in \mathcal{Z}(\mathfrak{R}), \tag{1.2}$$

for each $\mathfrak{x} \in [\mathcal{L}_1, \mathcal{L}_1]$ and $\mathfrak{y} \in [\mathcal{L}_2, \mathcal{L}_2]$. Since $\mathcal{L}_1, \mathcal{L}_2$ and $\mathfrak{R}$ satisfy the same differential identities [13, Theorem 3], the we have

$$[\mathfrak{F}(\mathfrak{x}), \mathfrak{F}(\mathfrak{y})] - [\mathfrak{x}, \mathfrak{y}] \in \mathcal{Z}(\mathfrak{R})$$

for each $\mathfrak{x}, \mathfrak{y} \in [\mathfrak{R}, \mathfrak{R}]$. Hence, application of Lemma 1 we find that $[\mathfrak{R}, \mathfrak{R}] \subseteq \mathcal{Z}(\mathfrak{R})$ that is, $\mathfrak{R}$ is commutative.

Theorem 3. *Let $\mathcal{A}$ be a prime Banach algebra and $\mathfrak{F}$ be a continuous linear skew derivation. Suppose that there are non-empty open subsets $\mathfrak{G}_1$ and $\mathfrak{G}_2$ of $\mathcal{A}$ such that $[\mathfrak{F}(\mathfrak{x}^m), \mathfrak{F}(\mathfrak{y}^n)] - [\mathfrak{x}^m, \mathfrak{y}^n] \in \mathcal{Z}(\mathcal{A})$ for each $\mathfrak{x} \in \mathfrak{G}_1$ and $\mathfrak{y} \in \mathfrak{G}_2$. Then $\mathcal{A}$ is commutative.*

Proof. We adopt the notation

$$\lambda(\mathfrak{x}, \mathfrak{y}, m, n) = [\mathfrak{F}(\mathfrak{x}^m), \mathfrak{F}(\mathfrak{y}^n)] - [\mathfrak{x}^m, \mathfrak{y}^n].$$

If $\mathfrak{F} = 0$, then $[\mathfrak{x}^m, \mathfrak{y}^n] \in \mathcal{Z}(\mathcal{A})$ for all $\mathfrak{x} \in \mathfrak{G}_1$ and $\mathfrak{y} \in \mathfrak{G}_2$. Then by [18, Theorem 2], $\mathcal{A}$ is commutative. Now we assume that $\mathfrak{F} \neq 0$ and for fix $\mathfrak{x} \in \mathfrak{G}_1$. For each m and n, we define the set

$$\Phi_{m,n} = \{\mathfrak{y} \in \mathcal{A} | \lambda(\mathfrak{x}, \mathfrak{y}, m, n) \notin \mathcal{Z}(\mathcal{A})\}.$$

We claim that each $\Phi_{m,n}$ is open in $\mathcal{A}$. To show $\Phi_{m,n}$ is open we have to show $\Phi^C_{m,n}$, the complement of $\Phi_{m,n}$ is closed. For this we take a sequence $(\mathfrak{z}_k) \in \Phi^C_{m,n}$, such that $\mathfrak{z}_k \to \mathfrak{z}$ as $k \to \infty$, we need to show that $\mathfrak{z} \in \Phi^C_{m,n}$. Since $\mathfrak{z}_k \in \Phi^C_{m,n}$, then

$$[\mathfrak{F}(\mathfrak{x}^m), \mathfrak{F}(\mathfrak{z}_k^n)] - [\mathfrak{x}^m, \mathfrak{z}_k^n] \in \mathcal{Z}(\mathcal{A}). \tag{1.3}$$

Taking limit on k, we obtain

$$\lim_{k\to\infty} ([\mathfrak{F}(\mathfrak{x}^m), \mathfrak{F}(\mathfrak{z}_k^n)] - [\mathfrak{x}^m, \mathfrak{z}_k^n]) \in \mathcal{Z}(\mathcal{A}) \tag{1.4}$$

Since $\mathfrak{F}$ is continuous, the last expression yields that

$$[\mathfrak{F}(\mathfrak{x}^m), \mathfrak{F}(\lim_{k\to\infty} (\mathfrak{z}_k^n))] - [\mathfrak{x}_m, \lim_{k\to\infty} (\mathfrak{z}_k^n)] = [\mathfrak{F}(\mathfrak{x}^m), \mathfrak{F}(\mathfrak{z}^n)] - [\mathfrak{x}^m, \mathfrak{z}^n] \in \mathcal{Z}(\mathcal{A}).$$

This implies that $\mathfrak{z} \in \Phi^C_{m,n}$, therefore $\Phi^C_{m,n}$ is closed and hence each $\Phi_{m,n}$ is open. If $\Phi_{m,n}$ is dense in $\mathcal{A}$ then, by the Baire category theorem, $\cap\{\Phi_{m,n}\}$ is also dense. But this would contradict the existence of $\mathfrak{G}_2$. Therefore for some integers $\mathfrak{r} = \mathfrak{r}(\mathfrak{x}) > 1$ and $\mathfrak{s} = \mathfrak{s}(\mathfrak{x}) > 1$, the set $\Theta_{\mathfrak{r},\mathfrak{s}}$ is not dense. Let Γ be a non-empty open set in the compliment of $\Theta_{\mathfrak{r},\mathfrak{s}}$, and $\mathfrak{q} \in \Gamma$. Take $\mathfrak{w} \in \mathcal{A}$. For all real $\mathfrak{t}$ sufficiently small, $\mathfrak{q} + \mathfrak{t}\mathfrak{w} \in \Gamma$, so $\lambda(\mathfrak{x}, \mathfrak{q} + \mathfrak{t}\mathfrak{w}, \mathfrak{r}, \mathfrak{s}) \in \mathcal{Z}(\mathcal{A})$ for each such $\mathfrak{t}$.

$$[\mathfrak{F}(\mathfrak{x})^{\mathfrak{r}}, \mathfrak{F}(\mathfrak{q} + \mathfrak{t}\mathfrak{w})^{\mathfrak{s}}] - [\mathfrak{x}^{\mathfrak{r}}, (\mathfrak{q} + \mathfrak{t}\mathfrak{w})^{\mathfrak{s}}] \in \mathcal{Z}(\mathcal{A}).$$

The above expression can be written as

$$\begin{aligned}
&[\mathfrak{F}(\mathfrak{x}^{\mathfrak{r}}), \mathfrak{C}_{\mathfrak{s},0}(\mathfrak{q}, \mathfrak{w})] - [\mathfrak{x}^{\mathfrak{r}}, \mathfrak{B}_{\mathfrak{s},0}(\mathfrak{q}, \mathfrak{w})] \\
&+([\mathfrak{F}(\mathfrak{x}^{\mathfrak{r}}), \mathfrak{C}_{\mathfrak{s}-1,1}(\mathfrak{q}, \mathfrak{w})] - [\mathfrak{x}^{\mathfrak{r}}), \mathfrak{B}_{\mathfrak{s}-1,1}(\mathfrak{q}, \mathfrak{w})])\mathfrak{t} \\
&+ \cdots\cdots \qquad\qquad + \cdots\cdots \\
&+([\mathfrak{F}(\mathfrak{x}^{\mathfrak{r}}), \mathfrak{C}_{1,\mathfrak{s}-1}(\mathfrak{q}, \mathfrak{w})] - [\mathfrak{x}^{\mathfrak{r}}), \mathfrak{B}_{1,\mathfrak{s}-a}(\mathfrak{q}, \mathfrak{w})])\mathfrak{t}^{\mathfrak{s}-1} \\
&+([\mathfrak{F}(\mathfrak{x}^{\mathfrak{r}}), \mathfrak{C}_{0,\mathfrak{s}}(\mathfrak{q}, \mathfrak{w})] - [\mathfrak{x}^{\mathfrak{r}}), \mathfrak{B}_{0,\mathfrak{s}}(\mathfrak{q}, \mathfrak{w})])\mathfrak{t}^{\mathfrak{s}},
\end{aligned}$$

where $\mathfrak{C}_{\sigma,\tau}(\mathfrak{q}, \mathfrak{w})$ denotes the sum of all terms in which $\mathfrak{q}$ appears exactly σ times and $\mathfrak{w}$ appears exactly τ times in the expansion of $(\mathfrak{q} + \mathfrak{t}\mathfrak{w})^{\mathfrak{s}}$, where σ and τ are nonnegative integers such that $\mathfrak{s} = \sigma + \tau$. The coefficient of $\mathfrak{t}^{\mathfrak{s}}$ in the polynomial expansions of $[\mathfrak{F}(\mathfrak{x}^{\mathfrak{r}}), \mathfrak{F}(\mathfrak{w}^{\mathfrak{s}})] - [\mathfrak{x}^{\mathfrak{r}}, \mathfrak{w}^{\mathfrak{s}}]$. Therefore we obtain $\lambda(\mathfrak{x}, \mathfrak{w}, \mathfrak{r}, \mathfrak{s}) \in \mathcal{Z}(\mathcal{A})$.

Therefore for $\mathfrak{x} \in \mathfrak{G}_1$ there are positive integers $\mathfrak{r}, \mathfrak{s}$ depending on $\mathfrak{w}$ such that for each $\mathfrak{w} \in \mathcal{A}$, $\lambda(\mathfrak{x}, \mathfrak{w}, \mathfrak{r}, \mathfrak{s}) \in \mathcal{Z}(\mathcal{A})$.

Next we show that for each $\mathfrak{y} \in \mathcal{A}$ there are integers $\mathfrak{M} = \mathfrak{M}(\mathfrak{y}) > 1$ and $\mathfrak{N} = \mathfrak{N}(\mathfrak{y}) > 1$ such that for each $\mathfrak{u} \in \mathcal{A}$ $\lambda(\mathfrak{u}, \mathfrak{y}, \mathfrak{M}, \mathfrak{N}) \in \mathcal{Z}(\mathcal{A})$. Fix $\mathfrak{y} \in \mathcal{A}$. For each positive integers k, l, set

$$\mathcal{Q}_{k,l} = \{\mathfrak{u} \in \mathcal{A} | \lambda(\mathfrak{u}, \mathfrak{y}, k, l) \notin \mathcal{Z}(\mathcal{A})\}.$$

Each $\mathcal{Q}_{k,l}$ is open. If each $\mathcal{Q}_{k,l}$ is dense then, by the Baire category theorem, so is $\cap\{\mathcal{Q}_{k,l}\}$. But this is contrary to what was shown earlier concerning the open set $\mathfrak{G}_1$. Hence there are integers $\mathfrak{M} > 1$ and $\mathfrak{N} > 1$ and a nonempty open set Ω in the $\mathcal{Q}^C_{\mathfrak{M},\mathfrak{N}}$. Let $\mathfrak{v} \in \Omega$ and $\mathfrak{h} \in \mathcal{A}$. For all real $\mathfrak{t}$ sufficiently small $\lambda(\mathfrak{v} + \mathfrak{t}\mathfrak{h}), \mathfrak{y}, \mathfrak{M}, \mathfrak{N}) \in \mathcal{Z}(\mathcal{A})$. Arguing as above, we see that $\lambda(\mathfrak{h}, \mathfrak{y}, \mathfrak{M}, \mathfrak{N}) \in \mathcal{Z}(\mathcal{A})$ for each $\mathfrak{h} \in \mathcal{A}$.

Now consider $\Delta_{p,q}$, $p > 1$ and $q > 1$, be the set of $\mathfrak{y} \in \mathcal{A}$ such that for each $\mathfrak{w} \in \mathcal{A}$ $[\mathfrak{F}(\mathfrak{w}^p), \mathfrak{F}(\mathfrak{y}^q)] - [\mathfrak{w}^p, \mathfrak{y}^q] \in \mathcal{Z}(\mathcal{A})$. But what we have shown, the union of the sets $\Delta_{p,q}$ is $\mathcal{A}$. It is easy to see that $\Delta_{p,q}$ is closed. Again, by Baire category theorem, some Δ_{p_1,q_1}, $p_1, q_1 > 1$, must have a nonempty open subset η. For $\mathfrak{z} \in \mathcal{A}, \mathfrak{y}_0 \in \eta$ and all sufficiently small real $\mathfrak{t}$, $\lambda(\mathfrak{w}, (\mathfrak{y}_0 + \mathfrak{t}\mathfrak{z}), p_1, q_1) \in \mathcal{Z}(\mathcal{A})$. Hence by earlier arguments, we see that for each $\mathfrak{w}, \mathfrak{z} \in \mathcal{A}$, we have $\lambda(\mathfrak{w}, \mathfrak{z}, \mathfrak{r}, \mathfrak{s}) \in \mathcal{Z}(\mathcal{A})$ that is $[\mathfrak{F}(\mathfrak{w}^\mathfrak{r}), \mathfrak{F}(\mathfrak{z}^\mathfrak{s})] - [\mathfrak{w}^\mathfrak{r}, \mathfrak{z}^\mathfrak{s}] \in \mathcal{Z}(\mathcal{A})$, then, by Theorem 2, $\mathcal{A}$ is commutative.

As an immediate consequence of Theorem 3 is the following corollary.

Corollary 1. *Let $\mathcal{A}$ be a prime Banach algebra and $\mathcal{F}$ be a continuous linear derivation. Suppose that there are non-empty open subsets $\mathfrak{G}_1$ and $\mathfrak{G}_2$ of $\mathcal{A}$ such that $[\mathcal{F}(\mathfrak{x}^m), \mathcal{F}(\mathfrak{y}^n)] - [\mathfrak{x}^m, \mathfrak{y}^n] \in \mathcal{Z}(\mathcal{A})$ for each $\mathfrak{x} \in \mathfrak{G}_1$ and $\mathfrak{y} \in \mathfrak{G}_2$. Then $\mathcal{A}$ is commutative.*

Now, we conclude with the following Example.

Example 1. Let $\mathbb{C}$ be the field of complex numbers, let

$$\mathbb{M} = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} | x, y, z, w \in \mathbb{C} \right\}$$

be a noncommutative unital prime algebra of all 2×2 matrices over $\mathbb{C}$ with the usual matrix addition, and define matrix multiplication as follows: $\mathbb{A} \times_l \mathbb{B} = l\mathbb{A}\mathbb{B}$ for all $\mathbb{A}, \mathbb{B} \in \mathbb{M}$ where $l = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ and $|\lambda| > 1$. For $\mathbb{A} = (\mu_{ij}) \in \mathbb{M}$, define $||\mathbb{A}|| = max_l \sum_{i=1}^{2} |\mu_{ij}|$. Then $\mathbb{M}$ is a normed linear space. Now, define a map

$\mathfrak{F} : \mathbb{M} \to \mathbb{M}$ by

$$\mathfrak{F}\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix},$$

and

$$\alpha\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & -y \\ -z & w \end{pmatrix}$$

for every $\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathbb{M}$. Since $\mathbb{M}$ is finite-dimensional, it is easily verify that $\mathfrak{F}$ is a nonzero skew derivation associated with α-derivation on $\mathbb{M}$. observe that

$$\mathfrak{G}_1 = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} |t \in \mathbb{R} \right\} \text{ and } \mathfrak{G}_2 = \left\{ \begin{pmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{pmatrix} |t \in \mathbb{R} \right\}.$$

are open subsets of $\mathbb{M}$ such that $[\mathfrak{F}(\mathbb{A}^m), \mathfrak{F}(\mathbb{B}^n)] - [\mathbb{A}^m, \mathbb{B}^n] \in \mathcal{Z}(\mathcal{A})$ for all $\mathbb{A} \in \mathfrak{G}_1$ and $\mathbb{B} \in \mathfrak{G}_2$. Hence,it follows from Theorem 3 that $\mathbb{M}$ is not a Banach algebra under the defined norm.

References

[1] S. Ali, A. N. Khan: On commutativity of Banach algebras with derivations, Bull. Aust. Math. Soc. 91 (2015), 419-425.

[2] H. E. Bell, M. N. Daif: On commutativity and strong commutativity preserving maps, Canad. Math. Bull., 37 (1994), 443-447.

[3] F. F. Bonsall, J. Duncan: Complete Normed algebras, Springer-Verlag, New York, (1973).

[4] M. Brešar, C. R. Miers: Strong commutativity preserving mappings of semiprime rings, Canad. Math. Bull. 37 (1994), 457-460.

[5] C. L. Chuang: The additive subgroup generated by a polynomial, Israel J. Math. 59(1) (1987), 98-106.

[6] R. D. Giri A. R. Dhoble: Some commutativity theorems for rings, Publ. Math. Debrecen 41 (1-2) (1992), 35-40.

[7] I. N. Hertesin: Power maps in rings, Michigan Math. J. 8 (1961), 29-32.

[8] I. N. Hertesin: A remark on rings and algebras, Michigan Math. J. 10 (1963), 269-272.

[9] I. N. Hertesin: Two remarks on the commutativity of rings, Canad. J.Math. 7 (1955), 411-412.

[10] I. N. Hertesin: Topics in Ring Theory, Univ. of Chicago Press, Chicago, 1969.

[11] I. N. Hertesin: A commutative theorem, J. Algebra 38 (1976), 112-118.

[12] N. Jacobson: Structure of rings, Amer. Math. Soc. Solloq. Publ., 37, Amer Math. Soc. Providence, R. I., 1956.

[13] T. K. Lee: Semiprime rings with differential identities, Bull. Inst. Math. Acad. Sinica. 20(1) (1992), 27-38.

[14] J. S. Lin, C. K. Liu: Strong commutativity preserving maps on Lie ideals, Linear Algebra Appl. 428(2008), 1601-1609.

[15] M. A. Quadri, M. A. Khan, M. Ashraf: Some elementry commutativity theorems for rings, Math. Stident 56(1-4) (1988), 223-226.

[16] B. Yood: Commutativity theorems for Banach algebras, Michigan Math. J. 37(2) (1990), 203-210.

[17] B. Yood: On commutativity of unital Banach algebras, Bull. Londan Math. Soc. 23(2) (1991), 278-280.

[18] B. Yood: Some commutativity theorems for Banach algebras, Publ. Math. 45(1-2) (1994), 29-33.