Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **39** (2019) no. 2, 57–69. doi:10.1285/i15900932v39n2p57

The Ulam stability of non–linear Volterra integro–dynamic equations on time scales

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Received: 31.5.2018; accepted: 5.8.2019.

Abstract. This manuscript presents the Ulam stability results of non–linear Volterra integro–dynamic equation and its adjoint equation on time scales. First, we obtain the Ulam stability of adjoint equation by using the integrating factor method. Then, the Ulam stability of the corresponding equation is proved by means of the property of the exponential function and related results that are proved in adjoint equation with the help of integrating factor method. At the end, an example is given that shows the validity of our main results.

Keywords: Ulam stability, time scales, Volterra integro–dynamic equation, integrating factor

MSC 2000 classification: 34G20, 34N05, 45J05, 45M10

Introduction

In 1940, in a talk before the mathematics club at the university of Wisconsin, Ulam [25, 26] presented a famed question related to the stability of homomorphisms: "*With which requirements does an additive mapping near an approximate additive mapping exists?*".

This question was answered by Hyers [9], for the case of Banach spaces, by using direct method. So this interesting stability initiated by Ulam and Hyers is called Hyers–Ulam(HU) stability. In 1978, Rassias [20] extended HU stability concept by introducing new function variables and after that it famed for the Hyers–Ulam–Rassias(HUR) stability. For further details and discussions, we recommend the book by Jung [11].

At the end of 19th century, a large number of researchers contributed to the stability idea of Ulam's type for various types of differential equations. There are many advantages of Ulam's type stability in tackling problems, related to, optimization techniques, numerical analysis, control theory and many more, in such situations to get an exact solution is challenging. Obłoza sounds to be the first mathematician for investigating the HU stability of linear differential equations (DEs), (see [17, 18]). Thereafter, Alsina and Ger [2] published their results that holds the HU stability of the differential equation $\xi^{'}(t) = \xi(t)$. Jung in [10] investigated HU stability of first order linear DEs. Then Li and Shen [13] extended HU stability concept for DEs of second order. Jung and Roh [12], in 2016, generalized the HU stability concept for linear DEs with complex constant coefficients. For more details on HU stability, see [10, 13, 14, 15, 17, 18, 21, 22, 23, 24, 27, 28, 29, 30, 31, 32, 34, 35, 36, 37, 38, 41, 42, 43].

The theory of dynamic equations on time scales has been rising fast and has acknowledged a lot of interest in recent years. This theory was introduced by Hilger [8] in 1988, with the inspiration to provide a unification of continuous and discrete calculus. The time scale analysis is giving an opportunity to study the differential and difference equations in a unified way. For the basics of time scales calculus, the reader is referred to see the excellent monographs written by Bohner and Peterson [4, 5].
Nowadays, the Ulam's type stability idea is also getting attention for dynamic equations on time scales. András and Mészáros [3], in 2013, discussed the HU stability of some integral equations on time scale by using Picard operators. Recently, Shah and Zada [24] obtained very interesting results about the existence, uniqueness and stability of solution to mixed integral dynamic systems with instantaneous and non–instantaneous impulses on time scales. For more details on time scales, see [1, 3, 6, 7, 16, 19, 22, 23, 24, 33, 39, 40].

The utmost purpose of this manuscript is to find the Ulam stability for the following non–linear Volterra integro–dynamic equation

$$\varphi^{\Delta}(t) = p(t)\varphi(t) + \int_{t_0}^{t} \mathcal{K}(t, s, \varphi(s))\Delta s,\ t \in T_S^0 = [t_0, \infty)_{T_S}, \tag{0.1}$$

and its adjoint equation

$$\vartheta^{\Delta}(t) = -p(t)\vartheta^{\Theta}(t) + \int_{t_0}^{t} \mathcal{K}(t, s, \vartheta(s))\Delta s,\ t \in T_S^0, \tag{0.2}$$

where $p \in \mathcal{R}^{+}{}_{T_S^0}$ and $\mathcal{K}(t, s, z(s))$ is continuous operator on $\Gamma = \{(t, s, z) : 0 = t_0 \leq s \leq t < \infty,\ z \in \mathbb{R}\}$.

1 Preamble

In this section, we recall the main definitions and some basic notations of time scales calculus.

An arbitrary non–empty closed subset of real numbers T_S is called a time scale. The forward jump operator $\Theta : T_S \to T_S$, backward jump operator $\rho : T_S \to T_S$ and graininess operator $\mu : T_S \to [0,\infty)$, are defined by:

$$\Theta(s) = \inf\{t \in T_S : t > s\},\ \rho(s) = \sup\{t \in T_S : t < s\},\ \mu(s) = \Theta(s) - s,$$

respectively. An arbitrary $t \in T_S$ is called left scattered (resp. left dense) when $t > \rho(t)$ (resp. $t = \rho(t)$). While, in case of $t < \Theta(t)$ (resp. $\Theta(t) = t$), we call t is right scattered (resp. right dense). For a time scale T_S, the set of all limiting points T_S^z is called the derived set and illustrated as follows:

$$T_S^z = \begin{cases} T_S \backslash (\rho(\sup T_S), \sup T_S], & \text{if } \sup T_S < \infty, \\ T_S, & \text{if } \sup T_S = \infty. \end{cases}$$

Let $W : T_S \to \mathbb{R}$ be a real valued function, then $W^\Theta : T_S \to \mathbb{R}$ is defined as $W^\Theta(t) = W(\Theta(t)),\ \forall\ t \in T_S$. The real valued function $W : T_S \to \mathbb{R}$ is called right–dense continuous if it is continuous at every right–dense point on T_S and its left–sided limit exists at every left–dense point on T_S. The set of all right–dense(rd) continuous functions will be denoted by $\mathcal{C}_{RD}(T_S, \mathbb{R})$. The function $\mathcal{W} : T_S \to \mathbb{R}$ is called regressive (resp. positively regressive) if $1+\mu(t)\mathcal{W}(t) \neq 0$, (resp. $1 + \mu(t)\mathcal{W}(t) > 0$) $\forall\ t \in T_S^z$. The set of all right–dense continuous regressive functions (resp. right–dense continuous positively regressive functions) will be denoted by $\mathcal{R}_{T_S}$(resp. $\mathcal{R}^+{}_{T_S}$). If $\mathtt{G}, \mathcal{H} \in \mathcal{R}_{T_S}$, then for all $s \in T_S^z$,

$$\begin{aligned} (\mathtt{G} \oplus \mathcal{H})(s) &= \mathtt{G}(s) + \mathcal{H}(s) + \mu(s)\mathtt{G}(s)\mathcal{H}(s), \\ (\ominus \mathtt{G})(s) &= -\frac{\mathtt{G}(s)}{1 + \mu(s)\mathtt{G}(s)}, \\ \mathtt{G} \ominus \mathcal{H} &= \mathtt{G} \oplus (\ominus \mathcal{H}). \end{aligned}$$

The delta derivative of the function $W : T_S \to \mathbb{R}$ on $t \in T_S^z$, is given by

$$W^\Delta(t) = \lim_{s \to t,\ s \neq \Theta(t)} \frac{W(\Theta(t)) - W(s)}{\Theta(t) - s}.$$

For a rd–continuous function $W : T_S \to \mathbb{R}$, the $\Delta-$integral is defined to be

$$\int_a^b W(t)\Delta t = w(b) - w(a), \text{ for all } a, b\ \in\ T_S,$$

where w is the anti–derivative of W, i.e., $w^\Delta = W$ on T_S^z.

For $p \in \mathcal{R}_{T_S}$, the generalized exponential function is defined by

$$e_p(a, b) = \exp\left(\int_a^b \alpha_{\mu(s)} p(s)\Delta s\right) \text{ for all } a, b \in T_S,$$

while,

$$\alpha_{\mu(t)}p(t) = \begin{cases} \dfrac{Log(1+\mu(t)p(t))}{\mu(t)}, & \text{if } \mu(t) \neq 0, \\ p(t), & \text{if } \mu(t) = 0, \end{cases}$$

is the cylindrical transformation.

The unique solution to the dynamic equation $z^{\Delta}(t) = p(t)z(t),\ z(t_0) = 1,\ t \in T_S$, is given by $e_p(t, t_0)$.

Theorem 1. *[4] Let $p,\ q \in \mathcal{R}_{T_S}$, then:*

(1) $e_0(t,s) = 1$ *and* $e_p(t,t) = 1$.

(2) $e_p(\Theta(t), s) = (1 + \mu(t)p(t))e_p(t,s)$.

(3) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$.

(4) $(e_p(t,s))^{\Delta} = p(t)e_p(t,s)$.

(5) $e_p(t,s)e_p(s,r) = e_p(t,r)$.

(6) If $r, u, v \in T_S$*, then*

$$\int_r^u p(\eta)e_p(v, \Theta(\eta))\Delta\eta = e_p(v,r) - e_p(v,u).$$

Theorem 2. *(Properties of Differentiation)[4] Let* $\mathrm{g}, \mathrm{h} : T_S \to \mathbb{R}$ *be the differentiable functions at* $t \in {T_S}^z$*. Then*

(1) $(\mathrm{g}+\mathrm{h})^{\Delta}(t) = \mathrm{g}^{\Delta}(t) + \mathrm{h}^{\Delta}(t)$.

(2) For any constant c*,*

$$(c\mathrm{h})^{\Delta}(t) = c\mathrm{h}^{\Delta}(t).$$

(3) $(\mathrm{gh})^{\Delta}(t) = \mathrm{g}^{\Delta}(t)\mathrm{h}(t) + \mathrm{g}(\Theta(t))\mathrm{h}^{\Delta}(t) = \mathrm{g}(t)\mathrm{h}^{\Delta}(t) + \mathrm{g}^{\Delta}(t)\mathrm{h}(\Theta(t))$.

(4) If $\mathrm{g}(\Theta(t))\mathrm{g}(t) \neq 0$*, then*

$$\left(\frac{1}{\mathrm{g}}\right)^{\Delta}(t) = -\frac{\mathrm{g}^{\Delta}(t)}{\mathrm{g}(t)\mathrm{g}(\Theta(t))}.$$

(5) If $h(t)h(\Theta(t)) \neq 0$*, then*

$$\left(\frac{\mathrm{g}}{\mathrm{h}}\right)^{\Delta}(t) = \frac{\mathrm{g}^{\Delta}(t)\mathrm{h}(t) - \mathrm{h}^{\Delta}(t)\mathrm{g}(t)}{\mathrm{h}(t)\mathrm{h}(\Theta(t))}.$$

Theorem 3. *[4] Let $t_0 \in T_S$ and $u_0,\ v_0 \in \mathbb{R}$. The unique solutions of the initial value problems*

$$\vartheta^{\Delta}(t) = -p(t)\vartheta^{\Theta}(t) + g(t),\ \vartheta(t_0) = u_0,$$

and

$$\varphi^{\Delta}(t) = p(t)\varphi(t) + g(t),\ \varphi(t_0) = v_0,$$

are given by

$$\vartheta(t) = e_{\ominus p}(t, t_0)u_0 + \int_{t_0}^{t} e_{\ominus p}(t, \eta)g(\eta)\Delta\eta,$$

and

$$\varphi(t) = e_p(t, t_0)v_0 + \int_{t_0}^{t} e_p(t, \Theta(\eta))g(\eta)\Delta\eta.$$

Lemma 1. [7] *If $\lambda \in \mathcal{R}^{+}{}_{T_S}$, then for dynamic equation, $\psi^{\Delta}(t) = \lambda\psi(t)$ the following inequality holds*

$$e_\lambda(t, s) \leq e^{\lambda(t-s)},\ \forall\ t, s \in T_S.$$

2 Main Results

Now we state our major results. The first result is to establish the Ulam stablity of the adjoint Eq. (0.2) of Eq. (0.1).

Theorem 4. *For $p \in \mathcal{R}^{+}{}_{T_S^0}$ and a given $\epsilon > 0$, if a rd-continuous differentiable function $\vartheta : T_S^0 \to \mathbb{R}$ satisfies the inequality*

$$\left|\vartheta^{\Delta}(t) + p(t)\vartheta^{\Theta}(t) - \int_{t_0}^{t} \mathcal{K}(t, s, \vartheta(s))\Delta s\right| \leq \epsilon,\ \forall\ t \in T_S^0, \tag{2.1}$$

then there exists a solution $h \in \mathcal{C}_{RD}(T_S^0, \mathbb{R})$ of Eq. (0.2) such that

$$|\vartheta(t) - h(t)| \leq \epsilon \int_{t_0}^{t} |e_p(\upsilon, t)|\Delta\upsilon, \forall\ t \in T_S^0.$$

Proof. From (2.1), we get

$$-\epsilon \leq \vartheta^{\Delta}(t) + p(t)\vartheta^{\Theta}(t) - \int_{t_0}^{t} \mathcal{K}(t, s, \vartheta(s))\Delta s \leq \epsilon, \tag{2.2}$$

since $p \in \mathcal{R}^+{}_{T_S^0}$, it is obvious that the exponential function $e_p(t,t_0) > 0,\ \forall\ t \in T_S^0$. Multiplying (2.2) by $e_p(t,t_0)$, we have

$$\begin{aligned}&-\epsilon e_p(t,t_0) \le \vartheta^{\Delta}(t)e_p(t,t_0) + p(t)\vartheta^{\Theta}(t)e_p(t,t_0)\\&-e_p(t,t_0)\int_{t_0}^{t}\mathcal{K}(t,s,\vartheta(s))\Delta s \le e_p(t,t_0)\epsilon.\end{aligned}$$

Integrating from t_0 to t with respect to υ, we get

$$\begin{aligned}-\epsilon\int_{t_0}^{t}e_p(\upsilon,t_0)\Delta\upsilon \quad \le \quad & \int_{t_0}^{t}\Big(\vartheta^{\Delta}(\upsilon)e_p(\upsilon,t_0) + p(\upsilon)\vartheta^{\Theta}(\upsilon)e_p(\upsilon,t_0)\Big)\Delta\upsilon\\&-\int_{t_0}^{t}e_p(\upsilon,t_0)\int_{\upsilon_0}^{\upsilon}\mathcal{K}(t,s,\vartheta(s))\Delta s\Delta\upsilon \le \epsilon\int_{t_0}^{t}e_p(\upsilon,t_0)\Delta\upsilon.\end{aligned}$$

This implies

$$\begin{aligned}&-\epsilon\int_{t_0}^{t}e_p(\upsilon,t_0)\Delta\upsilon \le \int_{t_0}^{t}\Big(\vartheta(\upsilon)e_p(\upsilon,t_0)\Big)^{\Delta}\Delta\upsilon\\&-\int_{t_0}^{t}e_p(\upsilon,t_0)\int_{\upsilon_0}^{\upsilon}\mathcal{K}(t,s,\vartheta(s))\Delta s\Delta\upsilon \le \epsilon\int_{t_0}^{t}e_p(\upsilon,t_0)\Delta\upsilon.\end{aligned}$$

Equivalently,

$$\begin{aligned}&-\epsilon\int_{t_0}^{t}e_p(\upsilon,t_0)\Delta\upsilon \le \vartheta(t)e_p(t,t_0) - \vartheta(t_0)\\&-\int_{t_0}^{t}e_p(\upsilon,t_0)\int_{\upsilon_0}^{\upsilon}\mathcal{K}(t,s,\vartheta(s))\Delta s\Delta\upsilon \le \epsilon\int_{t_0}^{t}e_p(\upsilon,t_0)\Delta\upsilon.\end{aligned}$$

Multiplying both sides of the above inequality by $e_{\ominus p}(t,t_0)$, we get

$$\begin{aligned}&-\epsilon\int_{t_0}^{t}e_p(\upsilon,t_0)e_{\ominus p}(t,t_0)\Delta\upsilon \le \vartheta(t)e_p(t,t_0)e_{\ominus p}(t,t_0) - \vartheta(t_0)e_{\ominus p}(t,t_0)\\&-\int_{t_0}^{t}e_p(\upsilon,t_0)e_{\ominus p}(t,t_0)\int_{\upsilon_0}^{\upsilon}\mathcal{K}(t,s,\vartheta(s))\Delta s\Delta\upsilon \le \epsilon\int_{t_0}^{t}e_p(\upsilon,t_0)e_{\ominus p}(t,t_0)\Delta\upsilon.\end{aligned}$$

By using the property 3 and property 5 of Theorem 1, we get

$$\begin{aligned}-\epsilon\int_{t_0}^{t}e_p(\upsilon,t)\Delta\upsilon \quad \le \quad & \vartheta(t) - \vartheta(t_0)e_{\ominus p}(t,t_0)\\&-\int_{t_0}^{t}e_p(\upsilon,t)\int_{\upsilon_0}^{\upsilon}\mathcal{K}(t,s,\vartheta(s))\Delta s\Delta\upsilon \le \epsilon\int_{t_0}^{t}e_p(\upsilon,t)\Delta\upsilon.\end{aligned}$$

This implies

$$\left|\vartheta(t)-\vartheta(t_0)e_{\ominus p}(t,t_0)-\int_{t_0}^{t}e_p(\upsilon,t)\int_{\upsilon_0}^{\upsilon}\mathcal{K}(t,s,\vartheta(s))\Delta s\Delta\upsilon\right|\leq\left|\epsilon\int_{t_0}^{t}e_p(\upsilon,t)\Delta\upsilon\right|$$

$$\leq\ \epsilon\int_{t_0}^{t}|e_p(\upsilon,t)|\Delta\upsilon.$$

By putting $h(t)=\vartheta(t_0)e_{\ominus p}(t,t_0)+\int_{t_0}^{t}e_p(\upsilon,t)\int_{\upsilon_0}^{\upsilon}\mathcal{K}(t,s,\vartheta(s))\Delta s\Delta\upsilon$, it can be easily verified that $h(t)$ is a solution of Eq. (0.2). so we get,

$$|\vartheta(t)-h(t)|\leq\epsilon\int_{t_0}^{t}|e_p(\upsilon,t)|\Delta\upsilon,$$

which is the required inequality. $\boxed{QED}$

Remark 1. If $\epsilon\int_{t_0}^{t}|e_p(\upsilon,t)|\Delta\upsilon$ is a positive constant function, then Eq. (0.2) has HU stability on T_S^0 and if $\epsilon\int_{t_0}^{t}|e_p(\upsilon,t)|\Delta\upsilon$ is a positive increasing function, then Eq. (0.2) has HUR stability on T_S^0.

In the following theorem, we state about the Ulam stability of Eq. (0.1).

Theorem 5. *For $p\in\mathcal{R}^+{}_{T_S^0}$ and a given $\epsilon>0$, if a rd-continuous differentiable function $\varphi:T_S^0\to\mathbb{R}$ satisfies the inequality*

$$\left|\varphi^{\Delta}(t)-p(t)\varphi(t)-\int_{t_0}^{t}\mathcal{K}(t,s,\varphi(s))\Delta s\right|\leq\epsilon,\ \forall\ t\in T_S^0,\tag{2.3}$$

then there exists a solution $g\in\mathcal{C}_{RD}(T_S^0,\mathbb{R})$ of Eq. (1.1) such that

$$|\varphi(t)-g(t)|\leq\epsilon\int_{t_0}^{t}|e_p(t,\Theta(\upsilon))|\Delta\upsilon,\forall\ t\in T_S^0.$$

Proof. Replacing $\varphi(t)$ in (2.3) by the formula,

$$\varphi^{\Theta}(t)=\varphi(t)+\mu(t)\varphi^{\Delta}(t),$$

we get

$$\left|(1+\mu(t)p(t))\varphi^{\Delta}(t)-p(t)\varphi^{\Theta}(t)-\int_{t_0}^{t}\mathcal{K}(t,s,\varphi(s))\Delta s\right|\leq\epsilon,\ \forall\ t\in T_S^0,$$

since $p\in\mathcal{R}^+{}_{T_S^0}$, we obtain that $1+\mu(t)p(t)>0$, multiply both sides of the above inequality by $\frac{1}{1+\mu(t)p(t)}$, we get

$$\left|\varphi^{\Delta}(t)+(\ominus p)(t)\varphi^{\Theta}(t)-\frac{1}{1+\mu(t)p(t)}\int_{t_0}^{t}\mathcal{K}(t,s,\varphi(s))\Delta s\right|\leq\epsilon\frac{1}{1+\mu(t)p(t)},\ \forall\ t\in T_S^0,$$

Note that $p \in \mathcal{R}^+{}_{T_S^0}$ implies $\ominus p \in \mathcal{R}^+{}_{T_S^0}$. By multiplying both sides of the above inequality by $e_{\ominus p}(t, t_0)$, we get

$$\Big| \varphi^\Delta(t) e_{\ominus p}(t, t_0) + (\ominus p)(t) e_{\ominus p}(t, t_0) \varphi^\Theta(t) - \frac{e_{\ominus p}(t, t_0)}{1 + \mu(t)p(t)} \int_{t_0}^{t} \mathcal{K}(t, s, \varphi(s))\Delta s \Big| \leq \epsilon \frac{e_{\ominus p}(t, t_0)}{1 + \mu(t)p(t)}, \ \forall\ t \in T_S^0.$$

By using the property 2 and property 3 of Theorem 1 in above inequality, we get

$$\Big| \varphi^\Delta(t) e_{\ominus p}(t, t_0) + (\ominus p)(t) e_{\ominus p}(t, t_0) \varphi^\Theta(t) - e_{\ominus p}(\Theta(t), t_0) \int_{t_0}^{t} \mathcal{K}(t, s, \varphi(s))\Delta s \Big| \leq \epsilon e_{\ominus p}(\Theta(t), t_0), \ \forall\ t \in T_S^0.$$

By the same calculation and process as in above Theorem 4, we can obtain

$$\begin{aligned} \Big| \varphi(t) - \varphi(t_0) e_p(t, t_0) - \int_{t_0}^{t} e_p(t, \Theta(\upsilon)) \int_{\upsilon_0}^{\upsilon} \mathcal{K}(t, s, \varphi(s))\Delta s \Delta\upsilon \Big| &\leq \Big| \epsilon \int_{t_0}^{t} e_p(t, \Theta(\upsilon))\Delta\upsilon \Big| \\ &\leq \epsilon \int_{t_0}^{t} |e_p(t, \Theta(\upsilon))|\Delta\upsilon. \end{aligned}$$

By putting $g(t) = \varphi(t_0)e_p(t, t_0) + \int_{t_0}^{t} e_p(t, \Theta(\upsilon)) \int_{\upsilon_0}^{\upsilon} \mathcal{K}(t, s, \varphi(s))\Delta s\Delta\upsilon$. It can be easily checked that $g(t)$ is a solution of Eq. (0.1), so we have

$$|\varphi(t) - g(t)| \leq \epsilon \int_{t_0}^{t} |e_p(t, \Theta(\upsilon))|\Delta\upsilon, \forall\ t \in T_S^0,$$

which is the required inequality. $\boxed{QED}$

Remark 2. If $\epsilon \int_{t_0}^{t} |e_p(t, \Theta(\upsilon))|\Delta\upsilon$ is a positive constant function, then Eq. (0.1) has HU stability on T_S^0 and if $\epsilon \int_{t_0}^{t} |e_p(t, \Theta(\upsilon))|\Delta\upsilon$ is a positive increasing function, then Eq. (1.1) has HUR stability on T_S^0.

Example 1. Consider the following non–linear Volterra integro–dynamic equation

$$\varphi^\Delta(t) = \varphi(t) + \int_{t_0}^{t} e_q(t, s) e_q(s, \varphi(s))\Delta s, \ t \in T_S^0, \tag{2.4}$$

and its adjoint equation

$$\vartheta^\Delta(t) = -\vartheta^\Theta(t) + \int_{t_0}^{t} e_q(t, s) e_q(s, \vartheta(s))\Delta s, \ t \in T_S^0, \tag{2.5}$$

where $p(t) = 1$ is a positively regressive constant function, $q \in \mathcal{C}_{RD}(T_S^0, \mathbb{R})$ and $\mathcal{K}(t, s, \varphi(s)) = e_q(t, s)e_q(s, \varphi(s))$. If $\varphi, \vartheta \in \mathcal{C}_{RD}(T_S^0, \mathbb{R})$ satisfies the following inequalities

$$\left|\vartheta^{\Delta}(t) + \vartheta^{\Theta}(t) - \int_{t_0}^{t} \mathcal{K}(t, s, \vartheta(s))\Delta s\right| \leq \frac{1}{2} = \epsilon, \ \forall\ t \in T_S^0, \tag{2.6}$$

$$\left|\varphi^{\Delta}(t) - \varphi(t) - \int_{t_0}^{t} \mathcal{K}(t, s, \varphi(s))\Delta s\right| \leq 1 = \epsilon, \ \forall\ t \in T_S^0, \tag{2.7}$$

then concerning to inequality (2.6), by using Theorem 4, $\forall\ t \in T_S^0$, we get

$$\begin{aligned}
|\vartheta(t) - h(t)| &\leq \frac{1}{2}\int_{t_0}^{t} |e_p(\upsilon, t)|\Delta\upsilon \\
&\leq \frac{1}{2}\int_{t_0}^{t} |e^{p(\upsilon - t)}|\Delta\upsilon \\
&= \frac{1}{2}e^{-t}\int_{t_0}^{t} e^{\upsilon}\Delta\upsilon \\
&= \frac{1}{2}e^{-t}(e^{t} - e^{t_0}) \\
&\leq \frac{1}{2}.
\end{aligned}$$

So by Remark 1, Eq. (2.5) is HU stable. Now concerning to inequality (2.7), by using Theorem 5, $\forall\ t \in T_S^0$, we get

$$\begin{aligned}
|\varphi(t) - g(t)| &\leq \int_{t_0}^{t} |e_p(t, \Theta(\upsilon))|\Delta\upsilon \\
&= \int_{t_0}^{t} |e_{\ominus p}(\Theta(\upsilon), t)\Delta|\upsilon \\
&= \int_{t_0}^{t} (1 + \mu(t) \ominus p(t))e_{\ominus p}(\upsilon, t)\Delta\upsilon \\
&= \int_{t_0}^{t} \left(1 + \mu(t)\big(\frac{-1}{1 + \mu(t)}\big)\right)e_p(t, \upsilon)\Delta\upsilon \\
&\leq \left(1 - \big(\frac{\mu(t)}{1 + \mu(t)}\big)\right)\int_{t_0}^{t} e^{p(t-\upsilon)}\Delta\upsilon \\
&= \left(1 - \big(\frac{\mu(t)}{1 + \mu(t)}\big)\right)e^{t}\int_{t_0}^{t} e^{-\upsilon}\Delta\upsilon \\
&= \left(1 - \big(\frac{\mu(t)}{1 + \mu(t)}\big)\right)e^{t}(-e^{-t} + e^{-t_0})
\end{aligned}$$

$$\begin{aligned}
&= \left(1-\big(\frac{\mu(t)}{1+\mu(t)}\big)\right)(e^{t-t_0}-1)\\
&\leq \left(1-\big(\frac{\mu(t)}{1+\mu(t)}\big)\right)(e^{t-t_0})\\
&= \left(e^{t-t_0}-\big(e^{t-t_0}\frac{\mu(t)}{1+\mu(t)}\big)\right)\\
&\leq e^t e^{-t_0}.
\end{aligned}$$

As e^t is a positive increasing function. So by Remark 2, Eq. (2.4) is HUR stable.

3 Conclusion

This manuscript is about the establishment of Ulam stability results of non–linear Volterra integro–dynamic equation and its adjoint equation on time scales. Our results were proved by using the integrating factor method. In fact, our results are important, when finding exact solution is quite difficult, and hence they are important in approximation theory etc [22, 23, 24].

Conflict of Interests

The authors declare that there is no conflict of interest regarding the publication of this paper.

Competing interest

The authors declare that they have no competing interest regarding this research work.

Author's contributions

All the authors contributed equally and significantly in writing this paper. All the authors read and approved the final manuscript.

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