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On (n, k)-quasi class Q Operators

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Abstract.
Let T be a bounded linear operator on a complex Hilbert space H. In this paper we introduce a new class of operators: (n, k)-quasi class Q operators, superclass of (n, k)-quasi paranormal operators. An operator T is said to be (n, k)-quasi class Q if it satisfies

$$\|T(T^k x)\|^2 \leq \frac{1}{n+1}\left(\|T^{1+n}(T^k x)\|^2 + n\|T^k x\|^2\right),$$

for all $x \in H$ and for some nonnegative integers n and k. We prove the basic structural properties of this class of operators. It will be proved that If T has a no non-trivial invariant subspace, then the nonnegative operator

$$D = T^{*k}\left(T^{*(1+n)}T^{(1+n)} - \frac{n+1}{n}T^*T + I\right)T^k$$

is a strongly stable contraction. In section 4, we give some examples which compare our class with other known classes of operators and as a consequence we prove that (n, k)-quasi class Q does not have SVEP property. In the last section we also characterize the (n, k)-quasi class Q composition operators on Fock spaces.

Keywords: (n, k)-quasi class Q, (n, k)-quasi paranormal operators, SVEP property, Fock space, composition operators.

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Introduction

Throughout this paper, let H be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let $L(H)$ denote the C^* algebra of all bounded operators on H. For $T \in L(H)$, we denote by $\ker(T)$ the null space and by $T(H)$ the range of T. The null operator and the identity on H will be denoted by O and I, respectively. If T is an operator, then T^* is its adjoint, and $\|T\| = \|T^*\|$.

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We shall denote the set of all complex numbers by $\mathbb{C}$, the set of all positive integers by $\mathbb{N}$, the set of all nonnegative integers by $\mathbb{N}_0$ and the complex conjugate of a complex number λ by $\overline{\lambda}$. The closure of a set M will be denoted by $\overline{M}$ and we shall henceforth shorten $T-\lambda I$ to $T-\lambda$. An operator $T \in L(H)$ is a positive operator, $T \geq O$, if $\langle Tx, x\rangle \geq 0$ for all $x \in H$.

We write $\sigma(T)$, $\sigma_p(T)$, $\sigma_a(T)$, and $r(T)$ for the spectrum, point spectrum, approximate point spectrum and spectral radius for operator T, respectively. It is well known that $r(T) \leq \|T\|$. The operator T is called normaloid if $r(T) = \|T\|$.

An operator $T \in L(H)$, is said to be paranormal [8], if

$$\|Tx\|^2 \leq \|T^2x\|$$

for any unit vector x in H. An operator $T \in L(H)$, is said to be quasi$-$paranormal operator if

$$\|T^2x\|^2 \leq \|T^3x\|\|Tx\|,$$

for all $x \in H$. Mecheri, [11] introduced a new class of operators called k-quasi paranormal operators. An operator T is called $k-$quasi paranormal if

$$\|T^{k+1}x\|^2 \leq \|T^{k+2}x\|\|T^kx\|,$$

for all $x \in H$, where $k \in \mathbb{N}_0$.

J. T. Yuan and G. X. Ji [14] introduced a new class of operators called (n,k)-quasi paranormal operators: An operator $T \in L(H)$ is said to be (n,k)-quasi paranormal operators if

$$\|T(T^kx)\| \leq \|T^{1+n}(T^kx)\|^{\frac{1}{1+n}}\|T^kx\|^{\frac{n}{n+1}},$$

for all $x \in H$.

1 Main results

Now we introduce the class of (n,k)-quasi class Q operators defined as follows:

Definition 1 An operator T is said to be of the (n,k)-quasi class Q if

$$\|T(T^kx)\|^2 \leq \frac{1}{n+1}\left(\|T^{1+n}(T^kx)\|^2 + n\|T^kx\|^2\right),$$

for all $x \in H$ and for some nonnegative integers n and k.

A $(1,k)$-quasi class Q operator is a k-quasi class Q operator:

$$\|T^{k+1}x\|^2 \leq \frac{1}{2}\left(\|T^{k+2}x\|^2 + \|T^kx\|^2\right);$$

$(1,1)$-quasi class Q operator is a quasi class Q operator: $\|T^2x\|^2 \le \frac{1}{2}\left(\|T^3x\|^2 + \|Tx\|^2\right)$; $(1,0)$-quasi class Q operator is a class Q operator, Duggal, Kubrusly, Levan [5]: $\|Tx\|^2 \le \frac{1}{2}\left(\|T^2x\|^2 + \|x\|^2\right)$; $(n,0)$-quasi class Q operator is a n-class Q operator

$$\|Tx\|^2 \le \frac{1}{n+1}\left(\|T^{1+n}x\|^2 + n\|x\|^2\right).$$

Yuan and Ji [14, Lemma 2.2] prove that an operator $T \in L(H)$ is of the (n,k)-quasi paranormal if and only if

$$T^{*k}\left(T^{*(1+n)}T^{(1+n)} - (n+1)\lambda^n T^*T + n\lambda^{n+1}I\right)T^k \ge O, \text{ for all } \lambda > 0.$$

Theorem 1. *An operator $T \in L(H)$ is of the (n,k)-quasi class Q, if and only if*

$$T^{*k}\left(T^{*(1+n)}T^{(1+n)} - (n+1)T^*T + nI\right)T^k \ge O,$$

where k and n are nonnegative integer numbers.

Proof. Since T is of the (n,k)-quasi class Q, then an application of the quadratic inequality implies

$$(n+1)\|T(T^kx)\|^2 \le \left(\|T^{1+n}(T^kx)\|^2 + n\|T^kx\|^2\right),$$

for all $x \in H$, where $k,n \in \mathbb{N}_0$. Then,

$$\left\langle T^{*k}\left(T^{*(1+n)}T^{(1+n)} - (n+1)T^*T + nI\right)T^kx, x\right\rangle \ge 0$$

for all $x \in \mathcal{H}$. The last relation is equivalent to

$$T^{*k}\left(T^{*(1+n)}T^{(1+n)} - (n+1)T^*T + nI\right)T^k \ge O.$$

QED

Lemma 1([4], page 17) For positive real numbers $a > 0$ and $b > 0$,

$$\lambda a + \mu b \ge a^\lambda b^\mu$$

holds for $\lambda > 0$ and $\mu > 0$ such that $\lambda + \mu = 1$.

Lemma 2 If T is an $(n,k)-$quasi paranormal operator, then T is an (n,k)-quasi class Q, operator.

Proof. Let T be an operator of $(n,k)-$quasi paranormal operator. Then, we have

$$\begin{aligned}&\|T(T^kx)\|^2\\ &\quad\leq \|T^{1+n}(T^kx)\|^{\frac{2}{1+n}}\|T^kx\|^{\frac{2n}{n+1}}\\ &\quad\leq \frac{1}{1+n}\|T^{1+n}(T^kx)\|^2+\frac{n}{n+1}\|T^kx\|^2\end{aligned}$$

so, T is an (n,k)-quasi class Q operator. QED

An operator $T\in L(H)$, is said to belong to $k-$quasi class $\mathcal{A}_n$ operator if

$$T^{*k}\left(|T^{n+1}|^{\frac{2}{n+1}}-|T|^2\right)T^k\geq O$$

for some nonnegative integer numbers n and k, [15].

From [15, Theorem 2.2] if T is a $k-$quasi class $\mathcal{A}_n$ operator, then T is an $(n,k)-$quasi paranormal operator, from the above theorem T is an (n,k)-quasi class Q operator.

If T is an (n,k)-quasi class Q operator, then T is an $(n,k+1)$-quasi class Q operator. The converse is not true, as it can be seen below.

Example 1 Consider the unilateral weighted shift operators as an infinite dimensional Hilbert space operator. Recall that given a bounded sequence of a positive numbers $\alpha:\alpha_1,\alpha_2,\alpha_3,\alpha_4,...$ (called weights) the unilateral weighted shift W_α associated with weight α is the operator on $H=l_2$ defined by $W_\alpha e_m=\alpha_m e_{m+1}$ for all $m\geq 1$, where $\{e_m\}_{m=1}^\infty$ is the canonical orthonormal basis on l_2.

$$\mathbf{W}_\alpha=\begin{pmatrix}0&0&0&0&0&\dots\\ \alpha_1&0&0&0&0&\dots\\ 0&\alpha_2&0&0&0&\dots\\ 0&0&\alpha_3&0&0&\dots\\ 0&0&0&\alpha_4&0&\dots\\ \vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}$$

Let $\text{diag}(\{\alpha_m\}_{m=1}^\infty)=\text{diag}(\alpha_1,\alpha_2,\alpha_3,...)$ denote an infinite diagonal matrix on l_2. Then,

$$\begin{aligned}&W_\alpha^{*k}\left(W_\alpha^{*(n+1)}W_\alpha^{(n+1)}-(n+1)W_\alpha^*W_\alpha+n\right)W_\alpha^k\\ &=\text{diag}(\{\alpha_m^2\alpha_{m+1}^2\cdot...\cdot\alpha_{m+k-2}^2\alpha_{m+k-1}^2\alpha_{m+k}^2\alpha_{m+k+1}^2\cdot...\cdot\alpha_{m+k+n-1}^2\alpha_{m+k+n}^2\}_{m=1}^\infty)\\ &-(n+1)\text{diag}(\{\alpha_m^2\alpha_{m+1}^2\cdot...\cdot\alpha_{m+k-2}^2\alpha_{m+k-1}^2\alpha_{m+k}^2\}_{m=1}^\infty)\\ &+n\text{diag}(\{\alpha_m^2\alpha_{m+1}^2\cdot...\cdot\alpha_{m+k-1}^2\}_{m=1}^\infty)\end{aligned}$$

Then,

$$\alpha_m^2\alpha_{m+1}^2\cdot\ldots\cdot\alpha_{m+k-1}^2\left(\alpha_{m+k}^2\alpha_{m+k+1}^2\cdot\ldots\cdot\alpha_{m+k+n-1}^2\alpha_{m+k+n}^2-(n+1)\alpha_{m+k}^2+n\right)\geq 0$$

Thus, W_α is an (n,k)-quasi class Q operator, if and only if,

$$\alpha_{m+k}^2\alpha_{m+k+1}^2\cdot\ldots\cdot\alpha_{m+k+n-1}^2\alpha_{m+k+n}^2-(n+1)\alpha_{m+k}^2+n\geq 0,$$

for $m\geq 1$.

If $\alpha_2=2$ and $\alpha_m=1$ for $m\geq 3$, then W_α is a $(2,2)$-quasi class Q operator but it is not a $(2,1)$-quasi class Q operator.

Since $(n,k)-$quasi paranormal is not a normaloid operator [14, Example 2.3], then (n,k)-quasi class Q is not a normaloid operator.

Theorem 2. *Let $T\in L(H)$. If $\lambda^{-\frac{1}{2}}T$ is an operator of the (n,k)-quasi class Q, then T is of the (n,k)-quasi paranormal for all $\lambda>0$.*

Proof. Let $\lambda^{-\frac{1}{2}}T$ be an operator of (n,k)-quasi class Q, then

$$(\lambda^{-\frac{1}{2}}T)^{*k}\left((\lambda^{-\frac{1}{2}}T)^{*(n+1)}(\lambda^{-\frac{1}{2}}T)^{(n+1)}-(n+1)(\lambda^{-\frac{1}{2}}T)^*(\lambda^{-\frac{1}{2}}T)+nI\right)(\lambda^{-\frac{1}{2}}T)^k\geq O$$

$$\lambda^{-\frac{k}{2}}T^{*k}\left(\lambda^{-(n+1)}T^{*(n+1)}T^{(n+1)}-(n+1)\lambda^{-1}T^*T+nI\right)\lambda^{-\frac{k}{2}}T^k\geq O,$$

$$\frac{1}{\lambda^{k+n+1}}T^{*k}\left(T^{*(n+1)}T^{(n+1)}-(n+1)\lambda^nT^*T+n\lambda^{(n+1)}\right)T^k\geq O,$$

$$T^{*k}\left(T^{*(n+1)}T^{(n+1)}-(n+1)\lambda^nT^*T+n\lambda^{(n+1)}\right)T^k\geq O$$

for all $\lambda>0$.

By this it is proved that the operator T is an $(n,k)-$quasiparanormal operator. QED

Theorem 3. *Let T be a Hilbert space operator. If $\|T\|\leq\sqrt{\frac{n}{n+1}}$,(so T is contraction) then T is an (n,k)-quasi class Q operator.*

Proof. From $\|T\|\leq\sqrt{\frac{n}{n+1}}$, we have $\|T\|^2\leq\frac{n}{n+1}$. Then,

$$O\leq nI-(n+1)T^*T\leq T^{*(1+n)}T^{(n+1)}-(n+1)T^*T+nI,$$

therefore

$$O\leq T^{*k}\left(T^{*(1+n)}T^{(n+1)}-(n+1)T^*T+nI\right)T^k$$

so T is of the (n,k)-quasi class Q operator. QED

Corollary 1 Let T be a Hilbert space operator. If $T^{n+1}=O$ then T is $(n,0)$-quasi class Q operator if and only if $\|T\|\leq\sqrt{\frac{n}{n+1}}$.

Theorem 4. *The following statements are equivalent:*
(1) T is an (n,k)-quasi class Q, operator
(2)

$$T=\begin{pmatrix}A & B\\ O & C\end{pmatrix}\qquad on\quad H=\overline{T^k(H)}\oplus\ker(T^{*k}),$$

where $A^{*(1+n)}A^{(1+n)}-(n+1)A^*A+nI\geq O$, and $C^k=O$. Furthermore, $\sigma(T)=\sigma(A)\cup\{0\}$.

Proof. The equivalence being evident in the case in which T has a dense range, we consider the case in which T does not have a dense range.
(1) $\Rightarrow$ (2) Consider the matrix representation of T with respect to the decomposition $H=\overline{T^k(H)}\oplus\ker(T^{*k})$:

$$T=\begin{pmatrix}A & B\\ O & C\end{pmatrix}.$$

Let P be the projection onto $\overline{T^k(H)}$. Since T is an (n,k)-quasi class Q, operator, we have

$$P\left(T^{*(1+n)}T^{(1+n)}-(n+1)T^*T+nI\right)P\geq O.$$

Therefore

$$A^{*(1+n)}A^{(1+n)}-(n+1)A^*A+nI\geq O.$$

Let $x=\begin{pmatrix}x_1\\ x_2\end{pmatrix}\in H=\overline{T^k(H)}\oplus\ker(T^{*k})$. Then,

$$\langle C^kx_2,x_2\rangle=\left\langle T^k(I-P)x,(I-P)x\right\rangle=\left\langle(I-P)x,T^{*k}(I-P)x\right\rangle=0,$$

thus $C^k=O$.

By [10, Corollary 7], $\sigma(A)\cup\sigma(C)=\sigma(T)\cup\vartheta$, where ϑ is the union of the holes in $\sigma(T)$, which happen to be a subset of $\sigma(A)\cap\sigma(C)$.

The operator C being nilpotent, $\sigma(A)\cup\sigma(C)$ has no interior points, and this by [7, Corollary (state corollary number)] implies $\sigma(T)=\sigma(A)\cup\{0\}$.

(2) $\Rightarrow$ (1) Suppose $T=\begin{pmatrix}A & B\\ O & C\end{pmatrix}$ on $H=\overline{T^k(H)}\oplus\ker(T^{*k})$, where $A^{*(1+n)}A^{(1+n)}-(n+1)A^*A+nI\geq O$ and $C^k=O$.
Since

$$T^k=\begin{pmatrix}A^k & \sum_{j=0}^{k-1}A^jBC^{k-1-j}\\ O & O\end{pmatrix}$$

we have

$$
\begin{aligned}
&T^{*k}\left(T^{*(1+n)}T^{(1+n)}-(n+1)T^{*}T+nI\right)T^{k}\\
&=\begin{pmatrix} A^{*k} & O\\ (\sum_{j=0}^{k-1}A^{j}BC^{k-1-j})^{*} & O\end{pmatrix}\begin{pmatrix} D & E\\ E^{*} & F\end{pmatrix}\begin{pmatrix} A^{k} & \sum_{j=0}^{k-1}A^{j}BC^{k-1-j}\\ O & O\end{pmatrix}\\
&=\begin{pmatrix} A^{*k}DA^{k} & A^{*k}D\sum_{j=0}^{k-1}A^{j}BC^{k-1-j}\\ (\sum_{j=0}^{k-1}A^{j}BC^{k-1-j})^{*}DA^{k} & (\sum_{j=0}^{k-1}A^{j}BC^{k-1-j})^{*}D\sum_{j=0}^{k-1}A^{j}BC^{k-1-j}\end{pmatrix}
\end{aligned}
$$

where

$$D=A^{*(1+n)}A^{(1+n)}-(n+1)A^{*}A+n$$

$$C=A^{*(1+n)}\sum_{j=0}^{n}A^{j}BC^{n-j}-(n+1)A^{*}B$$

$$F=(\sum_{j=0}^{n}A^{j}BC^{n-j})^{*}(\sum_{j=0}^{n}A^{j}BC^{n-j})+C^{*(1+n)}C^{(1+n)}-(n+1)(B^{*}B+CC^{*})+n$$

Let $v=x\oplus y$ be a vector in $H=\overline{T^{k}(H)}\oplus\ker(T^{*k})$, where $x\in\overline{T^{k}(H)}$ and $y\in\ker(T^{*k})$. Then

$$
\begin{aligned}
&\left\langle T^{*k}\left(T^{*(1+n)}T^{(1+n)}-(n+1)T^{*}T+nI\right)T^{k}v,v\right\rangle\\
&=\left\langle A^{*k}DA^{k}x,x\right\rangle+\left\langle A^{*k}D\sum_{j=0}^{k-1}A^{j}BC^{k-1-j}y,x\right\rangle+\left\langle(\sum_{j=0}^{k-1}A^{j}BC^{k-1-j})^{*}DA^{k}x,y\right\rangle\\
&+\left\langle(\sum_{j=0}^{k-1}A^{j}BC^{k-1-j})^{*}D\sum_{j=0}^{k-1}A^{j}BC^{k-1-j}y,y\right\rangle\\
&=\left\langle D(A^{k}x+\sum_{j=0}^{k-1}A^{j}BC^{k-1-j}y),(A^{k}x+\sum_{j=0}^{k-1}A^{j}BC^{k-1-j}y)\right\rangle
\end{aligned}
$$

Since $D=A^{*(1+n)}A^{(1+n)}-(n+1)A^{*}A+n\geq O$ we have

$$\left\langle T^{*k}\left(T^{*(1+n)}T^{(1+n)}-(n+1)T^{*}T+nI\right)T^{k}v,v\right\rangle\geq 0,$$

hence

$$T^{*k}\left(T^{*(1+n)}T^{(1+n)}-(n+1)T^{*}T+nI\right)T^{k}\geq O.$$

Thus, T is an (n,k)-quasi class Q, operator.

QED

Corollary 2 If T is an (n,k)-quasi class Q, operator and $T^k(H)$ is not dense range, then

$$T = \begin{pmatrix} A & B \\ O & C \end{pmatrix} \quad \text{on} \quad H = \overline{T^k(H)} \oplus \ker(T^{*k}),$$

where A is an n-class Q operator on $\overline{T^k(H)}$, and $C^k = O$.

Theorem 5. *If T is an (n,k)-quasi class Q operator and M is an invariant subspace for T, then the restriction $T|_M$ is also an (n,k)-quasi class Q operator.*

Proof. Let P be the projection onto M. Then $TP = PTP$, so that $T|_M = PTP$. Hence, for $x \in M$ we have

$$\begin{aligned} &\|(T|_M)((T|_M)^k x)\|^2 \\ &= \|(PTP)(PTP)^k x\|^2 = \|P(TT^k x)\|^2 \leq \|T(T^k x)\|^2 \\ &\leq \frac{1}{n+1}\left(\|T^{n+1}(T^k x)\|^2 + n\|T^k x\|^2\right) \\ &= \frac{1}{n+1}\left(\|(T|_M)^{n+1}((T|_M)^k x)\|^2 + n\|(T|_M)^k x\|^2\right). \end{aligned}$$

QED

Theorem 6. *If T is an invertible (n,k)-quasi class Q, operator, then the point approximate spectrum lies in the disc:*

$$\sigma_a(T) \subseteq \left\{\lambda \in \mathbb{C} : \frac{\sqrt{1+n}}{\|T^{-k-1}\|\sqrt{\|T^{n+k}\|^2 + n\|T^{k-1}\|^2}} \leq |\lambda| \leq \|T\|\right\}$$

Proof. Suppose T is an invertible (n,k)-quasi class Q, operator. Then we have

$$\begin{aligned} &\|x\|^2 \\ &= \|T^{-k-1}T^{k+1}x\|^2 \leq \|T^{-k-1}\|^2\|T^{k+1}x\|^2 \\ &\leq \|T^{-k-1}\|^2\|\frac{1}{n+1}\left(\|T^{n+1}(T^k x)\|^2 + n\|T^k x\|^2\right) \\ &\leq \|T^{-k-1}\|^2\|\frac{1}{n+1}\left(\|T^{n+k}\|^2\|Tx\|^2 + n\|T^{k-1}\|^2\|Tx\|^2\right) \end{aligned}$$

Hence,

$$\|Tx\|^2 \geq \frac{(1+n)\|x\|^2}{\|T^{-k-1}\|^2\left(\|T^{n+k}\|^2 + n\|T^{k-1}\|^2\right)}.$$

Suppose that $\lambda \in \sigma_a(T)$. Then, there exists a sequence $\{x_m\}$ such that $\|(T-\lambda)x_m\| \longrightarrow 0$ when $m \to \infty$. We have

$$\begin{aligned}\|Tx_m - \lambda x_m\| & \\ &\geq \|Tx_m\| - \|\lambda x_m\| \geq \|T\| - |\lambda| \\ &\geq \frac{(1+n)^{\frac{1}{2}}}{\|T^{-k-1}\| \left(\|T^{n+k}\|^2 + n\|T^{k-1}\|^2\right)^{\frac{1}{2}}} - |\lambda|\end{aligned}$$

So, when $m \to \infty$,

$$|\lambda| \geq \frac{\sqrt{1+n}}{\|T^{-k-1}\|\sqrt{\|T^{n+k}\|^2 + n\|T^{k-1}\|^2}}.$$

QED

2 (n,k)-Quasi Class Q Operators Which are Contractions

A contraction is an operator T such that $\|Tx\| \leq \|x\|$ for all $x \in H$. A proper contraction is an operator T such that $\|Tx\| < \|x\|$ for every nonzero $x \in H$. A strict contraction is an operator such that $\|T\| < 1$ (*i.e.*, $\sup_{x\neq 0} \frac{\|Tx\|}{\|x\|} < 1$). Obviously, every strict contraction is a proper contraction and every proper contraction is a contraction. An operator T is said to be completely non-unitary (c.n.u) if T restricted to every reducing subspace of H has no unitary part.

An operator T on H is uniformly stable, if the power sequence $\{T^m\}_{m=1}^{\infty}$ converges uniformly to the null operator (*i.e.*, $\|T^m\| \to O$). An operator T on H is strongly stable, if the power sequence $\{T^m\}_{m=1}^{\infty}$ converges strongly to the null operator (*i.e.*, $\|T^m x\| \to 0$, for every $x \in H$).

A contraction T is of class $C_{0\cdot}$ if T is strongly stable (*i.e.*, $\|T^m x\| \to 0$ and $\|Tx\| \leq \|x\|$ for every $x \in H$). If T^* is a strongly stable contraction, then T is of class $C_{\cdot 0}$. T is said to be of class $C_{1\cdot}$ if $\lim_{m\to\infty} \|T^m x\| > 0$ (equivalently, if $T^m x \not\to 0$ for every nonzero x in H). T is said to be of class $C_{\cdot 1}$ if $\lim_{m\to\infty} \|T^{*m} x\| > 0$ (equivalently, if $T^{*m} x \not\to 0$ for every nonzero x in H). We define the class $C_{\alpha\beta}$ for $\alpha, \beta = 0, 1$ by $C_{\alpha\beta} = C_{\alpha\cdot} \cap C_{\cdot\beta}$. These are the Nagy-Foiaş classes of contractions [13, p.72]. All combinations are possible leading to classes C_{00}, C_{01}, C_{10} and C_{11}. In particular, T and T^* are both strongly stable contractions if and only if T is a C_{00} contraction. Uniformly stable contractions are of class C_{00}. In the proof of the Theorems 3.1 and 3.2 are used similar techniques as in the proof of Theorems given in paper [6].

Theorem 7. *If T is a contraction of the (n,k)-quasi class Q operator, then the nonnegative operator*

$$D = T^{*k}\left(T^{*(1+n)}T^{(1+n)} - \frac{n+1}{n}T^*T + I\right)T^k$$

is a contraction whose power sequence $\{D^m\}_{m=1}^{\infty}$ converges strongly to a projection P and $T^{k+1}P = O$.

Proof. Suppose that T is a contraction of (n,k)-quasi class Q operator. Then

$$D = T^{*k}\left(T^{*(1+n)}T^{(1+n)} - \frac{n+1}{n}T^*T + I\right)T^k \geq O$$

Let $R = D^{\frac{1}{2}}$ be the unique nonnegative square root of D, then for every x in H and any nonnegative integer m, we have

$$\begin{aligned}
&\langle D^{m+1}x, x\rangle \\
&\quad = \|R^{m+1}x\|^2 \\
&\quad = \langle DR^m x, R^m x\rangle \\
&\quad = \left\langle T^{*(1+n+k)}T^{(1+n+k)}R^m x, R^m x\right\rangle \\
&\quad - \frac{n+1}{n}\left\langle T^{*k}T^*TT^kR^m x, R^m x\right\rangle + \left\langle T^{*k}T^kR^m x, R^m x\right\rangle \\
&\quad = \left\|T^{1+n}T^kR^m x\right\|^2 - \frac{n+1}{n}\left\|TT^kR^m x\right\|^2 + \left\|T^kR^m x\right\|^2 \\
&\quad \leq -\frac{1}{n}\left\|TT^kR^m x\right\|^2 + \|R^m x\|^2 \\
&\quad \leq \|R^m x\|^2 \\
&\quad = \langle D^m x, x\rangle.
\end{aligned}$$

Thus R (and so D) is a contraction (set $m = 0$), and $\{D^m\}_{m=1}^{\infty}$ is a decreasing sequence of nonnegative contractions. Then $\{D^m\}_{m=1}^{\infty}$ converges strongly to a projection P. Moreover

$$\begin{aligned}
&\frac{1}{n}\sum_{m=0}^{l}\|T^{k+1}R^m x\|^2 \\
&\qquad \leq \sum_{m=0}^{l}\left(\|R^m x\|^2 - \|R^{m+1}x\|^2\right) \\
&\qquad = \|x\|^2 - \|R^{l+1}x\|^2 \leq \|x\|^2,
\end{aligned}$$

for all nonnegative integers l and for every $x \in H$. Therefore $\|T^{k+1}R^m x\| \to 0$ as $m \to \infty$. Then we have

$$T^{k+1}Px = T^{k+1} \lim_{m\to\infty} D^m x = \lim_{m\to\infty} T^{k+1}R^{2m}x = 0,$$

for every $x \in H$. So that $T^{k+1}P = O$. QED

A subspace M of space H is said to be non-trivial invariant(alternatively, $T-$invariant) under T if $\{0\} \neq M \neq H$ and $T(M) \subseteq M$. A closed subspace $M \subseteq H$ is said to be a non-trivial hyperinvariant subspace for T if $\{0\} \neq M \neq H$ and is invariant under every operator $S \in L(H)$, which fulfills $TS = ST$.

Theorem 8. *Let T be a contraction of the (n, k)-quasi class Q operator. If T has a no non-trivial invariant subspace, then the nonnegative operator*

$$D = T^{*k}\left(T^{*(1+n)}T^{(1+n)} - \frac{n+1}{n}T^*T + I\right)T^k$$

is a strongly stable contraction.

Proof. We may assume that T is a non zero operator. Let T be a contraction of the (n, k)-quasi class Q operator. By the above theorem, we see that D is a contraction, $\{D^m\}_{m=1}^{\infty}$ converges strongly to a projection P, and $T^{k+1}P = O$. So, $PT^{*(k+1)} = O$. Suppose T has no non-trivial invariant subspaces. Since ker$P$ is a nonzero invariant subspace for T whenever $PT^{*(k+1)} = O$ and $T \neq O$, it follows that ker$P = H$. Hence $P = O$, and we see that $\{D^m\}_{m=1}^{\infty}$ converges strongly to the null operator O, so D is a strongly stable contraction. Since D is self-adjoint, $D \in C_{00}$. QED

Corollary 3 Let T be a contraction of the (n, k)-quasi class Q operator. If T has no non-trivial invariant subspace, then both T and the nonnegative operators

$$D = T^{*k}\left(T^{*(1+n)}T^{(1+n)} - \frac{n+1}{n}T^*T + I\right)T^k$$

are proper contractions.

Proof. A self-adjoint operator T is a proper contraction if and only if T is a C_{00} contraction. QED

Definition 2 If the contraction T is a direct sum of the unitary and $C_{\cdot 0}$ (c.n.u) contractions, then we say that T has a **Wold-type decomposition**.

Definition 3 [7] An operator $T \in L(H)$ is said to have the Putnam-Fuglede commutativity property (**PF property** for short) if $T^*X = XJ$ for any $X \in L(K, H)$ and any isometry $J \in L(K)$ such that $TX = XJ^*$.

Lemma 3[6, 12] Let T be a contraction. The following conditions are equivalent:

(1) For any bounded sequence $\{x_m\}_{m\in\mathbb{N}\cup\{0\}} \subset H$ such that $Tx_{m+1} = x_m$ the sequence $\{\|x_m\|\}_{m\in\mathbb{N}\cup\{0\}}$ is constant,

(2) T has a **Wold-type decomposition**,

(3) T has the **PF property**.

Theorem 9. *Let T be a contraction and the (n,k)-quasi class Q operator. Then T has a **Wold-type decomposition**.*

Proof. In proof of the theorem we use similar techniques as in Theorems given in paper [12]. Since T is a contraction operator, the decreasing sequence $\{T^lT^{*l}\}_{l=1}^{\infty}$ converges strongly to a nonnegative contraction. We denote by

$$S = \left(\lim_{l\to\infty} T^lT^{*l}\right)^{\frac{1}{2}}.$$

The operators T and S are related by $T^*S^2T = S^2$, $\underline{O \le S \le I}$ and S is self-adjoint operator. By [9] there exists an isometry $V : \overline{S(H)} \to \overline{S(H)}$ such that $VS = ST^*$, and thus $SV^* = TS$, and $\|SV^lx\| \to \|x\|$ for every $x \in \overline{S(H)}$. The isometry V can be extended to an isometry on H, which we still denote by V.

For an $x \in \overline{S(H)}$, we can define $x_m = SV^mx$ for $m \in \mathbb{N}\cup\{0\}$. Then for all nonnegative integers l we have

$$T^lx_{m+l} = T^lSV^{l+m}x = SV^{*l}V^{l+m}x = SV^mx = x_m,$$

and for all $l \le m$ we have

$$T^lx_m = x_{m-l}.$$

Since T is an (n,k)-quasi class Q operator and the nontrivial $x \in \overline{S(H)}$ we have

$$\begin{aligned} \|x_m\|^2 &= \|T^{k+1}x_{m+k+1}\|^2 \\ &\le \frac{1}{n+1}\left(\|T^{m+k+1}x_{m+k+1}\|^2 + n\|T^kx_{m+k+1}\|^2\right) \\ &= \frac{1}{n+1}\left(\|x_0\|^2 + n\|x_{m+1}\|^2\right) \end{aligned}$$

so

$$\|x_m\|^2 \le \frac{1}{n+1}\left(\|x_0\|^2 + n\|x_{m+1}\|^2\right).$$

Then

$$\begin{aligned}(\|x_m\|^2 - \|x_{m-1}\|^2) + (\|x_{m-1}\|^2 &- \|x_{m-2}\|^2) + ... + (\|x_1\|^2 - \|x_0\|^2) \\ &\leq n(\|x_{m+1}\|^2 - \|x_m\|^2)\end{aligned}$$

Put

$$b_m = \|x_m\|^2 - \|x_{m-1}\|^2,$$

and we have

$$nb_{m+1} \geq b_m + b_{m-1} + ... + b_1 \tag{2.1}$$

Since $x_m = Tx_{m+1}$, we have

$$\|x_m\| = \|Tx_{m+1}\| \leq \|x_{m+1}\| \text{ for every } m \in \mathbb{N},$$

then the sequence $\{\|x_m\|\}_{m\in\mathbb{N}\cup\{0\}}$ is increasing. From

$$SV^m = SV^*V^{m+1} = TSV^{m+1}$$

we have

$$\|x_m\| = \|SV^m x\| = \|TSV^{m+1}x\| \leq \|SV^{m+1}x\| \leq \|x\|,$$

for every $x \in \overline{S(H)}$ and $m \in \mathbb{N} \cup \{0\}$. Then $\{\|x_m\|\}_{m\in\mathbb{N}\cup\{0\}}$ is bounded. From this we have $b_m \geq 0$ and $b_m \to 0$ as $m \to \infty$.

It remains to check that all b_m equal zero. Suppose that there exists an integer $i \geq 1$ such that $b_i > 0$. Using inequality (2.1) we get $b_{i+1} \geq \frac{b_i}{n} > 0$, and it follows from an induction argument that $b_m \geq \frac{b_i}{n} > 0$ for all $m > i$. This is contradictory with that $b_m \to 0$ as $m \to \infty$. So $b_m = 0$ for all $m \in \mathbb{N}$ and thus $\|x_{m-1}\| = \|x_m\|$ for all $m \geq 1$. Thus the sequence $\{\|x_m\|\}_{m\in\mathbb{N}\cup\{0\}}$ is constant.

From Lemma 2, T has a **Wold-type decomposition**.

QED

3 Examples

In this section we will compare our class of operators with other known classes of operators. We will start from

Example 2 Let us consider the weighted shift operator $T : l_2(\mathbb{N}^+) \to l_2(\mathbb{N}^+)$, defined as follows:

$$T(x_1, x_2, \cdots) = (0, \alpha_1 x_1, \alpha_2 x_2, \cdots),$$

where $\alpha_n = \frac{1}{2^n}$, for every $n \geq 1$. This operator is (n,k)-quasi class Q, quasi-nilpotent but not quasi-hyponormal.

Proof. From the weighted shift operator definition we have that:

$$T(x) = (0, \alpha_1 x_1, \alpha_2 x_2, \cdots),$$

and

$$T^*(x_1, x_2, x_3, \cdots) = (\alpha_1 x_2, \alpha_2 x_3, \cdots).$$

Respectively, after some calculations we get that

$$T^n(x) = (\underbrace{0, 0, \cdots, 0}_{\text{n-times}}, \alpha_1\alpha_2\cdots\alpha_n x_1, \alpha_2\alpha_3\cdots\alpha_{n+1}x_2, \cdots),$$

and

$$T^{*n}(x) = (\alpha_1\alpha_2\cdots\alpha_n x_{n+1}, \alpha_2\alpha_3\cdots\alpha_{n+1}x_{n+2}, \cdots).$$

Now we obtain

$$(T^{*(n+k+1)}T^{(n+k+1)} - (n+1)T^{*(k+1)}T^{(k+1)} + nT^{*k}T^k)(x) =$$
$$([\alpha_1^2\alpha_2^2\cdots\alpha_{(n+k+1)}^2 - (n+1)\alpha_1^2\alpha_2^2\cdots\alpha_{(k+1)}^2 + n\alpha_1^2\alpha_2^2\cdots\alpha_k^2]x_1,$$
$$[\alpha_2^2\alpha_3^2\cdots\alpha_{(n+k+2)}^2 - (n+1)\alpha_2^2\alpha_3^2\cdots\alpha_{(k+2)}^2 + n\alpha_2^2\alpha_3^2\cdots\alpha_{(k+1)}^2]x_2, \cdots).$$

On the other hand

$$\langle(T^{*(n+k+1)}T^{(n+k+1)} - (n+1)T^{*(k+1)}T^{(k+1)} + nT^{*k}T^k)x, x\rangle =$$

$$[\alpha_1^2\alpha_2^2\cdots\alpha_{(n+k+1)}^2 - (n+1)\alpha_1^2\alpha_2^2\cdots\alpha_{(k+1)}^2 + n\alpha_1^2\alpha_2^2\cdots\alpha_k^2]||x_1||^2+$$

$$[\alpha_2^2\alpha_3^2\cdots\alpha_{(n+k+2)}^2 - (n+1)\alpha_2^2\alpha_3^2\cdots\alpha_{(k+2)}^2 + n\alpha_2^2\alpha_3^2\cdots\alpha_{(k+1)}^2]||x_2||^2 + \cdots =$$
$$\alpha_1^2\alpha_2^2\cdots\alpha_k^2[\alpha_{(k+1)}^2\cdots\alpha_{(n+k+1)}^2 - (n+1)\alpha_{(k+1)}^2 + n]||x_1||^2+$$
$$\alpha_2^2\alpha_3^2\cdots\alpha_{(k+1)}^2[\alpha_{(k+2)}^2\cdots\alpha_{(n+k+2)}^2 - (n+1)\alpha_{(k+2)}^2 + n]||x_1||^2 + \cdots \geq 0.$$

Because, from the definition of the weighted shift operator, we have

$$\alpha_{(k+1)}^2\cdots\alpha_{(n+k+1)}^2 - (n+1)\alpha_{(k+1)}^2 + n = n - \frac{n+1}{2^{2k+2}} + \frac{1}{2^{2k+2+2nk+n(n+3)}} \geq 0,$$

for every $k, n \in \mathbb{N}^+$.

Hence, it is proved that T is (n,k)-quasi class Q. After some calculations we get that

$$r(T) = 0,$$

from which it follows that $T-$ is quasi nilpotent. And finally it is not quasi-hyponormal, and this fact follows from the relation:

$$\alpha_n \nleq \alpha_{n+1}$$

and Proposition 3.4 in [2]. $\boxed{QED}$

Example 3 The (n,k)-quasi class Q, is significantly larger than the class of paranormal operators and does not have SVEP.

Proof. To prove the above assertion, we will take into consideration the operator T defined in the Example 1, with sequence weight $(\alpha_n) = \left(0, \sqrt{1-\frac{1}{3}}, \sqrt{1-\frac{1}{4}}, \cdots\right)$. The operator T is (n,k)-quasi class Q, if and only if(Example 1)

$$\alpha^2_{m+k}\alpha^2_{m+k+1}\cdot \ldots \cdot \alpha^2_{m+k+n-1}\alpha^2_{m+k+n} - (n+1)\alpha^2_{m+k} + n \geq 0,$$

for $m \geq 1$. If we substitute the weighted sequence (α_n), in the last relation we obtain:

$$\left(1-\frac{1}{m+k+1}\right)\cdot\left(1-\frac{1}{m+k+2}\right)\cdots\left(1-\frac{1}{m+k+n+1}\right)-n-1+\frac{n+1}{m+k+1}+n =$$

$$= \frac{n(n+1)}{(m+k+1)(m+k+n+1)} \geq 0.$$

T has its adjoint T^* which is a Fredholm operator. T has the SVEP at 0 if and only if

$$K(T^*) = \{x \in H/\text{there exists a sequence} \quad (y_n) \subset H \quad \text{and} \quad \delta > 0,$$

$$\text{for which } x = y_0, T^*(y_{n+1}) = y_n, ||y_n|| \leq \delta^n ||x||, n \in \mathbb{N}\},$$

is finite codimensional, (from Theorem 2.10 in [1]). But $K(T^*)$ does not contain any e_n. Hence, T does not have SVEP. On the other hand, we know that an $(n,k)-$quasi paranormal operator has SVEP, [14, Theorem 4.1]. Consequently, we have proved that T which is (n,k)-quasi class Q, is not an $(n,k)-$quasi paranormal operator. $\boxed{QED}$

4 On (n,k)-quasi class Q composition on Fock-spaces

Let $z=(z_1,z_2,...,z_m)$ and $w=(w_1,w_2,...,w_m)$ be point in $\mathbb{C}^m$, $\langle z,w\rangle=\sum_{k=1}^m z_k\overline{w_k}$ and $|z|=\sqrt{\langle z,z\rangle}$. The Fock space $\mathcal{F}_m^2$ is the Hilbert space of all holomorphic functions on $\mathbb{C}^m$ (entire functions) with inner product

$$\langle f,g\rangle=\frac{1}{(2\pi)^m}\int_{\mathbb{C}^m} f(z)\overline{g(z)}e^{-\frac{1}{2}|z|^2}dA(z),$$

here $dA(z)$ denotes Lebesgue measure on $\mathbb{C}^m$, and $\frac{1}{(2\pi)^m}e^{-\frac{1}{2}|z|^2}dA(z)$ is called Gaussian measure on $\mathbb{C}^m$. The sequence $\{e_m=\sqrt{\frac{1}{m!}}z^m\}_{m\in\mathbb{N}}$ forms an orthonormal basis for $\mathcal{F}_m^2$.

Since each point evaluation is a bounded linear functional on $\mathcal{F}_n^2$, for each $w\in\mathbb{C}^m$ there exists a unique function $u_w\in\mathcal{F}_m^2$ such that $\langle f,u_w\rangle=f(w)$ for all $f\in\mathcal{F}_m^2$. The reproducing kernel functions for the Fock space are given by $u_w(z)=e^{\frac{\langle z,w\rangle}{2}}$ and $\|u_w\|=e^{\frac{|w|^2}{4}}$.

For a given holomorphic mapping $\phi:\mathbb{C}^m\mapsto\mathbb{C}^m$, the composition operator $C_\phi:\mathcal{F}_m^2\mapsto\mathcal{F}_m^2$ is given by $C_\phi(f)=f\circ\phi$, $f\in\mathcal{F}_m^2$, so $(C_\phi f)(z)=f(\phi(z)$. The multiplication operator M_u induced by an entire function u on $\mathcal{F}_m^2$ is defined as $M_uf(z)=u(z)f(z)$ for an entire function f.

Lemma 4[3, Lemma 2]If $f(z)=Az+B$, where A is an $m\times m$ matrix with $\|A\|\le 1$ and B is an $m\times 1$ vector and if $\langle A\xi,B\rangle=0$ whenever $|A\xi|=|\xi|$ then $C_\phi^*=M_{u_b}C_\tau$, where $\tau(z)=A^*z$ and M_{u_b} is the multiplication by the kernel function u_b.

Theorem 10. *A composition operator C_ϕ is an (n,k)-quasi class Q operator on $\mathcal{F}_m^2$ if and only if*

$$M_{u_b\circ\tau^k}...M_{u_b\circ\tau^{n+k}}C_{\phi^{n+k+1}\circ\tau^{n+k+1}}-(n+1)M_{u_b\circ\tau^k}C_{\phi^{k+1}\circ\tau^{k+1}}+nC_{\phi^k\circ\tau^k}\ge 0$$

Proof. A composition operator C_ϕ is an (n,k)-quasi class Q operator on $\mathcal{F}_m^2$ if and only if

$$C_\phi^{*(1+n+k)}C_\phi^{(1+n+k)}-(n+1)C_\phi^{*(k+1)}C_\phi^{k+1}+nC_\phi^{*k}C_\phi^k\ge O. \tag{4.1}$$

By Lemma 4 we have

$$C_\phi^{*(n+k)}(C_\phi^*C_\phi)C_\phi^{(n+k)}=C_\phi^{*(n+k)}((M_{u_b}C_\tau)C_\phi)C_\phi^{(n+k)}.$$

Since $C_\phi C_\tau=C_{\tau\circ\phi}$ we have

$$C_\phi^{*(n+k)}(C_\phi^*C_\phi)C_\phi^{(n+k)}=C_\phi^{*(n+k)}(M_{u_b}C_{\phi\circ\tau})C_\phi^{(n+k)}=C_\phi^{*(n+k)}(M_{u_b}C_{\phi^{n+k+1}\circ\tau}).$$

Again by using Lemma 4, therefore

$$C_\phi^{*(n+k)}(C_\phi^* C_\phi)C_\phi^{(n+k)} = C_\phi^{*(n+k-1)} M_{u_b} C_\tau (M_{u_b} C_{\phi^{n+k+1}\circ\tau}).$$

Since

$$C_\tau M_{u_b} = M_{u_b\circ\tau} C_\tau$$

then

$$C_\phi^{*(n+k)}(C_\phi^* C_\phi)C_\phi^{(n+k)} = C_\phi^{*(n+k-1)} M_{u_b} M_{u_b\circ\tau} C_{\phi^{n+k+1}\circ\tau^2}.$$

Continuing this way we obtain

$$C_\phi^{*(n+k+1)} C_\phi^{(n+k+1)} = M_{u_b} M_{u_b\circ\tau} ... M_{u_b\circ\tau^{n+k}} C_{\phi^{n+k+1}\circ\tau^{n+k+1}}. \tag{4.2}$$

From relations (4.1) and (4.2) we have: C_ϕ is an (n,k)-quasi class Q operator on $\mathcal{F}_m^2$ if and only if

$$M_{u_b} M_{u_b\circ\tau} ... M_{u_b\circ\tau^{n+k}} C_{\phi^{n+k+1}\circ\tau^{n+k+1}} -$$
$$(n+1) M_{u_b} M_{u_b\circ\tau} ... M_{u_b\circ\tau^{k}} C_{\phi^{k+1}\circ\tau^{k+1}} + n M_{u_b} M_{u_b\circ\tau} ... M_{u_b\circ\tau^{k-1}} C_{\phi^{k}\circ\tau^{k}} \geq 0,$$

hence

$$M_{u_b\circ\tau^k} ... M_{u_b\circ\tau^{n+k}} C_{\phi^{n+k+1}\circ\tau^{n+k+1}} - (n+1) M_{u_b\circ\tau^{k}} C_{\phi^{k+1}\circ\tau^{k+1}} + n C_{\phi^{k}\circ\tau^{k}} \geq 0$$

QED

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