Note di Matematica
Note Mat. **40** (2020) no. 1, 27–43.
ISSN 1123-2536, e-ISSN 1590-0932
doi:10.1285/i15900932v40n1p27

On Fejér Type Inequalities For Products Two Convex Functions

Hüseyin BUDAK
Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-Turkey
hsyn.budak@gmail.com

Yonca BAKIŞ
Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-Turkey
yonca.bakis93@hotmail.com

Received: 21.9.2018; accepted: 24.1.2019.

Abstract. In this paper, we first obtain some new Fejér type inequalities for products of two convex mappings. Moreover, by applying these inequalities for Riemann-Liouville fractional integrals, we establish some Fejér type inequalities involving Riemann-Liouville fractional integrals. The most important feature of our work is that it contains Fejér type inequalities for both classical integrals and fractional integrals.

Keywords: Fejér type inequalities, convex function, fractional integrals.

MSC 2000 classification: primary 26D07, 26D10 secondary 26D15, 26A33

1 Introduction

Over the past three decades, the field of integral inequalities has undergone explosive growth. The many research paper related to some type inequalities such as Hermite-Hadamard, Ostrowski, Gruss, etc. have been written. Recently, many works have focused on Hermite-Hadamard type inequalities for products two convex functions. Moreover, in some of these paper authors obtained Hermite-Hadamard type inequalities contains fractional integrals. Inspired by these inequalities, we establish some Hermite-Hadamard-Fejér type inequalities for products two convex functions. That our paper contains Fejér type inequalities for both classical integrals and fractional integrals is the most important feature of our work.

The overall structure of the study takes the form of four sections including introduction. The remainder of this work is organized as follows: we first give classical Hermite-Hadamard and Fejér inequalities and present the definitions of Riemann-Liouville fractional integrals. Moreover, we give Hermite-Hadamard type inequalities for products two convex functions involving classical integrals or fractional integrals. In Section 2, we establish two integral inequalities of

Hermite-Hadamard-Fejér type which generalize some results obtained in earlier works. We give also some special cases of these inequalities. Utilizing the results established in Section 2, Fejér type inequalities for products two convex functions involving Riemann-Liouville fractional integrals are proved in Section 3. Finally, conclusions and future directions of research are discussed in Section 4.

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see, e.g.,[5], [19, p.137]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hermite-Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Over the years, many studies have focused on to establish generalization of the inequality (1.1) and to obtain new bounds for left hand side and right hand side of the inequality (1.1).

The weighted version of the inequalities (1.1), so-called Hermite-Hadamard-Fejér inequalities, was given by Fejér in [6] as follow:

Theorem 1. *$f : [a, b] \to \mathbb{R}$, be a convex function, then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx \leq \int_a^b f(x)w(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b w(x)dx \tag{1.2}$$

holds, where $w : [a, b] \to \mathbb{R}$ is non-negative, integrable, and symmetric about $x = \frac{a+b}{2}$ (i.e. $w(x) = w(a + b - x)$).

In [18], Pachpatte proved the following inequalities for products of convex functions:

Theorem 2. *Let f and g be real-valued, non-negative and convex functions on $[a, b]$. Then we have*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b), \tag{1.3}$$

and

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)g(x)dx + \frac{1}{6}M(a,b) + \frac{1}{3}N(a,b) \tag{1.4}$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$ *and* $N(a,b) = f(a)g(b) + f(b)g(a)$.

In recent years, the generalized versions of inequalities (1.3) and (1.4) for several convexity have been proved. For some of them please refer to ([3], [4], [8], [12], [22], [24], [25]). Moreover in [14], Latif and Alomari proved some inequalities for product of two co-ordinated convex function. Furthermore in [16] and [17], Ozdemir et al. gave some generalizations of results given by Latif and Alomari using the product of two coordinated s-convex maappings and product of two coordinated h-convex mappings, respectively. In [1], Budak and Sarıkaya proved Hermite-Hadamard type inequalities for products of two co-ordinated convex mappings via fractional integrals.

In the following, we give the definition of Riemann-Liouville fractional integrals. More details for Riemann-Liouville fractional integrals, one can consult ([7], [11], [15], [20]).

Definition 1. Let $f \in L_1[a,b]$. The Riemann-Liouville fractional integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{b}(t-x)^{\alpha-1}f(t)dt, \quad x < b$$

respectively. Here, Γ is the Gamma function defined by

$$\Gamma(x) = \int_{0}^{\infty} e^{-t}t^{x-1}dt$$

and $J_{a+}^{0}f(x) = J_{b-}^{0}f(x) = f(x)$.

It is remarkable that Sarikaya et al.[21] first gave the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 3. *Let* $f : [a,b] \to \mathbb{R}$ *be a positive function with* $a < b$ *and* $f \in L_1[a,b]$. *If* f *is a convex function on* $[a,b]$, *then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.5}$$

with $\alpha > 0$.

In [2], Chen proved the following fractional Hermite-Hadamard type inequalities for products of convex functions:

Theorem 4. *Let f and g real-valued, non-negative, convex functions on $[a,b]$. Then*

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)g(b)+J_{b-}^{\alpha}f(a)g(a)\right] \\ \leq\ &\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)M(a,b)+\frac{\alpha}{(\alpha+1)(\alpha+2)}N(a,b)\end{aligned} \tag{1.6}$$

where $M(a,b)$ and $N(a,b)$ are defined as in Theorem 2.

Theorem 5. *Let f and g real-valued, non-negative, convex functions on $[a,b]$. Then*

$$\begin{aligned}&2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\ \leq\ &\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)g(b)+J_{b-}^{\alpha}f(a)g(a)\right] \\ &+\frac{\alpha}{(\alpha+1)(\alpha+2)}M(a,b)+\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)N(a,b)\end{aligned} \tag{1.7}$$

where $M(a,b)$ and $N(a,b)$ are defined as in Theorem 2.

Moreover, some papers were dedicated on Hermite-Hadamard type inequalities for products of convex functions via other type fractional integrals. For example, please refer to ([10], [13], [23]).

On the other hand, Işcan gave following Lemma and using this Lemma he proved the following Fejer type inequalities for Riemann-Liouville fractional integrals in [9].

Lemma 1. *If $w:[a,b]\to\mathbb{R}$ is integrable and symmetric to $(a+b)/2$ with $a<b$, then*

$$J_{a+}^{\alpha}w(b)=J_{b-}^{\alpha}w(a)=\frac{1}{2}\left[J_{a+}^{\alpha}w(b)+J_{b-}^{\alpha}w(a)\right]$$

with $\alpha>0$.

Theorem 6. *Let $f:[a,b]\to\mathbb{R}$ be convex function with $0\leq a<b$ and $f\in L_1[a,b]$. If $w:[a,b]\to\mathbb{R}$ is non-negative,integrable and symmetric to $(a+b)/2$, then the following inequalities for fractional integrals hold*

$$\begin{aligned}f\left(\frac{a+b}{2}\right)\left[J_{a+}^{\alpha}w(b)+J_{b-}^{\alpha}w(a)\right]\ &\leq\ \left[J_{a+}^{\alpha}(fw)(b)+J_{b-}^{\alpha}(fw)(a)\right] \\ &\leq\ \frac{f(a)+f(b)}{2}\left[J_{a+}^{\alpha}w(b)+J_{b-}^{\alpha}w(a)\right]\end{aligned} \tag{1.8}$$

with $\alpha > 0$.

2 Fejér Type Inequalities for Products of Two Convex Functions

In this section, we present some Fejér type inequalities for products two convex functions.

Theorem 7. *Suppose that* $w : [a, b] \to \mathbb{R}$ *is non-negative, integrable, and symmetric about* $x = \frac{a+b}{2}$ *(i.e.* $w(x) = w(a + b - x)$*). If* $f, g : I \to \mathbb{R}$ *are two real-valued, non-negative and convex functions on* I*, then for any* $a, b \in I$*, we have*

$$\int_a^b f(x)g(x)w(x)dx \tag{2.1}$$

$$\leq \frac{M(a,b)}{(b-a)^2}\int_a^b (b-x)^2 w(x)dx + \frac{N(a,b)}{(b-a)^2}\int_a^b (b-x)(x-a)w(x)dx$$

where

$$M(a, b) = f(a)g(a) + f(b)g(b) \text{ and } N(a, b) = f(a)g(b) + f(b)g(a).$$

Proof. Since f and g are convex functions on $[a, b]$, then we have

$$f((1-t)a + tb) \leq (1-t) f(a) + tf(b) \tag{2.2}$$

and

$$g((1-t)a + tb) \leq (1-t) g(a) + tg(b). \tag{2.3}$$

By (2.2) and (2.3), we have

$$f((1-t)a + tb) g((1-t)a + tb) \tag{2.4}$$

$$\leq (1-t)^2 f(a)g(a) + t^2 f(b)g(b) + t(1-t)[f(a)g(b) + f(b)g(a)].$$

Multiplying both sides of (2.4) by $w((1-t)a + tb)$, then integrating the resulting inequality with respect to t from 0 to 1, we obtain

$$\begin{aligned}
&\int_0^1 f((1-t)a+tb)\,g((1-t)a+tb)\,w((1-t)a+tb)\,dt \qquad (2.5)\\
\leq\ & f(a)g(a)\int_0^1 (1-t)^2 w((1-t)a+tb)\,dt \qquad (2.6)\\
&+f(b)g(b)\int_0^1 t^2 w((1-t)a+tb)\,dt\\
&+[f(a)g(b)+f(b)g(a)]\int_0^1 t(1-t)\,w((1-t)a+tb)\,dt.
\end{aligned}$$

By change of variable $x=(1-t)a+tb$, we get

$$\begin{aligned}
&\int_0^1 f((1-t)a+tb)\,g((1-t)a+tb)\,w((1-t)a+tb)\,dt \qquad (2.7)\\
=\ & \frac{1}{b-a}\int_a^b f(x)g(x)w(x)dx.
\end{aligned}$$

Moreover, since w is symmetric about $\frac{a+b}{2}$, it is easily observe that

$$\begin{aligned}
\int_0^1 t^2 w((1-t)a+tb)\,dt &= \frac{1}{(b-a)^3}\int_a^b (x-a)^2 w(x)dx \qquad (2.8)\\
&= \frac{1}{(b-a)^3}\int_a^b (b-x)^2 w(x)dx
\end{aligned}$$

and

$$\int_0^1 (1-t)^2 w((1-t)a+tb)\,dt = \frac{1}{(b-a)^3}\int_a^b (b-x)^2 w(x)dx. \qquad (2.9)$$

We also have

$$\int_0^1 t(1-t)\,w((1-t)a+tb)\,dt = \frac{1}{(b-a)^3}\int_a^b (b-x)(x-a)\,w(x)dx. \qquad (2.10)$$

By substituting the equalities (2.7)-(2.10) in (2.5), then we have the following inequality

$$\begin{aligned} & \frac{1}{b-a}\int_a^b f(x)g(x)w(x)dx \qquad (2.11) \\ \leq \; & \frac{f(a)g(a)+f(b)g(b)}{(b-a)^3}\int_a^b (b-x)^2\, w(x)dx \\ & + \frac{f(a)g(b)+f(b)g(a)}{(b-a)^3}\int_a^b (b-x)(x-a)\, w(x)dx. \end{aligned}$$

If we multiply both sides of (2.11) by $(b-a)$, then we obtain the desired result.
QED

Remark 1. If we choose $w(x)=1$ for all $x\in[a,b]$ in Theorem 7, then the inequality (2.1) reduces to the inequality (1.3).

Remark 2. If we choose $g(x)=1$ for all $x\in[a,b]$ in Theorem 7, then we have

$$\int_a^b f(x)w(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b w(x)dx$$

which is the same the second inequality in (1.2).

Proof. For $g(x)=1$ for all $x\in[a,b]$, from the inequality (2.1), we get

$$\begin{aligned} & \int_a^b f(x)w(x)dx \qquad (2.12) \\ \leq \; & \frac{f(a)+f(b)}{(b-a)^2}\left[\int_a^b (b-x)^2\, w(x)dx + \int_a^b (b-x)(x-a)\, w(x)dx\right] \\ = \; & \frac{f(a)+f(b)}{(b-a)}\int_a^b (b-x)\, w(x)dx. \end{aligned}$$

Since w is symmetric about $x = \frac{a+b}{2}$, we have

$$\begin{aligned}\int_a^b (b-x)\,w(x)dx &= \int_a^{\frac{a+b}{2}} (b-x)\,w(x)dx + \int_{\frac{a+b}{2}}^b (b-x)\,w(x)dx \qquad (2.13)\\ &= \int_a^{\frac{a+b}{2}} (b-x)\,w(x)dx + \int_a^{\frac{a+b}{2}} (x-a)\,w(a+b-x)dx \\ &= (b-a)\int_a^{\frac{a+b}{2}} w(x)dx \\ &= \frac{(b-a)}{2}\int_a^b w(x)dx.\end{aligned}$$

Putting the equality (2.13) in (2.12), we obtain the required result. QED

Theorem 8. *Suppose that conditions of Theorem 7 hold, then we have the following inequality*

$$\begin{aligned}& 2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\int_a^b w(x)dx \qquad (2.14)\\ \leq & \int_a^b f(x)g(x)w(x)dx + \frac{M(a,b)}{(b-a)^2}\int_a^b (b-x)\,(x-a)\,w\,(x)\,dx \\ & + \frac{N(a,b)}{(b-a)^2}\int_a^b (b-x)^2\,w\,(x)\,dx\end{aligned}$$

where $M(a,b)$ and $N(a,b)$ are defined as in Theorem 7.

Proof. For $t \in [0,1]$, we can write

$$\frac{a+b}{2} = \frac{(1-t)a+tb}{2} + \frac{ta+(1-t)b}{2}.$$

Using the convexity of f and g, we have

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\
= & f\left(\frac{(1-t)a+tb}{2}+\frac{ta+(1-t)b}{2}\right) g\left(\frac{(1-t)a+tb}{2}+\frac{ta+(1-t)b}{2}\right) \\
\leq & \frac{1}{4}\left[f((1-t)a+tb)+f(ta+(1-t)b)\right]\left[g((1-t)a+tb)+g(ta+(1-t)b)\right] \\
= & \frac{1}{4}\left[f((1-t)a+tb)g((1-t)a+tb)+f(ta+(1-t)b)g(ta+(1-t)b)\right] \\
& +\frac{1}{4}\left[f((1-t)a+tb)g(ta+(1-t)b)+f(ta+(1-t)b)g((1-t)a+tb)\right].
\end{aligned}$$

For the second expression in the last equality, by using again the convexity of f and g, we obtain

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\
\leq & \frac{1}{4}\left[f((1-t)a+tb)g((1-t)a+tb)+f(ta+(1-t)b)g(ta+(1-t)b)\right] \\
& +\frac{1}{2}t(1-t)\left[f(a)g(a)+f(b)g(b)\right]+\frac{1}{4}\left[t^2+(1-t)^2\right]\left[f(a)g(b)+f(b)g(a)\right].
\end{aligned} \tag{2.15}$$

Multiplying both sides of (2.15) by $w((1-t)a+tb)$, then integrating the resulting inequality with respect to t from 0 to 1, we obtain

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \int_0^1 w((1-t)a+tb)\,dt \qquad (2.16)\\
\leq & \frac{1}{4}\int_0^1 \left[f((1-t)a+tb)g((1-t)a+tb)+f(ta+(1-t)b)g(ta+(1-t)b)\right] \\
& \times w((1-t)a+tb)\,dt \\
& +\frac{M(a,b)}{2}\int_0^1 t(1-t)\,w((1-t)a+tb)\,dt \\
& +\frac{N(a,b)}{4}\int_0^1 \left[t^2+(1-t)^2\right] w((1-t)a+tb)\,dt.
\end{aligned}$$

Using the identities (2.7)-(2.10) in (2.16), we obtain

$$f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \frac{1}{b-a} \int_a^b w(x)dx \tag{2.17}$$

$$\leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x)g(x)w(x)dx + \frac{1}{b-a}\int_a^b f(x)g(x)w(a+b-x)dx\right]$$

$$+\frac{M(a,b)}{2(b-a)^3}\int_a^b (b-x)(x-a)w(x)\,dx + \frac{N(a,b)}{2(b-a)^3}\int_a^b (b-x)^2 w(x)\,dx.$$

By multiplying the both sides of (2.17) by $2(b-a)$ then we obtain the desired result (2.14). $\boxed{QED}$

Remark 3. If we choose $w(x)=1$ for all $x\in[a,b]$ in Theorem 8, then the inequality (2.14) reduces to the inequality (1.4).

Corollary 1. *If we choose $g(x)=1$ for all $x\in[a,b]$ in Theorem 8, then we have*

$$2f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx \leq \int_a^b f(x)w(x)dx + \frac{f(a)+f(b)}{2}\int_a^b w(x)\,dx.$$

Proof. For $g(x)=1$ for all $x\in[a,b]$, from the inequality (2.14), we get

$$2f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx$$

$$\leq \int_a^b f(x)w(x)dx$$

$$+\frac{f(a)+f(b)}{(b-a)^2}\left[\int_a^b (b-x)(x-a)w(x)\,dx + \int_a^b (b-x)^2 w(x)\,dx\right]$$

$$= \int_a^b f(x)w(x)dx + \frac{f(a)+f(b)}{b-a}\int_a^b (b-x)w(x)\,dx.$$

By the equality (2.13), we obtain the desired result. $\boxed{QED}$

3 Some Results For Riemann-Liouville Fractional Integrals

In this section, we apply the inequalities obtained in Section 2 to Riemann-Liouville fractional integrals. Thus, we establish some Fejér types inequalities involving Riemann-Liouville fractional integrals.

Theorem 9. *Suppose that $w : [a,b] \to \mathbb{R}$ is non-negative, integrable, and symmetric about $x = \frac{a+b}{2}$ (i.e. $w(x) = w(a+b-x)$). If $f, g : I \to \mathbb{R}$ are two real-valued, non-negative and convex functions on I, then for any $a, b \in I$ and $\alpha > 0$, we have*

$$\begin{aligned} & J_{a+}^{\alpha}(fgw)(b) + J_{b-}^{\alpha}(fgw)(a) \\ \leq & \frac{M(a,b)}{(b-a)^2\Gamma(\alpha)} \int_a^b (b-x)^{\alpha-1}\left[(b-x)^2 + (x-a)^2\right] w(x)dx \\ & + \frac{2N(a,b)}{(b-a)^2\Gamma(\alpha)} \int_a^b (x-a)(b-x)^{\alpha} w(x)dx \end{aligned} \tag{3.1}$$

where Γ is the Gamma function.

Proof. Based on the assumption that w is non-negative, integrable and symmetric about $x = \frac{a+b}{2}$, it is obvious that $h(x) = \frac{1}{\Gamma(\alpha)}\left[(b-x)^{\alpha-1} + (x-a)^{\alpha-1}\right] w(x)$ is non-negative, integrable and symmetric about $x = \frac{a+b}{2}$. Thus, by using Theorem 7, we can write the following inequality

$$\begin{aligned} & \int_a^b f(x)g(x)h(x)dx \\ \leq & \frac{M(a,b)}{(b-a)^2} \int_a^b (b-x)^2 h(x)dx + \frac{N(a,b)}{(b-a)^2} \int_a^b (b-x)(x-a)h(x)dx, \end{aligned}$$

i.e.

$$\begin{aligned} & \frac{1}{\Gamma(\alpha)} \int_a^b (b-x)^{\alpha-1} f(x)g(x)w(x)dx \\ & + \frac{1}{\Gamma(\alpha)} \int_a^b (x-a)^{\alpha-1} f(x)g(x)w(x)dx \end{aligned} \tag{3.2}$$

$$\leq \frac{M(a,b)}{(b-a)^2\Gamma(\alpha)}\int_a^b (b-x)^2\left[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right]w(x)dx$$
$$+\frac{N(a,b)}{(b-a)^2\Gamma(\alpha)}\int_a^b (b-x)(x-a)\left[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right]w(x)dx.$$

From the Definition 1, we have

$$\frac{1}{\Gamma(\alpha)}\int_a^b (b-x)^{\alpha-1}f(x)g(x)w(x)dx \tag{3.3}$$
$$+\frac{1}{\Gamma(\alpha)}\int_a^b (x-a)^{\alpha-1}f(x)g(x)w(x)dx$$
$$=J_{a+}^{\alpha}(fgw)(b)+J_{b-}^{\alpha}(fgw)(a).$$

Moreover, since w is symmetric about $x=\frac{a+b}{2}$, we have

$$\int_a^b (b-x)^2\left[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right]w(x)dx \tag{3.4}$$
$$=\int_a^b (b-x)^{\alpha+1}w(x)dx+\int_a^b (b-x)^2(x-a)^{\alpha-1}w(x)dx$$
$$=\int_a^b (b-x)^{\alpha+1}w(x)dx+\int_a^b (x-a)^2(b-x)^{\alpha-1}w(x)dx$$
$$=\int_a^b (b-x)^{\alpha-1}\left[(b-x)^2+(x-a)^2\right]w(x)dx$$

and similarly

$$\int_a^b (b-x)(x-a)\left[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right]w(x)dx \tag{3.5}$$
$$=\int_a^b (x-a)(b-x)^{\alpha}w(x)dx+\int_a^b (b-x)(x-a)^{\alpha}w(x)dx$$

$$= \int_a^b (x-a)(b-x)^{\alpha} w(x)dx + \int_a^b (x-a)(b-x)^{\alpha} w(x)dx$$

$$= 2\int_a^b (x-a)(b-x)^{\alpha} w(x)dx.$$

If we substitute the equalities (3.3)-(3.5) in (3.2), then we obtain the desired inequality (3.1). QED

Remark 4. If we choose $w(x) = 1$ for all $x \in [a, b]$ in Theorem 9, then the inequality (3.1) reduces to the inequality (1.6).

Remark 5. If we choose $\alpha = 1$ in Theorem 9, then the inequality (3.1) reduces to the inequality (2.1).

Remark 6. If we choose $g(x) = 1$ for all $x \in [a, b]$ in Theorem 9, then we have

$$J_{a+}^{\alpha}(gw)(b) + J_{b-}^{\alpha}(fw)(a) \leq \frac{f(a)+f(b)}{2}\left[J_{a+}^{\alpha}w(b) + J_{b-}^{\alpha}w(a)\right]$$

which is the same the second inequality in (1.8).

Proof. For $g(x) = 1$ for all $x \in [a, b]$, from the inequality (3.1) and using Lemma 1, we get

$$J_{a+}^{\alpha}(gw)(b) + J_{b-}^{\alpha}(fw)(a) \tag{3.6}$$

$$\leq \frac{f(a)+f(b)}{(b-a)^2\Gamma(\alpha)}$$

$$\times \left[\int_a^b (b-x)^{\alpha-1}\left[(b-x)^2+(x-a)^2\right]w(x)dx + 2\int_a^b (x-a)(b-x)^{\alpha} w(x)dx\right]$$

$$= \frac{f(a)+f(b)}{(b-a)^2\Gamma(\alpha)}$$

$$\left[\int_a^b (b-x)^{\alpha-1}\left[(b-x)^2+(x-a)^2+2(x-a)(b-x)\right]w(x)dx\right]$$

$$= \frac{f(a)+f(b)}{\Gamma(\alpha)}\int_a^b (b-x)^{\alpha-1} w(x)dx$$

$$= \frac{f(a)+f(b)}{2}\left[J_{a+}^{\alpha}w(b)+J_{b-}^{\alpha}w(a)\right]$$

which completes the proof. $\boxed{QED}$

Theorem 10. *Suppose that conditions of Theorem 9 hold, then we have the following inequality*

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\left[J_{a+}^{\alpha}w(b)+J_{b-}^{\alpha}w(a)\right] \tag{3.7}$$

$$\leq J_{a+}^{\alpha}\left(fgw\right)(b)+J_{b-}^{\alpha}\left(fgw\right)(a)$$

$$+\frac{2M(a,b)}{(b-a)^{2}\Gamma(\alpha)}\int_{a}^{b}(x-a)(b-x)^{\alpha}w(x)dx$$

$$+\frac{N(a,b)}{(b-a)^{2}\Gamma(\alpha)}\int_{a}^{b}(b-x)^{\alpha-1}\left[(b-x)^{2}+(x-a)^{2}\right]w(x)dx.$$

Proof. From the assumption that w is non-negative, integrable and symmetric about $x=\frac{a+b}{2}$, we have $h(x)=\frac{1}{\Gamma(\alpha)}\left[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right]w(x)$ is non-negative, integrable and symmetric about $x=\frac{a+b}{2}$. Thus, by using Theorem 8, we can write the following inequality

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\int_{a}^{b}h(x)dx$$

$$\leq \int_{a}^{b}f(x)g(x)h(x)dx+\frac{M(a,b)}{(b-a)^{2}}\int_{a}^{b}(b-x)(x-a)h(x)dx$$

$$+\frac{N(a,b)}{(b-a)^{2}}\int_{a}^{b}(b-x)^{2}h(x)dx.$$

That is, we have

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\frac{1}{\Gamma(\alpha)}\int_{a}^{b}\left[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right]w(x)$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_a^b f(x)g(x) \left[(b-x)^{\alpha-1} + (x-a)^{\alpha-1} \right] w(x)dx$$

$$+ \frac{M(a,b)}{(b-a)^2 \Gamma(\alpha)} \int_a^b (b-x)(x-a) \left[(b-x)^{\alpha-1} + (x-a)^{\alpha-1} \right] w(x)dx$$

$$+ \frac{N(a,b)}{(b-a)^2 \Gamma(\alpha)} \int_a^b (b-x)^2 \left[(b-x)^{\alpha-1} + (x-a)^{\alpha-1} \right] w(x)dx.$$

From the Definition 1 and using the identities (3.3)-(3.5), we have the desired result (3.7). QED

Remark 7. If we choose $w(x) = 1$ for all $x \in [a,b]$ in Theorem 10, then the inequality (3.7) reduces to the inequality (1.7).

Remark 8. If we choose $\alpha = 1$ in Theorem 10, then the inequality (3.7) reduces to the inequality (2.14).

Corollary 2. *If we choose $g(x) = 1$ for all $x \in [a,b]$ in Theorem 10, then we have*

$$2f\left(\frac{a+b}{2}\right) \left[J_{a+}^{\alpha} w(b) + J_{b-}^{\alpha} w(a) \right]$$

$$\leq J_{a+}^{\alpha} (fw)(b) + J_{b-}^{\alpha} (fw)(a) + \frac{f(a)+f(b)}{2} \left[J_{a+}^{\alpha} w(b) + J_{b-}^{\alpha} w(a) \right]$$

Proof. The proof is obvious from the inequality (3.6). QED

4 Concluding Remarks

In this paper, we present some Fejér type inequalities for products two convex functions. The obtained results are also applied to Riemann-Liouville fractional integrals. For further investigations we propose to consider the Fejér type inequalities for products other type convex functions or for other fractional integral operators.

References

[1] H. Budak and M. Z. Sarıkaya, *Hermite-Hadamard type inequalities for products of two co-ordinated convex mappings via fractional integrals*, Journal of Applied Mathematics and Statistics, 58(4), 2019, 11-30.

[2] F. Chen, *A note on Hermite-Hadamard inequalities for products of convex functions via Riemann-Liouville fractional integrals*, Ital. J. Pure Appl. Math., 33, (2014), 299-306.

[3] F. Chen, *A note on Hermite-Hadamard inequalities for products of convex functions*, Journal of Applied Mathematics, vol. 2013, Article ID 935020, 5 pages, 2013.

[4] F. Chen, S. Wu, *Several complementary inequalities to inequalities of Hermite-Hadamard type for s-convex functions*, J. Nonlinear Sci. Appl., 9 (2016), 705–716.

[5] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.

[6] L. Fejer, *Über die Fourierreihen,* II. Math. Naturwiss. Anz Ungar. Akad. Wiss., 24 (1906), 369–390. (Hungarian).

[7] R. Gorenflo, F. Mainardi, *Fractional calculus: integral and differential equations of fractional order*, Springer Verlag, Wien (1997), 223-276.5.

[8] N. N. Hue and D. Q. Huy, "Some inequalities of the Hermite-Hadamard type for product of two functions, Journal of New Theory (2016): 26-37

[9] I. Iscan, *Hermite-Hadamard-Fejer type inequalities for convex functions via fractional integrals*, Studia Universitatis Babeş-Bolyai Mathematica, 60(3) (2015), 355-366.

[10] I. Işcan and M. Kunt, *Hermite-Hadamard type inequalities for product of GA-convex functions via Hadamard fractional integrals*, Stud. Univ. Babeş-Bolyai Math. 62(4), 2017, 451–459.

[11] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier Sci. B.V., Amsterdam, 2006.

[12] U.S. Kırmacı, M.K. Bakula, M.E. Özdemir, J. Pečarić, *Hadamard-tpye inequalities for s-convex functions*, Appl. Math. Comput. 193 (2007) 26–35.

[13] M Kunt, Işcan and N Yazici, *Hermite-Hadamard type inequalities for product od harmonically convex functions via Riemann-Liouville fractional integrals*, Journal of Mathematical Analysis, 7(4), 2016, 74-82.

[14] M. A. Latif and M. Alomari, *Hadamard-type inequalities for product two convex functions on the co-ordinates*, Int Math Forum. 2009;4(47):2327–2338.

[15] S. Miller and B. Ross, *An introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, USA, 1993, p.2.

[16] M. E. Ozdemir, M. A. Latif, A. O. Akdemir, *On some Hadamard-type inequalities for product of two s-convex functions on the co-ordinates*, J.Inequal. Appl. 21 (2012) 1–13.

[17] M. E. Ozdemir, M. A. Latif, A. O. Akdemir, *On some Hadamard-type inequalities for product of two h-convex functions on the co-ordinates*, Turkish Journal of Science 1 (2016): 41-58.

[18] B. G. Pachpatte, *On some inequalities for convex functions*, RGMIA Res. Rep. Coll. 6 (E) (2003).

[19] J.E. Pečarić, F. Proschan and Y.L. Tong, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, Boston, 1992.

[20] I. Podlubni, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[21] M.Z. Sarikaya, E. Set, H. Yaldiz and N., Basak, *Hermite -Hadamard's inequalities for fractional integrals and related fractional inequalities*, Mathematical and Computer Modelling, DOI:10.1016/j.mcm.2011.12.048, 57 (2013) 2403–2407.

[22] E. Set, M.E. Özdemir, S.S. Dragomir, *On the Hermite-Hadamard inequality and other integral inequalities involving two functions*, J. Inequal. Appl. (2010) 9. Article ID 148102.

[23] E. Set, J. Choi and B. Çelik, *New Hermite-Hadamard type inequalities for product of different convex functions involving certain fractional integral operators*, J. Math. Computer Sci., 18 (2018), 29–36.

[24] Y. Wu, F. Qi, and D.-W. Niu, *Integral inequalities of Hermite–Hadamard type for the product of strongly logarithmically convex and other convex functions*, Maejo International Journal of Science and Technology 9 (2015), no. 3, 394–402.

[25] H.-P. Yin, F. Qi, *Hermite-Hadamard type inequalities for the product of* (α, m)*-convex functions*, J. Nonlinear Sci. Appl. 8 (2015) 231–236.