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Existence and multiplicity results for a doubly anisotropic problem with sign-changing nonlinearity

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Abstract. We consider in this paper the following problem

$$\begin{cases} -\sum\limits_{i=1}^{N} \partial_i \left[|\partial_i u|^{p_i-2} \partial_i u \right] - \sum\limits_{i=1}^{N} \partial_i \left[|\partial_i u|^{q_i-2} \partial_i u \right] = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Where Ω is a bounded regular domain in $\mathbb{R}^N$, $1 < p_1 \le p_2 \le ... \le p_N$ and $1 < q_1 \le q_2 \le ... \le q_N$, we will also assume that f is a continuous function, that have a finite number of zeroes, changing sign between them.

Keywords: Anisotropic problem, exitence and mutiplicity, variational methods.

MSC 2000 classification: primary 35J66, secondary 35J35

1 Introduction

$$\begin{cases} -\sum\limits_{i=1}^{N} \partial_i \left[|\partial_i u|^{p_i-2} \partial_i u \right] - \sum\limits_{i=1}^{N} \partial_i \left[|\partial_i u|^{q_i-2} \partial_i u \right] = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Where Ω is a bounded regular domain in $\mathbb{R}^N$, we will assume that f fulfill some suitable hypotheses, $1 < p_1 \le p_2 \le ... \le p_N$ and $1 < q_1 \le q_2 \le ... \le q_N$.

We will often use the notation

$$L_{(p_i)}u = \sum_{i=1}^{N} \partial_i \left[|\partial_i u|^{p_i-2} \partial_i u \right],$$

There is a huge literature related to the anisotropic operator, when considered with a linear, non-linear or singular terms we invite the reader to see [1, 4, 5, 6, 7, 9, 13, 25]

f is supposed to be such that

(H1)f is a continuous function such that $f(0) \geq 0$, and there are $0 < a_1 < b_1 < a_2 < ... < b_{m-1} < a_m$ the zeroes of f such that

$$\begin{cases} f \leq 0 & in\ (a_k, b_k) \\ f \geq 0 & in\ (b_k, a_{k+1}) \end{cases}$$

(H2) $\int_{a_k}^{a_{k+1}} f(t)dt > 0; \ \ \forall k = 1, 2, .., m-1.$

These kind of hypotheses, was introduced by different authors in the some early works [3, 8, 11], with the aim to study the problem

$$\begin{cases} -\triangle u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

More recently, the results obtained there was generalized in [2] for the $p\&q$-laplacian, that is

$$\begin{cases} -\triangle_p u - \triangle_q u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and in the case of the ϕ-laplacian in [18], where the considered problem

$$\begin{cases} -div(\phi(|\nabla u|)\nabla u) = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

ϕ being a function fulfilling some suitable conditions.

Observe that the anisotropic operator, and the doubly anisotropic operator considered in this paper cannot be obtained as a particular case of the previous cited, and his its own structure, as we will present in this paper.

In the whole paper C will denote a constant that may change from line to line.

2 Preliminary results

Problem (1.1) is associated to the following anisotropic Sobolev spaces

$$W^{1,(p_i)}(\Omega) = \left\{ v \in W^{1,1}(\Omega) ; \partial_i v \in L^{p_i}(\Omega) \right\}$$

and

$$W_0^{1,(p_i)}(\Omega) = W^{1,(p_i)}(\Omega) \cap W_0^{1,1}(\Omega)$$

endowed by the usual norm

$$\|v\|_{W_0^{1,(p_i)}(\Omega)} = \sum_{i=1}^{N} \|\partial_i v\|_{L^{p_i}(\Omega)} .$$

As we are dealing with a doubly anisotropic operator, the natural functional space is

$$X = W_0^{1,(p_i)}(\Omega) \cap W_0^{1,(q_i)}(\Omega)$$

endowed with the norm

$$\|v\|_X = \|v\|_{W_0^{1,(p_i)}(\Omega)} + \|v\|_{W_0^{1,(q_i)}(\Omega)} .$$

Definition 1. We will say that $u \in W_0^{1,(p_i)}(\Omega)$ is a weak solution to (1.1) if and only if

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i-2} \partial_i u \partial_i \varphi + \sum_{i=1}^{N} \int_\Omega |\partial_i u|^{q_i-2} \partial_i u \partial_i \varphi = \lambda \int_\Omega f(u)\varphi \qquad \forall \varphi \in W_0^{1,(p_i)}(\Omega) .$$

We will also use very often the following indices

$$\frac{1}{\overline{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i}$$

and

$$\overline{p}^* = \frac{N\overline{p}}{N - \overline{p}}, \; p_\infty = \max\{p_N, \overline{p}^*\}$$

without loss of generality we will assume that $\overline{p}^* \leq \overline{q}^*$

The following Sobolev type inequalities will be often used in this paper, we refer to the early works [23], [16] and [20].

Theorem 1. *There exists a positive constant C, depending only on Ω, such that for every $v \in W_0^{1,(p_i)}(\Omega)$, we have*

$$\|v\|_{L^{\overline{p}^*}(\Omega)}^{p_N} \leq C \sum_{i=1}^{N} \|\partial_i v\|_{L^{p_i}(\Omega)}^{p_i}, \tag{2.1}$$

$$\|v\|_{L^r(\Omega)} \leq C \sum_{i=1}^{N} \|\partial_i v\|_{L^{p_i}(\Omega)} \quad \forall r \in [1, \overline{p}^*] \tag{2.2}$$

$$\|v\|_{L^r(\Omega)} \leq C \prod_{i=1}^{N} \|\partial_i v\|_{L^{p_i}(\Omega)}^{\frac{1}{N}} \quad \forall r \in [1, \overline{p}^*] \tag{2.3}$$

and $\forall v \in W_0^{1,(p_i)}(\Omega) \cap L^\infty(\Omega)$, $\overline{p} < N$

$$\left(\int_\Omega |v|^r \right)^{\frac{N}{p}-1} \leq C \prod_{i=1}^{N} \left(\int_\Omega |\partial_i v|^{p_i} |v|^{t_i p_i} \right)^{\frac{1}{p_i}}, \tag{2.4}$$

for every r and t_j chosen such a way to have

$$\begin{cases} \frac{1}{r} = \frac{\gamma_i(N-1)-1+\frac{1}{p_i}}{t_i+1} \\ \sum_{i=1}^{N} \gamma_i = 1. \end{cases}$$

We also have the following algebraic inequalities :

- There exists a $C > 0$ not depending on $\rho \in (0,1)$ such that for given $\sigma_i > 0$, $i = 1, 2...N$ we have

$$\sum_{i=1}^{N} \sigma_i = \rho \Longrightarrow \sum_{i=1}^{N} \frac{\sigma_i^{p_i}}{p_i} \geq C \rho^{p_N} \tag{2.5}$$

- For $p_i \geq 2$

$$C |a-b|^{p_i} \leq \left(|a|^{p_i-2} a - |b|^{p_i-2} b \right)(a-b) \tag{2.6}$$

- For $1 < p_i \leq 2$

$$C \frac{|a-b|^2}{(|a|+|b|)^{2-p_i}} \leq \left(|a|^{p_i-2} a - |b|^{p_i-2} b \right)(a-b) \tag{2.7}$$

In view of applying the above inequalities, allthrough this pper we will suppose that all the p_i are neither $p_i \geq 2$ nor $1 < p_i \leq 2$ and the same for the q_i for $i = 1, ..., N$.

Lemma 1. *Let $g \in C(\mathbb{R})$ be a continuous function and $s_0 > 0$ be such that*

$$\begin{array}{ll} g(s) \geq 0 & \text{if } s \in (-\infty, 0) \\ g(s) \leq 0 & \text{if } s \in [s_0, +\infty) \end{array}$$

then if u is a solution of

$$\begin{cases} -\sum\limits_{i=1}^{N} \partial_i \left[|\partial_i u|^{p_i - 2} \partial_i u \right] - \sum\limits_{i=1}^{N} \partial_i \left[|\partial_i u|^{q_i - 2} \partial_i u \right] = \lambda g(u) \\ \qquad u = 0 \qquad\qquad\qquad \text{on } \partial\Omega \end{cases} \tag{2.8}$$

it verifies $u \geq 0$ a.e.in Ω, $u \in L^\infty(\Omega)$ and $\|u\|_{L^\infty} < s_0$.

Proof. We recall that $u = u^+ - u^-$ where $u^- = \max(-u, 0)$ and $u^+ = \max(0, u)$; as $\partial_i u^- = \begin{cases} -\partial_i u & \text{if } u < 0 \\ 0 & \text{if } u \geq 0 \end{cases}$ we have that $u^- \in W_0^{1,(p_i)}(\Omega)$ whenever $u \in W_0^{1,(p_i)}(\Omega)$. Using u^- as a test function in (2.8) we obtain

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i - 2} \partial_i u \partial_i u^- + \sum_{i=1}^{N} \int_\Omega |\partial_i u|^{q_i - 2} \partial_i u \partial_i u^- = \int_\Omega g(u) u^-$$

that is

$$\sum_{i=1}^{N} \int_{\Omega \cap [u<0]} |\partial_i u|^{p_i} + \sum_{i=1}^{N} \int_{\Omega \cap [u<0]} |\partial_i u|^{q_i} = \int_{\Omega \cap [u<0]} g(u) u$$

by the definition of g, $g(u)u \leq 0$ when $u < 0$ so

$$\sum_{i=1}^{N} \int_{\Omega \cap [u<0]} |\partial_i u|^{p_i} + \sum_{i=1}^{N} \int_{\Omega \cap [u<0]} |\partial_i u|^{q_i} \leq 0$$

and thus necessarily the set $(\Omega \cap [u < 0])$ is a null measure set, and so $u = u^+ \geq 0$.

On the other hand observe that $\partial_i (u - s_0)^+ = \begin{cases} +\partial_i u & \text{if } u > s_0 \\ 0 & \text{if } u \leq s_0 \end{cases}$ we have that $(u - s_0) \in X$ whenever $u \in X$. Using $(u - s_0)^+$ as test function in (2.8) we obtain

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i - 2} \partial_i u \partial_i (u - s_0)^+ + \sum_{i=1}^{N} \int_\Omega |\partial_i u|^{q_i - 2} \partial_i u \partial_i (u - s_0)^+ = \int_\Omega g(u) (u - s_0)^+$$

that is

$$\sum_{i=1}^{N}\int\limits_{\Omega\cap[u>s_0]}|\partial_i u|^{p_i}+\sum_{i=1}^{N}\int\limits_{\Omega\cap[u>s_0]}|\partial_i u|^{q_i}=\int\limits_{\Omega\cap[u>s_0]}g(u)\,(u-s_0)$$

by the definition of $g, g(u)\,(u-s_0)\leq 0$

$$\sum_{i=1}^{N}\int\limits_{\Omega\cap[u>s_0]}|\partial_i u|^{p_i}+\sum_{i=1}^{N}\int\limits_{\Omega\cap[u>s_0]}|\partial_i u|^{q_i}\leq 0$$

and thus necessarily the set $(\Omega\cap[u>s_0])$ is a null measure set, and so $u\leq s_0$. QED

3 Existence and multiplicity results

For each $k=1,2,..,m-1$, consider the following problem

$$\begin{cases} -\sum\limits_{i=1}^{N}\partial_i\left[|\partial_i u|^{p_i-2}\,\partial_i u\right]-\sum\limits_{i=1}^{N}\partial_i\left[|\partial_i u|^{p_i-2}\,\partial_i u\right]=\lambda f_k(u) \\ u=0 \qquad\qquad \text{on } \partial\Omega \end{cases} \tag{3.1}$$

where

$$f_k(s)=\begin{cases} f(0) & \text{if } s\leq 0 \\ f(s) & \text{if } 0\leq s\leq a_k \\ 0 & s>a_k \end{cases}$$

Proposition 1. *There exists $\overline{\lambda}>0$ such that for every $\lambda\in(\overline{\lambda},+\infty)$, problem (3.1) posses a nonnegative solution $u=u_{k,\lambda}$ such that $\|u_k\|_{L^\infty}\leq a_k$.*

Proof. As a direct consequence of Lemma 1 we have $\|u_k\|_{L^\infty}\leq a_k$.

Let

$$\Phi_{k,\lambda}(u):=\sum_{i=1}^{N}\frac{1}{p_i}\int\limits_{\Omega}|\partial_i u|^{p_i}+\sum_{i=1}^{N}\frac{1}{q_i}\int\limits_{\Omega}|\partial_i u|^{q_i}-\lambda\int\limits_{\Omega}F_k(u),$$

where $F_k(t)=\int\limits_0^t f_k(s)ds$, the set $\boldsymbol{C}_{k,\lambda}$ of the critical points of $\Phi_{k,\lambda}(u)$ corresponds to the set of solution to (3.1). Observe that as f_k is a bounded function we have

$m_k |t| \leq \int\limits_0^t m_k ds \leq F_k(t) \leq \int\limits_0^t M_k ds \leq M_k |t|$, thus

$$\begin{aligned}\Phi_{k,\lambda}(u) &= \sum_{i=1}^N \frac{1}{p_i} \int_\Omega |\partial_i u|^{p_i} + \sum_{i=1}^N \frac{1}{q_i} \int_\Omega |\partial_i u|^{q_i} - \lambda \int_\Omega F_k(u) \\ &\geq \sum_{i=1}^N \frac{1}{p_i} \int_\Omega |\partial_i u|^{p_i} + \sum_{i=1}^N \frac{1}{q_i} \int_\Omega |\partial_i u|^{q_i} - \lambda M_k \int_\Omega |u|,\end{aligned}$$

by Hölder inequality we obtain that

$$\Phi_{k,\lambda}(u) \geq \sum_{i=1}^N \frac{1}{p_i} \int_\Omega |\partial_i u|^{p_i} + \sum_{i=1}^N \frac{1}{q_i} \int_\Omega |\partial_i u|^{q_i} - \lambda M_k \|u\|_{L^{\overline{p}^*}},$$

by Sobolev inequality

$$\Phi_{k,\lambda}(u) \geq \sum_{i=1}^N \frac{1}{p_i} \int_\Omega |\partial_i u|^{p_i} + \sum_{i=1}^N \frac{1}{q_i} \int_\Omega |\partial_i u|^{q_i} - \lambda M_k C \sum_{i=1}^N \|\partial_i u\|_{L^{p_i}},$$

as $p_N \geq p_i$ for every i

$$\Phi_{k,\lambda}(u) \geq \frac{1}{p_N} \sum_{i=1}^N \|\partial_i u\|_{L^{p_i}}^{p_i} + \frac{1}{q_N} \sum_{i=1}^N \|\partial_i u\|_{L^{q_i}}^{q_i} - \lambda M_k C \sum_{i=1}^N \|\partial_i u\|_{L^{p_i}},$$

by the fact that

$$\|u\|_X \to +\infty \Rightarrow \|\partial_i u\|_{L^{p_i}} \to +\infty \text{ or } \|\partial_i u\|_{L^{q_i}} \to +\infty \text{ for some } i,$$

we obtain the coercivity of $\Phi_{k,\lambda}(u)$ that is

$$\Phi_{k,\lambda}(u) \to +\infty \text{ when } \|u\|_X \to +\infty.$$

On the other hand, as $\Phi_{k,\lambda}(u)$ continuous it is also lower semi continuous, and thus by Weirstrass theorem, it is also possible to show that a Palais Smail séquence $\{u_n\}_n$ converges strongly, indeed as $\{u_n\}_n$ is bounded in X

$$u_n \rightharpoonup u \text{ weakly in } X,$$

thus

$$u_n \to u \text{ strongly in } L^r(\Omega) \text{ for evry } 1 \leq r < \overline{p}^*,$$

and in particular

$$\int_\Omega |u_n| \to \int_\Omega |u|\,;$$

using $(u_n - u)$ as test function in (3.1) we obtain

$$\sum_{i=1}^N \int_\Omega |\partial_i u|^{p_i-2}\,\partial_i u \partial_i(u_n-u) + \sum_{i=1}^N \int_\Omega |\partial_i u|^{q_i-2}\,\partial_i u \partial_i(u_n-u) = \lambda \int_\Omega f_k(u)(u_n-u)$$

which gives

$$\begin{aligned} &\sum_{i=1}^N \left[\int_\Omega \left(\left(|\partial_i u_n|^{p_i-2}\,\partial_i u_n - |\partial_i u|^{p_i-2}\,\partial_i u \right) \partial_i(u_n-u) + \partial_i u_n\,|\partial_i u|^{p_i-2}\,\partial_i u - |\partial_i u|^{p_i} \right) \right] \\ + &\sum_{i=1}^N \left[\int_\Omega \left(\left(|\partial_i u_n|^{q_i-2}\,\partial_i u_n - |\partial_i u|^{q_i-2}\,\partial_i u \right) \partial_i(u_n-u) + \partial_i u_n\,|\partial_i u|^{q_i-2}\,\partial_i u - |\partial_i u|^{q_i} \right) \right] \\ = &\ \lambda \int_\Omega f_k(u)(u_n-u), \end{aligned}$$

that is

$$\begin{aligned} &\sum_{i=1}^N \left[\int_\Omega \left(|\partial_i u_n|^{p_i-2}\,\partial_i u_n - |\partial_i u|^{p_i-2}\,\partial_i u \right) \partial_i(u_n-u) + \int_\Omega \left(\partial_i u_n\,|\partial_i u|^{p_i-2}\,\partial_i u - |\partial_i u|^{p_i} \right) \right] \\ + &\sum_{i=1}^N \left[\int_\Omega \left(|\partial_i u_n|^{q_i-2}\,\partial_i u_n - |\partial_i u|^{q_i-2}\,\partial_i u \right) \partial_i(u_n-u) + \int_\Omega \left(\partial_i u_n\,|\partial_i u|^{q_i-2}\,\partial_i u - |\partial_i u|^{q_i} \right) \right] \\ = &\ \lambda \int_\Omega f_k(u)(u_n-u) \end{aligned}$$

by inequality (2.6) for $p_i, q_i > 2$

$$\sum_{i=1}^N \int_\Omega |\partial_i(u_n-u)|^{p_i} + \sum_{i=1}^N \int_\Omega \left(\partial_i u_n\,|\partial_i u|^{p_i-2}\,\partial_i u - |\partial_i u|^{p_i} \right) + \sum_{i=1}^N \int_\Omega |\partial_i(u_n-u)|^{q_i} +$$

$$+ \sum_{i=1}^N \int_\Omega \left(\partial_i u_n\,|\partial_i u|^{q_i-2}\,\partial_i u - |\partial_i u|^{q_i} \right) \le \lambda C \int_\Omega f_k(u)(u_n-u)$$

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega}|\partial_i(u_n-u)|^{p_i}+\sum_{i=1}^{N}\int_{\Omega}|\partial_i(u_n-u)|^{q_i}\\
\leq\ &\lambda C\int_{\Omega}f_k(u)(u_n-u)-\sum_{i=1}^{N}\int_{\Omega}\left(\partial_i u_n\,|\partial_i u|^{p_i-2}\,\partial_i u-|\partial_i u|^{p_i}\right)\\
-\ &\sum_{i=1}^{N}\int_{\Omega}\left(\partial_i u_n\,|\partial_i u|^{q_i-2}\,\partial_i u-|\partial_i u|^{q_i}\right),
\end{aligned}$$

by the weak convergence of $\{u_n\}_n$

$$-\sum_{i=1}^{N}\int_{\Omega}\left(\partial_i u_n\,|\partial_i u|^{p_i-2}\,\partial_i u-|\partial_i u|^{p_i}\right)-\sum_{i=1}^{N}\int_{\Omega}\left(\partial_i u_n\,|\partial_i u|^{q_i-2}\,\partial_i u-|\partial_i u|^{q_i}\right)=o(1)$$

thus

$$\begin{aligned}
\sum_{i=1}^{N}\int_{\Omega}|\partial_i(u_n-u)|^{p_i}+\sum_{i=1}^{N}\int_{\Omega}|\partial_i(u_n-u)|^{q_i}\ &\leq\ \lambda C\int_{\Omega}f_k(u)(u_n-u)+o(1)\\
&\leq\ \lambda C M_k\int_{\Omega}(u_n-u)+o(1)
\end{aligned}$$

as $u_n\to u$ strongly in $L^r(\Omega)$ for every $1\leq r<\overline{p}^*$, we conclude that

$$\|u_n-u\|_X\to 0.$$

The same result can be obtained for the cases ($p_i<2$ and $q_i>2$) and ($p_i<2$ and $q_i<2$) by using in a smililar way inequality (2.7) instead of (2.6), which ends the proof. QED

Theorem 2. *There exists $\overline{\lambda}>0$ such that for every $\lambda\in(\overline{\lambda},+\infty)$, problem (1.1) posses at least $(m-1)$ nonnegative solutions u_i such that $u_i\in X$ and $a_i\leq\|u_i\|_{L^\infty}\leq a_{i+1}$.*

Proof. Let u be a solution of (3.1), so by lemma1 it is necessarily such that , $u\in L^\infty(\Omega)$ and $0\leq u<a_{k-1}$ a.e.in Ω thus $f_{k-1}(u)=f(u)$ and then u is also a solution to (1.1). To prove the last part of the theorem we claim that for each $k\in\{2,...m\}$ there is $\lambda_k>0$,such that for all $\lambda>\lambda_k$ we have $u_{k,\lambda}\notin\boldsymbol{C}_{k-1,\lambda}$ where $\Phi_{k,\lambda}(u_{k,\lambda})=\min_{v\in X}\Phi_{k,\lambda}(v)$, first let $\delta>0$ and consider

$$\Omega_\delta=\{x\in\Omega,\ dist(x,\partial\Omega)<\delta\},$$

and

$$\alpha_k = F(a_k) - \max_{0<s<a_{k-1}} |F(s)| = F(a_k) - C_k,$$

by hypothesis (H2) $\alpha_k > 0$. Consider $w_\delta \in C_0^\infty(\Omega)$ such that

$$0 \le w_\delta \le a_k,$$

and

$$w_\delta = a_k, \text{ when } x \in \Omega \setminus \Omega_\delta,$$

we have

$$\int_\Omega F(w_\delta) \ge \int_\Omega F(a_k) - 2C_k |\Omega_\delta|,$$

which yields to

$$\int_\Omega F(w_\delta) - \int_\Omega F(u) \ge \alpha_k |\Omega| - 2C_k |\Omega_\delta|,$$

since $|\Omega_\delta| \to 0$ as $\delta \to 0$ there must exit a δ such that

$$\beta_k = \alpha_k |\Omega| - 2C_k |\Omega_\delta| > 0$$

for that δ we put $w_\delta = w$, we have

$$\begin{aligned}
\Phi_{k,\lambda}(w) \quad - \quad & \Phi_{k-1,\lambda}(u_{k-1,\lambda}) = \sum_{i=1}^N \frac{1}{p_i} \int_\Omega |\partial_i w|^{p_i} + \sum_{i=1}^N \frac{1}{q_i} \int_\Omega |\partial_i w|^{q_i} - \lambda \int_\Omega F_k(w) \\
- \quad & \sum_{i=1}^N \frac{1}{p_i} \int_\Omega |\partial_i u_{k-1,\lambda}|^{p_i} - \sum_{i=1}^N \frac{1}{q_i} \int_\Omega |\partial_i u_{k-1,\lambda}|^{q_i} + \lambda \int_\Omega F_k(u_{k-1,\lambda}) \\
\le \quad & \sum_{i=1}^N \frac{1}{p_i} \int_\Omega |\partial_i w|^{p_i} + \sum_{i=1}^N \frac{1}{q_i} \int_\Omega |\partial_i w|^{q_i} - \lambda \int_\Omega (F_k(w) - F_k(u_{k-1,\lambda})) \\
\le \quad & \sum_{i=1}^N \frac{1}{p_i} \int_\Omega |\partial_i w|^{p_i} + \sum_{i=1}^N \frac{1}{q_i} \int_\Omega |\partial_i w|^{q_i} - \lambda \beta_k,
\end{aligned}$$

for λ large enough we have

$$\Phi_{k,\lambda}(w) - \Phi_{k-1,\lambda}(u_{k-1,\lambda}) < 0$$

that is

$$\Phi_{k,\lambda}(w) < \Phi_{k-1,\lambda}(u_{k-1,\lambda})$$

so

$$\Phi_{k,\lambda}(u_{k,\lambda}) \leq \Phi_{k,\lambda}(w) < \Phi_{k-1,\lambda}(u_{k-1,\lambda})$$

so we have proved that $u_{k,\lambda}$ and $u_{k-1,\lambda}$ are two distinct solutions to (1.1). Now suppose by contradiction that

$$0 \leq u_{k,\lambda} < a_{k-1}$$

we necessarily would have

$$\Phi_{k-1,\lambda}(u_{k-1,\lambda}) \leq \Phi_{k-1,\lambda}(u_{k,\lambda}) = \Phi_{k,\lambda}(u_{k,\lambda})$$

wich is a contradiction, and in conclusion

$$a_{k-1} < \|u_{k,\lambda}\|_{L^\infty} \leq a_k.$$

which ends the proof. QED

Remark 1. Obviously, and under the same conditions on f, all the results obtained here are still valid for the following simply anisotropic problem:

$$\begin{cases} -\sum_{i=1}^{N} \partial_i \left[|\partial_i u|^{p_i - 2} \partial_i u \right] = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

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