Note di Matematica
Note Mat. **39** (2019) no. 1, 65–87.

ISSN 1123-2536, e-ISSN 1590-0932
doi:10.1285/i15900932v39n1p65

Lie algebra representations and 2-index 4-variable 1-parameter Hermite polynomials

H. M. Srivastava
Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada.
Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan, Republic of China.
harimsri@math.uvic.ca

Ghazala Yasmin
Department of Applied Mathematics, Aligarh Muslim University, Aligarh-202002, India.
ghazala30@gmail.com

Abdulghani Muhy
Department of Applied Mathematics, Aligarh Muslim University, Aligarh-202002, India.
muhyi2007@gmail.com

Received: 25.6.2018; accepted: 16.3.2019.

Abstract. This paper is an attempt to stress the usefulness of multi-variable special functions by expressing them in terms of the corresponding Lie algebra or Lie group. The problem of framing the 2-index 4-variable 1-parameter Hermite polynomials (2I4V1PHP) into the context of the irreducible representations $\uparrow_{\omega,\mu}$ of $\mathcal{G}(0,1)$ and $\uparrow'_{\omega,\mu}$ of $\mathcal{K}_5$ is considered. Certain relations involving 2I4V1PHP $H_{m,n}(x,y,z,u;\rho)$ are obtained using the approach adopted by Miller. Certain examples involving other forms of Hermite polynomials are derived as special cases. Further, some properties of the 2I4V1PHP $H_{m,n}(x,y,z,u;\rho)$ are obtained by using a quadratic combination of four operators defined on a Lie algebra of endomorphisms of a vector space.

Keywords: 2-index 4-variable 1-parameter Hermite polynomials, Lie group, Lie algebra, representation theory, implicit formulae.

MSC 2000 classification: primary 33C50, 22E05, secondary 33C80

1 Introduction

The representations of the Lie algebras generate in a natural way all known classical special polynomials. This allows one to extremely simplify the theory of orthogonal polynomials by expressing them in terms of the corresponding Lie algebra or Lie group. The interplay between special functions, Lie theory and differential equations provides a sturdy tool for the development of mathematical physics. One of the numerous consequences of the relationship between special functions and Lie theory is to study and investigate the properties of special functions (see [19, 20]).

Special functions are matrix elements of basis vectors for unitary irreducible representations of low-dimensional Lie groups. A detailed study of these groups and their Lie algebras leads to a unified treatment of an important proportion of special function theory, especially that part of the theory which is most useful in mathematical physics.

Orthogonal polynomials and special functions play an important role in developing numerical and analytical methods in mathematics, physics and engineering. Over the past decades, this area of research has received ever-increasing attention and has gained a growing momentum in modern topics such as computational probability, numerical analysis, computational fluid dynamics, data assimilation, image and signal processing etc.

The theory of generalized and multi-variable special functions serves as an analytical foundation for the majority of several problems in mathematical physics. The use of generalized functions often facilitates the analysis by permitting complex expression to be represented more simply in terms of some generalized functions. Hermite polynomials play a fundamental role in the extension of the classical special functions to the generalized and multi-variable case (see [6, 7, 17, 18]).

The Hermite polynomials appear in probability, such as the Edgeworth series, in numerical analysis as Gaussian quadrature, in combinatorics as an example of an Appell sequence, obeying the umbral calculus, in finite element methods as shape functions for beams, in physics, where they give rise to the eigenstates of the quantum harmonic oscillator and in systems theory in connection with nonlinear operations on Gaussian noise. Recently, an increasing interest has grown related to the Lie-theoretical representations of multi-variable Hermite polynomials (see [8, 14, 13, 21, 22]).

We consider the 2- index 4-variable 1-parameter Hermite polynomials (2I4V1PHP) defined by means of the generating function [3]:

$$exp(xs + ys^2 + zt + ut^2 + \rho st) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} H_{m,n}(x, y, z, u; \rho)\frac{s^m\ t^n}{m!\ n!}. \tag{1.1}$$

On differentiating equation (1.1) w.r.t. x, y, z, u, s, t, ρ the following pure and differential recurrence relations for the 2I4V1PHP $H_{m,n}(x, y, z, u; \rho)$ are obtained:

$$\begin{aligned}
&\frac{\partial}{\partial x}H_{m,n}(x,y,z,u;\rho) = mH_{m-1,n}(x,y,z,u;\rho),\\
&\frac{\partial}{\partial y}H_{m,n}(x,y,z,u;\rho) = m(m-1)H_{m-2,n}(x,y,z,u;\rho),\\
&\frac{\partial}{\partial z}H_{m,n}(x,y,z,u;\rho) = nH_{m,n-1}(x,y,z,u;\rho),\\
&\frac{\partial}{\partial u}H_{m,n}(x,y,z,u;\rho) = n(n-1)H_{m,n-2}(x,y,z,u;\rho),\\
&\frac{\partial}{\partial \rho} = mnH_{m-1,n-1}(x,y,z,u;\rho)
\end{aligned} \tag{1.2}$$

and

$$\begin{aligned}
H_{m,n}(x,y,z,u;\rho) = xH_{m,n}(x,y,z,u;\rho) + 2ymH_{m-1,n}(x,y,z,u;\rho)\\
+\rho nH_{m,n-1}(x,y,z,u;\rho),
\end{aligned} \tag{1.3}$$

$$\begin{aligned}
H_{m,n}(x,y,z,u;\rho) = zH_{m,n}(x,y,z,u;\rho) + 2unH_{m,n-1}(x,y,z,u;\rho)\\
+\rho mH_{m-1,n}(x,y,z,u;\rho).
\end{aligned} \tag{1.4}$$

For suitable values of the variables and parameters, the 2I4V1PHP $H_{m,n}(x,y,z,u;\rho)$ reduce to other Hermite polynomials, we note the following relations as special cases of 2I4V1PHP $H_{m,n}(x,y,z,u;\rho)$

(1)
$$H_{m,n}(x,0,z,0;\rho) = h_{m,n}(x,z|\rho), \tag{1.5}$$

where $h_{m,n}(x,z|\rho)$ denotes the incomplete 2-index 2-variable 1-parameter Hermite polynomials (i2I2V1PHP) defined by the generating function [4] (see also [10]):

$$exp(xs+zt+\rho st) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} h_{m,n}(x,z|\rho)\frac{s^m\ t^n}{m!\ n!}. \tag{1.6}$$

(2)
$$H_{m,n}(x,y,0,0;0) = H_m(x,y), \tag{1.7}$$

where $H_m(x,y)$ denotes the 2-variable Hermite Kampé de Fériet polynomials (2VHKdFP) defined by the generating function [2]:

$$exp(xs+ys^2) = \sum_{m=0}^{\infty} H_m(x,y)\frac{s^m}{m!}. \tag{1.8}$$

(3)
$$H_{m,n}(2x,-y,0,0;0)=H_m(x,y), \tag{1.9}$$

where $H_m(x,y)$ denotes the 2-variable generalized Hermite polynomials (2VGHP) defined by the generating function [5]:

$$exp(2xs-ys^2)=\sum_{m=0}^{\infty}H_m(x,y)\frac{s^m}{m!}. \tag{1.10}$$

(4)
$$H_{m,n}(2x,-1,0,0;0)=H_m(x), \tag{1.11}$$

where $H_m(x)$ denotes the Hermite polynomials (HP) defined by the generating function [1, 11]:

$$exp(2xs-s^2)=\sum_{m=0}^{\infty}H_m(x)\frac{s^m}{m!}. \tag{1.12}$$

In view of the recurrence relations, the 2I4V1PHP $H_{m,n}(x,y,z,u;\rho)$ are suitable to be framed within the context of the representations of the harmonic oscillator Lie algebras $\mathcal{G}(0,1)$ and $\mathcal{K}_5$ [12]. The theory of group representations and its connection to special functions give a powerful tool to the development of mathematical physics. Special functions arise as basis vectors and matrix elements corresponding to local multiplier representations of Lie group.

The harmonic oscillator Lie group $G(0,1)$ [12] (see also[15, chapter 8]) is the set of all 4×4 matrices of the form

$$\mathsf{g}(a,b,c,\tau)=\begin{pmatrix}1 & ce^{\tau} & a & \tau\\ 0 & e^{\tau} & b & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \qquad a,b,c,\tau\in\mathbb{C}, \tag{1.13}$$

where the group operation is matrix multiplication. The Lie algebra $\mathcal{G}(0,1)$ of $G(0,1)$ can be identified with the space of 4×4 matrices of the form

$$\alpha=\begin{pmatrix}1 & x_2 & x_4 & x_3\\ 0 & x_3 & x_1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, \qquad x_1,x_2,x_3,x_4\in\mathbb{C}, \tag{1.14}$$

with Lie product $[\alpha,\beta]=\alpha\beta-\beta\alpha,\quad \alpha,\beta\in\mathcal{G}(0,1)$.

The matrices

$$\mathfrak{J}^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathfrak{J}^- = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$\mathfrak{J}^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{E} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{1.15}$$

with commutation relations

$$[\mathfrak{J}^3, \mathfrak{J}^{\pm}] = \pm\mathfrak{J}^{\pm}, \quad [\mathfrak{J}^+, \mathfrak{J}^-] = -\mathcal{E}, \quad [\mathcal{E}, \mathfrak{J}^{\pm}] = [\mathcal{E}, \mathfrak{J}^3] = \Theta, \tag{1.16}$$

where Θ is the 4×4 zero matrix, form a basis for $\mathcal{G}(0,1)$.

The 5-dimensional complex Lie group K_5 [12] is the set of all 5×5 matrices of the form

$$\mathsf{g}(q,a,b,c,\tau) = \begin{pmatrix} 1 & ce^{\tau} & be^{-\tau} & 2a - bc & \tau \\ 0 & e^{\tau} & 2qe^{-\tau} & b - 2qc & 0 \\ 0 & 0 & e^{-\tau} & -c & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad q,a,b,c,\tau \in \mathbb{C}, \tag{1.17}$$

where the group operation is matrix multiplication. The Lie algebra $\mathcal{K}_5$ of K_5 is 5-dimensional complex Lie algebra with basis $\mathfrak{J}^{\pm}, \mathfrak{J}^3, \mathcal{E}, \mathcal{Q}$ and commutation relations [12, p.299]:

$$\begin{gathered} [\mathfrak{J}^3, \mathfrak{J}^{\pm}] = \pm\mathfrak{J}^{\pm}, \quad [\mathfrak{J}^3, \mathcal{Q}] = 2\mathcal{Q}, \\ [\mathfrak{J}^-, \mathfrak{J}^+] = \mathcal{E}, \quad [\mathfrak{J}^-, \mathcal{Q}] = 2\mathfrak{J}^+, \quad [\mathfrak{J}^+, \mathcal{Q}] = \Theta, \\ [\mathfrak{J}^{\pm}, \mathcal{E}] = [\mathfrak{J}^3, \mathcal{E}] = [\mathcal{Q}, \mathcal{E}] = \Theta. \end{gathered} \tag{1.18}$$

It is clear that the set of group elements with $q = 0$ forms a subgroup of K_5 isomorphic to $G(0,1)$ [12, p.9].

In this article, certain formulae involving the 2I4V1PHP $H_{m,n}(x,y,z,u;\rho)$ are established using the representation theory of the Lie groups $G(0,1)$ and K_5. In Section 2, certain implicit formulae involving 2I4V1PHP $H_{m,n}(x,y,z,u;\rho)$ are derived by making use of the irreducible representations $\uparrow_{\omega,\mu}$ of the Lie algebra $\mathcal{G}(0,1)$. In Section 3, certain implicit formulae involving 2I4V1PHP $H_{m,n}(x,y,z,u;\rho)$ are derived by making use of the irreducible representations $\uparrow'_{\omega,\mu}$ of $\mathcal{K}_5$. In Section 4, certain examples involving various forms of Hermite polynomials are obtained. In concluding section, differential and pure recurrence relations and differential equations for the 2I4V1PHP $H_{m,n}(x,y,z,u;\rho)$ are derived through a formalism of double index sequences.

2 Lie algebra $\mathcal{G}(0,1)$ and implicit formulae

First, we derive certain formulae involving 2I4V1PHP $H_{m,n}(x,y,z,u;\rho)$ by framing them into the context of the representation $\uparrow_{\omega,\mu}$ of the Lie algebra $\mathcal{G}(0,1)$. The irreducible representation $\uparrow_{\omega,\mu}$ of the Lie algebra $\mathcal{G}(0,1)$ is defined for each $\omega,\mu\in\mathbb{C}$ such that $\mu\neq 0$. The spectrum S of this representation is

$$S=\{\,-\omega+r : r\quad \text{a non-negative integer}\,\}$$

and there is a basis $\{\,f_{m,n} : m,n\in S\,\}$ for the representation space V with the properties

$$\begin{gathered}
\mathtt{J}^3 f_{m,n}=mf_{m,n},\qquad \mathtt{E}f_{m,n}=\mu f_{m,n},\\
\mathtt{J}^+ f_{m,n}=\mu f_{m+1,n},\qquad \mathtt{J}^- f_{m,n}=(m+\omega)f_{m-1,n},\\
\mathtt{C}_{0,1}f_{m,n}=(\mathtt{J}^+\mathtt{J}^- -\mathtt{E}\mathtt{J}^3)f_{m,n}=\mu\omega f_{m,n}.
\end{gathered}\tag{2.1}$$

The complex constant ω is clearly irrelevant as far as the study of special functions is concerned. Hence without loss of generality, we can assume $\omega=0$. Also, there is no loss of generality for special function theory if we set $\mu=1$.

The operators $\mathtt{J}^+,\mathtt{J}^-,\mathtt{J}^3,\mathtt{E}$ satisfy the following commutation relations

$$[\mathtt{J}^3,\mathtt{J}^\pm]=\pm\mathtt{J}^\pm,\quad [\mathtt{J}^-,\mathtt{J}^+]=\mathtt{E},\quad [\mathtt{E},\mathtt{J}^\pm]=[\mathtt{E},\mathtt{J}^3]=0.\tag{2.2}$$

In order to derive the implicit formulae involving the 2I4V1PHP $H_{m,n}(x,y,z,u;\rho)$, it is convenient to find the differential operators, whose eigen functions are of the form

$$f_{m,n}(x,y,z,u,s,t;\rho)=\Psi_{m,n}(x,y,z,u;\rho)s^m t^n.\tag{2.3}$$

There are a number of possible solutions of equation (2.2). In view of the recurrence relations (1.2) and (1.3), the operators take the form

$$\begin{aligned}
\mathtt{J}^3&=x\frac{\partial}{\partial x}+2y\frac{\partial}{\partial y}+\rho\frac{\partial}{\partial \rho},\\
\mathtt{J}^+&=2ys\frac{\partial}{\partial x}+\rho s\frac{\partial}{\partial z}+xs,\\
\mathtt{J}^-&=\frac{1}{s}\frac{\partial}{\partial x},\\
\mathtt{E}&=1.
\end{aligned}\tag{2.4}$$

The operators $\mathtt{J}^+,\mathtt{J}^-,\mathtt{J}^3,\mathtt{E}$ satisfy commutation relations (2.2) and are defined on $\mathcal{F}$, the space of all functions analytic in a neighborhood of the point $(x^0,y^0,z^0,u^0,t^0,s^0,\rho^0)=(1,1,1,1,1,1,1)$. In order to obtain a realization of the representation $\uparrow_{0,1}$ of $\mathcal{G}(0,1)$ by operators (2.4) acting on $\mathcal{F}$, the non- zero functions $f_{m,n}(x,y,z,u,s,t;\rho) = \Psi_{m,n}(x,y,z,u;\rho)s^m t^n$ are obtained such that

$$\begin{aligned} \mathtt{J}^3 f_{m,n} &= m f_{m,n}, \qquad \mathtt{E} f_{m,n} = f_{m,n}, \\ \mathtt{J}^+ f_m &= f_{m+1,n}, \qquad \mathtt{J}^- f_{m,n} = m f_{m-1,n}, \\ \mathtt{C}_{0,1} f_{m,n} &= (\mathtt{J}^+\mathtt{J}^- - \mathtt{E}\mathtt{J}^3) f_{m,n} = 0, \end{aligned} \tag{2.5}$$

for all $m \geq 0$.

Again, we take the function $f_{m,n}(x,y,z,u,s,t;\rho) = \Psi_{m,n}(x,y,z,u;\rho)s^m t^n$ such that

$$\begin{aligned} \mathtt{J}^{3'} f_{m,n} &= n f_{m,n}, \qquad \mathtt{E}' f_{m,n} = f_{m,n}, \\ \mathtt{J}^{+'} f_m &= f_{m,n+1}, \qquad \mathtt{J}^{-'} f_{m,n} = n f_{m,n-1}, \\ \mathtt{C}'_{0,1} f_{m,n} &= (\mathtt{J}^+\mathtt{J}^- - \mathtt{E}'\mathtt{J}^3) f_{m,n} = 0, \end{aligned} \tag{2.6}$$

for all $n \geq 0$.

Also, in view of the recurrence relations (1.2) and (1.3), the operators $\mathtt{J}^{+'}, \mathtt{J}^{-'}, \mathtt{J}^{3'}, \mathtt{E}'$ take the form

$$\begin{aligned} \mathtt{J}^{3'} &= z\frac{\partial}{\partial z} + 2u\frac{\partial}{\partial u} + \rho\frac{\partial}{\partial \rho}, \\ \mathtt{J}^{+'} &= 2ut\frac{\partial}{\partial z} + \rho t\frac{\partial}{\partial x} + zt, \\ \mathtt{J}^{-'} &= \frac{1}{t}\frac{\partial}{\partial z}, \\ \mathtt{E}' &= 1 \end{aligned} \tag{2.7}$$

and note that these operators satisfy the commutation relations identical to (2.2).

In term of the conditions (2.5) and (2.6), the function $\Psi_{m,n}(x,y,z,u;\rho)$, $m,n \geq 0$ satisfies the following equations:

$$\begin{aligned} &\left(2y\frac{\partial}{\partial x} + \rho\frac{\partial}{\partial z} + x\right)\Psi_{m,n}(x,y,z,u;\rho) = \Psi_{m+1,n}(x,y,z,u;\rho), \\ &\frac{\partial}{\partial x}\Psi_{m,n}(x,y,z,u;\rho) = \Psi_{m-1,n}(x,y,z,u;\rho), \\ &\left(2y\frac{\partial^2}{\partial x^2} + \rho\frac{\partial^2}{\partial x \partial z} - 2y\frac{\partial}{\partial y} - \rho\frac{\partial}{\partial \rho}\right)\Psi_{m,n}(x,y,z,u;\rho) = 0, \ m = 0,1,2,... \end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
&\left(2u\frac{\partial}{\partial z}+\rho\frac{\partial}{\partial x}+z\right)\Psi_{m,n}(x,y,z,u;\rho)=\Psi_{m,n+1}(x,y,z,u;\rho),\\
&\frac{\partial}{\partial z}\Psi_{m,n}(x,y,z,u;\rho)=\Psi_{m,n-1}(x,y,z,u;\rho),\\
&\left(2u\frac{\partial^2}{\partial z^2}+\rho\frac{\partial^2}{\partial x\partial z}-2u\frac{\partial}{\partial u}-\rho\frac{\partial}{\partial \rho}\right)\Psi_{m,n}(x,y,z,u;\rho)=0,\ \ n=0,1,2,...
\end{aligned} \tag{2.9}$$

respectively.

Further, it is observed that the polynomials $H_{m,n}(x,y,z,u;\rho)$ are the solutions of equations (2.8) and (2.9). In fact, the choice

$$\Psi_{m,n}(x,y,z,u;\rho)=H_{m,n}(x,y,z,u;\rho)$$

satisfies equations (2.8) and (2.9), for all $m,n\in S$. Thus, we conclude that the functions $f_{m,n}(x,y,z,u,s,t;\rho)=H_{m,n}(x,y,z,u;\rho)s^mt^n$ form a basis for a realization of the representation $\uparrow_{0,1}$ of the Lie algebra $\mathcal{G}(0,1)$. This representation of $\mathcal{G}(0,1)$ can be extended to a local multiplier representation $T(\mathsf{g}),\mathsf{g}\in G(0,1)$ defined on $\mathcal{F}$.

According to the result [12, Theorem 1.10], the differential operators given by (2.4) generate a Lie algebra which is the algebra of generalized Lie derivatives of a multiplier representation $T(g)$ of $G(0,1)$. A simple computation using equations (2.4) gives

$$\begin{aligned}
&[T(exp\ b\mathfrak{J}^+)f](x,y,z,u,s,t;\rho)\\
&\qquad=exp\big(xsb+ys^2b^2\big)f\left(x\left(1+\frac{2ysb}{x}\right),y,z\left(1+\frac{\rho sb}{z}\right),u,s,t;\rho\right),\\
&[T(exp\ c\mathfrak{J}^-)f](x,y,z,u,s,t;\rho)=f\left(x\left(1+\frac{c}{sx}\right),y,z,u,s,t;\rho\right),\\
&[T(exp\ \tau\mathfrak{J}^3)f](x,y,z,u,s,t;\rho)=f(x\ exp(\tau),y\ exp(2\tau),z,u,s,t;\rho\ exp(\tau)),\\
&[T(exp\ a\mathcal{E})f](x,y,z,u,s,t;\rho)=exp(a)f(x,y,z,u,s,t;\rho),
\end{aligned} \tag{2.10}$$

for $f\in\mathcal{F}$. If $\mathsf{g}\in G(0,1)$ has the parameters (a,b,c,τ), then

$$\mathsf{g}=exp(b\mathfrak{J}^+)\ exp(c\mathfrak{J}^-)\ exp(\tau\mathfrak{J}^3)\ exp(a\mathcal{E})$$

and consequently

$$T(\mathsf{g})f=T(exp(b\mathfrak{J}^+))T(exp(c\mathfrak{J}^-))T(exp(\tau\mathfrak{J}^3))T(exp(a\mathcal{E}))f. \tag{2.11}$$

An explicit computations gives

$$[T(\mathsf{g})f](x,y,z,u,s,t;\rho) = exp\big(a + xsb + ys^2b^2\big)$$
$$f\left(x\left(1+\frac{2ysb}{x}+\frac{c}{sx}\right)\ exp(\tau), y\ exp(2\tau), z\left(1+\frac{\rho sb}{z}\right), u, s, t; \rho\ exp(\rho)\right). \quad (2.12)$$

The implicit formula involving the polynomials $H_{m,n}(x,y,z,u;\rho)$ is established by proving the following result:

Theorem 1. *For all b, c, x, y, z, u, ρ, $s \in \mathbb{C}$, the following implicit summation formula involving the polynomials $H_{m,n}(x,y,z,u;\rho)$ holds true:*

$$exp\big(xsb + ys^2b^2 - k\tau\big)$$
$$H_{k,n}\left(x\left(1+\frac{2ysb}{x}+\frac{c}{sx}\right)\ exp(\tau), y\ exp(2\tau), z\left(1+\frac{\rho sb}{z}\right), u; \rho\ exp(\rho)\right)$$
$$= \sum_{l=0}^{\infty} c^{k-l} L_l^{(k-l)}(-bc) H_{l,n}(x,y,z,u;\rho)\ s^{l-k}, \quad k,l \geq n; \quad k,n = 0,1,2,.... \quad (2.13)$$

Proof. Every function f in $\mathcal{F}$ has a unique power series expansion, which is convergent for all $x,y,z,u \in \mathbb{C}$. Thus, for fixed n the function $f_{m,n}(x,y,z,u,s,t;\rho) = H_{m,n}(x,y,z,u;\rho)s^m t^n,\ m \geq 0$ form an analytic basis for $\mathcal{F}$. With respect to this analytic basis, the matrix element are defined by

$$[T(\mathsf{g})f_{k,n}](x,y,z,u,s,t;\rho) = \sum_{l=0}^{\infty} \beta_{lk}(\mathsf{g}) f_{l,n}(x,y,z,u,s,t;\rho), \quad \mathsf{g} \in G(0,1);$$
$$k,n = 0,1,2,... \quad (2.14)$$

which in view of equation (2.12) gives

$$exp\big(a + xsb + ys^2b^2\big)$$
$$H_{k,n}\left(x\left(1+\frac{2ysb}{x}+\frac{c}{sx}\right)\ exp(\tau), y\ exp(2\tau), z\left(1+\frac{\rho sb}{z}\right), u; \rho\ exp(\rho)\right)$$
$$= \sum_{l=0}^{\infty} \beta_{lk}(\mathsf{g}) H_{l,n}(x,y,z,u;\rho)\ s^{l-k}, \quad k,l \geq n; \quad k,n = 0,1,2,.... \quad (2.15)$$

The matrix elements $\beta_{lk}(\mathsf{g})$ [12, p. 87 (4.26)] can be written as:

$$\beta_{lk}(\mathsf{g}) = exp(a + k\tau) c^{k-l} L_l^{(k-l)}(-bc), \quad k,l \geqslant 0, \quad (2.16)$$

where the functions L_l^n are the associated Laguerre polynomials [1].

Finally, substitution of $\beta_{lk}(\mathsf{g})$ given by equation (2.16) in equation (2.15) yields the assertion (2.13) of Theorem 1. $\boxed{QED}$

Remark 1. Taking $b = 0$ in equation (2.13) and making use of the limit [12, p.88]:

$$c^n L_l^{(n)}(bc)|_{b=0} = \begin{cases} \binom{n+l}{n} c^n & ; \ n \geqslant 0, \\ 0 & ; \ n < 0, \end{cases} \tag{2.17}$$

we obtain the following consequence of Theorem 1

Corollary 1. *For all* c, x, y, z, u, ρ, $s \in \mathbb{C}$, *the following generating relation involving the polynomials* $H_{m,n}(x, y, z, u; \rho)$ *holds true:*

$$exp\big(-k\tau\big) H_{k,n}\left(x\left(1 + \frac{c}{sx}\right) exp(\tau), y \ exp(2\tau), z, u; \rho \ exp(\rho)\right)$$
$$= \sum_{l=0}^{k} (-c)^{k-l} \binom{k}{k-l} H_{l,n}(x, y, z, u; \rho) \ s^{l-k}, \quad k, l \geq n; \quad k, n = 0, 1, 2, \ldots. \tag{2.18}$$

Remark 2. Taking $c = 0$ in equation (2.13) and making use of the limit [12, p.88]:

$$c^n L_l^{(n)}(bc)|_{c=0} = \begin{cases} 0 & ; \ n > 0, \\ \frac{(-b)^{-n}}{(-n)!} & ; \ n \leq 0, \end{cases} \tag{2.19}$$

we obtain the following consequence of Theorem 1

Corollary 2. *For all* b, x, y, z, u, ρ, $s \in \mathbb{C}$, *the following generating relation involving the polynomials* $H_{m,n}(x, y, z, u; \rho)$ *holds true:*

$$exp\big(xsb + ys^2b^2 - k\tau\big)$$
$$H_{k,n}\left(x\left(1 + \frac{2ysb}{x}\right) exp(\tau), y \ exp(2\tau), z\left(1 + \frac{\rho sb}{z}\right), u; \rho \ exp(\rho)\right)$$
$$= \sum_{l=0}^{\infty} \frac{b^{l-k}}{(l-k)!} H_{l,n}(x, y, z, u; \rho) \ s^{l-k}, \quad k, l \geq n; \quad k, n = 0, 1, 2, \ldots. \tag{2.20}$$

Similarly, we can establish the following result corresponding to the operators (2.7).

Theorem 2. *For all* b', c', x, y, z, u, ρ, $t \in \mathbb{C}$, *the following implicit summation formula involving the polynomials* $H_{m,n}(x, y, z, u; \rho)$ *holds true:*

$$exp\big(ztb' + ut^2b'^2 - k\tau\big)$$
$$H_{m,k}\left(x\left(1 + \frac{\rho tb'}{x}\right), y, z\left(1 + \frac{2utb'}{z} + \frac{c'}{tz}\right) exp(\tau), u \ exp(2\tau); \rho \ exp(\rho)\right)$$
$$= \sum_{l=0}^{\infty} c'^{k-l} L_l^{(k-l)}(-b'c') H_{m,l}(x, y, z, u; \rho) \ t^{l-k},$$
$$k, l \geq m; \quad k, m = 0, 1, 2, \ldots. \tag{2.21}$$

Corollary 3. *For all c', x, y, z, u, ρ, $t \in \mathbb{C}$, the following generating relation involving the polynomials $H_{m,n}(x,y,z,u;\rho)$ holds true:*

$$exp\big(-k\tau\big)H_{m,k}\left(x,y,z\left(1+\frac{c'}{tz}\right) exp(\tau), u\ exp(2\tau); \rho\ exp(\rho)\right)$$
$$= \sum_{l=0}^{k}(-c')^{k-l}\binom{k}{k-l}H_{m,l}(x,y,z,u;\rho)\ t^{l-k}, \quad k,l \geq m; \quad k,m = 0,1,2,... \tag{2.22}$$

Corollary 4. *For all b', x, y, z, u, ρ, $t \in \mathbb{C}$, the following generating relation involving the polynomials $H_{m,n}(x,y,z,u;\rho)$ holds true:*

$$exp\big(ztb' + ut^2b'^2 - k\tau\big)$$
$$H_{m,k}\left(x\left(1+\frac{\rho tb'}{x}\right), y, z\left(1+\frac{2utb'}{z}\right) exp(\tau), u\ exp(2\tau); \rho\ exp(\rho)\right)$$
$$= \sum_{l=0}^{\infty}\frac{b'^{l-k}}{(l-k)!}H_{m,l}(x,y,z,u;\rho)\ t^{l-k}, \ k,l \geq m; \quad k,m = 0,1,2,.... \tag{2.23}$$

3 Lie algebra $\mathcal{K}_5$ and implicit formulae

In this section, we derive the implicit formulae involving 2I4V1PHP $H_{m,n}(x,y,z,u;\rho)$ by using the representation $\uparrow'_{\omega,\mu}$ of the Lie algebra $\mathcal{K}_5$. The irreducible representation $\uparrow'_{\omega,\mu}$ of the Lie algebra $\mathcal{K}_5$ is defined for each $\omega, \mu \in \mathbb{C}$ such that $\mu \neq 0$. The spectrum S of this representation is

$$S = \{\, -\omega + r : r \text{ a non-negative integer}\, \}$$

and there is a basis $\{\, f_{m,n} : m,n \in S\, \}$ for the representation space V with the properties

$$\begin{gathered} \mathtt{J}^3 f_{m,n} = m f_{m,n}, \qquad \mathtt{E} f_{m,n} = \mu f_{m,n}, \\ \mathtt{J}^+ f_{m,n} = \mu f_{m+1,n}, \qquad \mathtt{J}^- f_{m,n} = (m+\omega) f_{m-1,n}, \\ Q f_{m,n} = \mu f_{m+2,n}. \end{gathered} \tag{3.1}$$

Here, we set $\omega = 0$ and $\mu = 1$.

The operators $\mathtt{J}^+$, $\mathtt{J}^-$, $\mathtt{J}^3$, $\mathtt{E}$, Q satisfy the following commutation relations

$$\begin{gathered} [\mathtt{J}^3, \mathtt{J}^\pm] = \pm \mathtt{J}^\pm, \quad [\mathtt{J}^-, \mathtt{J}^+] = \mathtt{E}, \\ [\mathtt{J}^\pm, \mathtt{E}] = [Q, \mathtt{E}] = [\mathtt{J}^3, \mathtt{E}] = [\mathtt{J}^+, Q] = 0, \\ [\mathtt{J}^3, Q] = 2Q, [\mathtt{J}^-, Q] = 2\mathtt{J}^+. \end{gathered} \tag{3.2}$$

In order to derive the implicit formulae involving the 2I4V1PHP $H_{m,n}(x,y,z,u;\rho)$, it is convenient to find the differential operators, whose eigen functions are of the form

$$f_{m,n}(x,y,z,u,s,t;\rho) = \Phi_{m,n}(x,y,z,u;\rho)s^m t^n. \tag{3.3}$$

Using the recurrence relations (1.2) and (1.3), the operators $\mathtt{J}^+$, $\mathtt{J}^-$, $\mathtt{J}^3$, $\mathtt{E}$, Q are obtained as follows:

$$\begin{aligned}
\mathtt{J}^3 &= x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + \rho\frac{\partial}{\partial \rho},\\
\mathtt{J}^+ &= 2ys\frac{\partial}{\partial x} + \rho s\frac{\partial}{\partial z} + xs,\\
\mathtt{J}^- &= \frac{1}{s}\frac{\partial}{\partial x},\\
\mathtt{E} &= 1,\\
Q &= 2xys^2\frac{\partial}{\partial x} + 2x\rho s^2\frac{\partial}{\partial z} + \rho^2 s^2\frac{\partial}{\partial u} + 2y\rho s^2\frac{\partial}{\partial \rho} + 2ys^3\frac{\partial}{\partial s} + (x^2+2y)s^2.
\end{aligned} \tag{3.4}$$

The operators $\mathtt{J}^+$, $\mathtt{J}^-$, $\mathtt{J}^3$, $\mathtt{E}$, Q satisfy commutation relations (3.2) and are defined on $\mathcal{F}$. Further, to obtain a realization of the representation $\uparrow'_{0,1}$ of $\mathcal{K}_5$ by operators (3.4) acting on $\mathcal{F}$, the non- zero functions $f_{m,n}(x,y,z,u,s,t;\rho) = \Phi_{m,n}(x,y,z,u;\rho)s^m t^n$ are obtained such that

$$\begin{gathered}
\mathtt{J}^3 f_{m,n} = m f_{m,n}, \qquad \mathtt{E} f_{m,n} = f_{m,n},\\
\mathtt{J}^+ f_{m,n} = f_{m+1,n}, \qquad \mathtt{J}^- f_{m,n} = m f_{m-1,n},\\
Q f_{m,n} = f_{m+2,n}.
\end{gathered} \tag{3.5}$$

for all $m \geq 0$.

Again, we take the function $f_{m,n}(x,y,z,u,s,t;\rho) = \Phi_{m,n}(x,y,z,u;\rho)s^m t^n$ such that

$$\begin{gathered}
\mathtt{J}'^3 f_{m,n} = n f_{m,n}, \qquad \mathtt{E}' f_{m,n} = f_{m,n},\\
\mathtt{J}'^+ f_{m,n} = f_{m,n+1}, \qquad \mathtt{J}'^- f_{m,n} = n f_{m,n-1},\\
Q f_{m,n} = f_{m,n+2}.
\end{gathered} \tag{3.6}$$

for all $n \geq 0$.

Also, using the recurrence relations (1.2) and (1.3), the operators $\mathtt{J}^{+'}$, $\mathtt{J}^{-'}$, $\mathtt{J}^{3'}$, $\mathtt{E}'$, Q

can be obtained as

$$
\begin{aligned}
\mathtt{J}^{3'} &= z\frac{\partial}{\partial z} + 2u\frac{\partial}{\partial u} + \rho\frac{\partial}{\partial \rho},\\
\mathtt{J}^{+'} &= 2ut\frac{\partial}{\partial z} + \rho t\frac{\partial}{\partial x} + zt,\\
\mathtt{J}^{-'} &= \frac{1}{t}\frac{\partial}{\partial z},\\
\mathtt{E}' &= 1,\\
Q &= 2zut^2\frac{\partial}{\partial z} + 2z\rho t^2\frac{\partial}{\partial x} + \rho^2 s^2\frac{\partial}{\partial y} + 2u\rho t^2\frac{\partial}{\partial \rho} + 2ut^3\frac{\partial}{\partial t} + (z^2+2u)t^2
\end{aligned}
\tag{3.7}
$$

and note that these operators satisfy the commutation relations identical to (3.2).

In term of the conditions (3.5) and (3.6), the function $\Phi_{m,n}(x,y,z,u;\rho)$, $m,n \geq 0$ satisfies the following equations

$$
\begin{aligned}
&\left(2y\frac{\partial}{\partial x} + \rho\frac{\partial}{\partial z} + x\right)\Phi_{m,n}(x,y,z,u;\rho) = \Phi_{m+1,n}(x,y,z,u;\rho),\\
&\frac{\partial}{\partial x}\Phi_{m,n}(x,y,z,u;\rho) = \Phi_{m-1,n}(x,y,z,u;\rho),\\
&\left(2xy\frac{\partial}{\partial x} + 2x\rho\frac{\partial}{\partial z} + \rho^2\frac{\partial}{\partial u} + 2y\rho\frac{\partial}{\partial \rho} + x^2 + 2y(m+1)\right)\Phi_{m,n}(x,y,z,u;\rho)\\
&\qquad = \Phi_{m+2,n}(x,y,z,u;\rho), \quad m = 0,1,2,...
\end{aligned}
\tag{3.8}
$$

and

$$
\begin{aligned}
&\left(2u\frac{\partial}{\partial z} + \rho\frac{\partial}{\partial x} + z\right)\Phi_{m,n}(x,y,z,u;\rho) = \Phi_{m,n+1}(x,y,z,u;\rho),\\
&\frac{\partial}{\partial z}\Phi_{m,n}(x,y,z,u;\rho) = \Phi_{m,n-1}(x,y,z,u;\rho),\\
&\left(2\ zu\frac{\partial}{\partial z} + 2z\rho\frac{\partial}{\partial x} + \rho^2\frac{\partial}{\partial y} + 2u\rho\frac{\partial}{\partial \rho} + z^2 + 2u(n+1)\right)\Phi_{m,n}(x,y,z,u;\rho)\\
&\qquad = \Phi_{m,n+2}(x,y,z,u;\rho), \ n = 0,1,2,...,
\end{aligned}
\tag{3.9}
$$

respectively.

The solutions of equations (3.8) and (3.9) are the polynomials $H_{m,n}(x,y,z,u;\rho)$. In fact, the choice $\Phi_{m,n}(x,y,z,u;\rho) = H_{m,n}(x,y,z,u;\rho)$ satisfies equations (3.8) and (3.9), for all $m,n \in S$.

Thus, we conclude that the functions

$$
f_{m,n}(x,y,z,u,s,t;\rho) = H_{m,n}(x,y,z,u;\rho)s^m t^n
$$

form a basis for a realization of the representation $\uparrow'_{0,1}$ of the Lie algebra $\mathcal{K}_5$. This infinitesimal representation can be extended to a local multiplier representation $T'(\mathsf{g}), \mathsf{g} \in K_5$ defined on the space $\mathcal{F}$.

Proceeding on the same argument as in the case of the representation $\uparrow_{0,1}$, the extended forms of the groups generated by the operators (3.4) are derived as follows:

$$\begin{aligned}
&[T'(exp\ b\mathfrak{J}^+)f](x,y,z,u,s,t;\rho)\\
&\qquad = exp\big(xsb + ys^2b^2\big) f\left(x\left(1+\frac{2ysb}{x}\right), y, z\left(1+\frac{\rho sb}{z}\right), u, s, t; \rho\right),\\
&[T'(exp\ c\mathfrak{J}^-)f](x,y,z,u,s,t;\rho) = f\left(x\left(1+\frac{c}{sx}\right), y, z, u, s, t; \rho\right),\\
&[T'(exp\ \tau\mathfrak{J}^3)f](x,y,z,u,s,t;\rho) = f(x\ exp(\tau), y\ exp(2\tau), z, u, s, t; \rho\ exp(\tau)), \qquad (3.10)\\
&[T'(exp\ a\mathcal{E})f](x,y,z,u,s,t;\rho) = exp(a) f(x,y,z,u,s,t;\rho),\\
&[T'(exp\ q\mathcal{Q})f](x,y,z,u,s,t;\rho) = \gamma^{-\frac{1}{2}} exp\left(\frac{x^2}{4y}\left(\frac{1}{\gamma}-1\right)\right)\\
&\qquad f\left(x\gamma^{-\frac{1}{2}}, y, \left(z + \frac{x\rho/2y}{\gamma}\right), u\ exp\left(\frac{\rho^2}{4y}\left(\frac{1}{\gamma}-1\right)\right), s\gamma^{-\frac{1}{2}}, t; \rho\gamma^{-\frac{1}{2}}\right),
\end{aligned}$$

where $\gamma = 1 - 4ys^2q$ and $f \in \mathcal{F}$. If $\mathsf{g} \in K_5$ has the parameters (q, a, b, c, τ), then

$$\mathsf{g} = exp(q\mathcal{Q})\ exp(b\mathfrak{J}^+)\ exp(c\mathfrak{J}^-)\ exp(\tau\mathfrak{J}^3)\ exp(a\mathcal{E})$$

and consequently

$$T'(\mathsf{g})f = T'(exp(q\mathcal{Q}))T'(exp(b\mathfrak{J}^+))T'(exp(c\mathfrak{J}^-))T'(exp(\tau\mathfrak{J}^3))T'(exp(a\mathcal{E}))f. \qquad (3.11)$$

An explicit computations gives

$$\begin{aligned}
&[T'(\mathsf{g})f](x,y,z,u,s,t;\rho) = \gamma^{-\frac{1}{2}} exp\big(\gamma^{-\frac{1}{2}}\varphi + a\big) f\left(x\gamma^{-\frac{1}{2}}\left(1+\frac{2ysb}{x}+\frac{c\gamma}{sx}\right) exp(\tau),\right.\\
&\left. y\ exp(2\tau), \left(z + \frac{\rho x/4y}{\gamma} + \rho sb\gamma^{-\frac{1}{2}}\right), u\ exp\left(\frac{\rho^2}{4y}\left(\frac{1}{\gamma}-1\right)\right), s\gamma^{-\frac{1}{2}}, t; \rho\gamma^{-\frac{1}{2}}\ exp(\tau)\right) \qquad (3.12)
\end{aligned}$$

where $\varphi = x^2s^2q + xsb + ys^2b^2$.

The matrix elements $\mathcal{C}_{lk}(\mathsf{g})$ of $T'(g)$ with respect to the analytic basis $\{f_{m,n} : m, n \in S\}$ are uniquely determined by $\uparrow'_{0,1}$ of $\mathcal{K}_5$ and are defined by

$$[T(\mathsf{g})f_{k,n}](x,y,z,u,s,t;\rho) = \sum_{l=0}^{\infty} \mathcal{C}_{lk}(\mathsf{g}) f_{l,n}(x,y,z,u,s,t;\rho),\ \ \mathsf{g} \in K_5;$$
$$k, n = 0, 1, 2, \ldots. \qquad (3.13)$$

The implicit formula involving the polynomials $H_{m,n}(x,y,z,u;\rho)$ is established by proving the following result

Theorem 3. *For all a, b, c, q, x, y, z, u, ρ, $s \in \mathbb{C}$, the following implicit summation formula involving the polynomials $H_{m,n}(x,y,z,u;\rho)$ holds true:*

$$\gamma^{-\frac{(k+1)}{2}} exp\big(\gamma^{-\frac{1}{2}}\varphi + a\big) H_{k,n}\Bigg(x\gamma^{-\frac{1}{2}}\left(1+\frac{2ysb}{x}+\frac{c\gamma}{sx}\right) exp(\tau), y\ exp(2\tau),$$
$$\left(z+\frac{\rho x/4y}{\gamma}+\rho sb\gamma^{-\frac{1}{2}}\right), u\ exp\left(\frac{\rho^2}{4y}\left(\frac{1}{\gamma}-1\right)\right); \rho\gamma^{-\frac{1}{2}}\ exp(\tau)\Bigg)$$
$$= \sum_{l=0}^{\infty} \mathcal{C}_{lk}(\mathrm{g}) H_{l,n}(x,y,z,u;\rho)\ s^{l-k}, \quad k,l \geq n; \quad k,n = 0,1,2,\dots. \tag{3.14}$$

Proof. In view of equation (3.12), the proof of (3.14) is direct use of relation (3.13). QED

Remark 3. Taking $q = 0$ in equation (3.14) and making use of the corresponding matrix element [12, p. 309(9.29)]:

$$\mathcal{C}_{lk}(\mathrm{g}) = exp(a+k\tau)c^{(k-l)} L_l^{k-l}(-bc), \quad l,k \geq 0, \tag{3.15}$$

the following consequence of Theorem 3 is deduced

Corollary 5. *For all b, c, x, y, z, u, ρ, $s \in \mathbb{C}$, the following generating relation involving the polynomials $H_{m,n}(x,y,z,u;\rho)$ holds true:*

$$exp\big(xsb + ys^2b^2 - k\tau\big)$$
$$H_{k,n}\left(x\left(1+\frac{2ysb}{x}+\frac{c}{sx}\right) exp(\tau), y\ exp(2\tau), \left(z+\frac{\rho x}{4y}+\rho sb\right), u; \rho\ exp(\tau)\right)$$
$$= \sum_{l=0}^{k} c^{(k-l)} L_l^{k-l}(-bc) H_{l,n}(x,y,z,u;\rho)\ s^{l-k},$$
$$k,l \geq n; \quad k,n = 0,1,2,\dots. \tag{3.16}$$

Remark 4. Taking $a = c = \tau = 0$ in equation (3.14) and making use of the corresponding matrix element [12, p. 309]:

$$\mathcal{C}_{lk}(\mathrm{g}) = \begin{cases} 0 & ;\ 0 \leq l \leq k, \\ \frac{(-q)^{\left(\frac{l-k}{2}\right)}}{(l-k)!} H_{l-k}\left(\frac{b}{2(-q)^{\frac{1}{2}}}\right) & ;\ l \geq k \geq 0, \end{cases} \tag{3.17}$$

the following consequence of Theorem 3 is deduced

Corollary 6. *For all q, b, x, y, z, u, ρ, $s \in \mathbb{C}$, the following generating relation involving the polynomials $H_{m,n}(x, y, z, u; \rho)$ holds true:*

$$\gamma^{\frac{-(k+1)}{2}}\ exp\Big(\frac{x^2s^2q + xsb + ys^2b^2}{\gamma}\Big)$$
$$H_{k,n}\bigg(x\gamma^{-\frac{1}{2}}\Big(1 + \frac{2ysb}{x}\Big), y, \Big(z + \frac{\rho x}{4y\gamma}\Big), u\ exp\Big(\frac{\rho^2}{4y}\Big(\frac{1}{\gamma} - 1\Big)\Big); \rho\gamma^{\frac{1}{2}}\bigg)$$
$$= \sum_{l=0}^{\infty} \frac{(-q)^{\left(\frac{l-k}{2}\right)}}{(l-k)!} H_{l-k}\Big(\frac{b}{2(-q)^{\frac{1}{2}}}\Big) H_{l,n}(x, y, z, u; \rho)\ s^{l-k},$$
$$l \geq k \geq 0; \quad k, n = 0, 1, 2, ... \tag{3.18}$$

Similarly, we can establish the following results corresponding to the operators (3.7).

Theorem 4. *For all a', b', c', q', x, y, z, u, ρ, $t \in \mathbb{C}$, the following implicit summation formula involving the polynomials $H_{m,n}(x, y, z, u; \rho)$ holds true:*

$$\gamma'^{-\frac{(k+1)}{2}} exp\big(\gamma'^{-\frac{1}{2}}\varphi + a\big) H_{m,k}\bigg(\Big(x + \frac{\rho z/4u}{\gamma'} + \rho t b'\gamma'^{-\frac{1}{2}}\Big), y\ exp\Big(\frac{\rho^2}{4u}\Big(\frac{1}{\gamma'} - 1\Big)\Big),$$
$$z\gamma'^{-\frac{1}{2}}\Big(1 + \frac{2utb'}{z} + \frac{c'\gamma'}{tz}\Big)\ exp(\tau), u\ exp(2\tau); \rho\gamma'^{-\frac{1}{2}}\ exp(\tau)\bigg)$$
$$= \sum_{l=0}^{\infty} \mathcal{C}'_{lk}(\mathrm{g}) H_{m,l}(x, y, z, u; \rho)\ t^{l-k}, \quad k, l \geq m; \quad k, m = 0, 1, 2, ..., \tag{3.19}$$

where $\gamma' = 1 - 4ut^2q'$.

Corollary 7. *For all b', c', x, y, z, u, ρ, $t \in \mathbb{C}$, the following generating relation involving the polynomials $H_{m,n}(x, y, z, u; \rho)$ holds true:*

$$exp\big(ztb' + ut^2b'^2 - k\tau\big)$$
$$H_{m,k}\bigg(\Big(x + \frac{\rho z}{4u} + \rho t b'\Big), y, z\Big(1 + \frac{2utb'}{z} + \frac{c'}{tz}\Big)\ exp(\tau), u\ exp(2\tau); \rho\ exp(\tau)\bigg)$$
$$= \sum_{l=0}^{k} c'^{(k-l)} L_l^{k-l}(-b'c') H_{m,l}(x, y, z, u; \rho)\ t^{l-k}, \quad k, l \geq m; \quad k, m = 0, 1, 2, ... \tag{3.20}$$

Corollary 8. *For all q', b', x, y, z, u, ρ, $t \in \mathbb{C}$, the following generating relation involving the polynomials $H_{m,n}(x, y, z, u; \rho)$ holds true:*

$$\gamma'^{\frac{-(k+1)}{2}}\ exp\Big(\frac{z^2t^2q' + ztb' + ut^2b'^2}{\gamma'}\Big)$$
$$H_{m,k}\bigg(\Big(x + \frac{\rho z}{4u\gamma'}\Big), y\ exp\Big(\frac{\rho^2}{4u}\Big(\frac{1}{\gamma'} - 1\Big)\Big), z\gamma'^{-\frac{1}{2}}\Big(1 + \frac{2utb'}{z}\Big), u; \rho\gamma'^{\frac{1}{2}}\bigg))$$
$$= \sum_{l=0}^{\infty} \frac{(-q')^{\left(\frac{l-k}{2}\right)}}{(l-k)!} H_{l-k}\Big(\frac{b'}{2(-q')^{\frac{1}{2}}}\Big) H_{l,m}(x, y, z, u; \rho)\ t^{l-k},$$

$$l \geq k \geq 0; \quad k, m = 0, 1, 2, ... \tag{3.21}$$

In the next section, the summation formulae for the i2I2V1PHP $h_{m,n}(x, z|\rho)$, 2VHKdFP $H_m(x, y)$ and 2VHP $H_m(x, y)$ are obtained as special cases of the results derived in this section.

4 Examples

We consider the following examples

Example 1. Taking $y = u = 0$ in equations (2.13), (2.18) and (2.20) and using relation (1.5) in the resultant equations, the following implicit summation formulae involving i2I2V1PHP $h_{m,n}(x, z|\rho)$ are obtained

$$exp\big(xsb - k\tau\big)\ h_{k,n}\left(x\left(1 + \frac{c}{sx}\right)\ exp(\tau), z\left(1 + \frac{\rho sb}{z}\right); \rho\ exp(\rho)\right)$$
$$= \sum_{l=0}^{\infty} c^{k-l} L_l^{(k-l)}(-bc) h_{l,n}(x, z; \rho)\ s^{l-k}, \quad k, l \geq n; \quad k, n = 0, 1, 2, ..., \tag{4.1}$$

$$exp\big(-k\tau\big)\ h_{k,n}\left(x\left(1 + \frac{c}{sx}\right)\ exp(\tau), z; \rho\ exp(\rho)\right)$$
$$= \sum_{l=0}^{k} (-c)^{k-l} \binom{k}{k-l} h_{l,n}(x, z; \rho)\ s^{l-k}, \quad k, l \geq n; \quad k, n = 0, 1, 2, ..., \tag{4.2}$$

$$exp\big(xsb - k\tau\big) h_{k,n}\left(x\ exp(\tau), z\left(1 + \frac{\rho sb}{z}\right); \rho\ exp(\rho)\right)$$
$$= \sum_{l=0}^{\infty} \frac{b^{l-k}}{(l-k)!} h_{l,n}(x, z; \rho)\ s^{l-k}, \quad k, l \geq n; \quad k, n = 0, 1, 2, ... \tag{4.3}$$

Example 2. Taking $z = u = \rho = 0$ in equations (2.13), (2.18), (2.20), (3.14) and (3.18) and using relation (1.7) in the resultant equations, the following implicit summation formulae involving 2VHKdFP $H_m(x, y)$ are obtained

$$exp\big(xsb + ys^2b^2 - k\tau\big) H_k\left(x\left(1 + \frac{2ysb}{x} + \frac{c}{sx}\right)\ exp(\tau), y\ exp(2\tau)\right)$$
$$= \sum_{l=0}^{\infty} c^{k-l} L_l^{(k-l)}(-bc) H_l(x, y)\ s^{l-k}, \quad k, = 0, 1, 2, ..., \tag{4.4}$$

$$exp\big(-k\tau\big) H_k\left(x\left(1 + \frac{c}{sx}\right)\ exp(\tau), y\ exp(2\tau)\right)$$
$$= \sum_{l=0}^{k} (-c)^{k-l} \binom{k}{k-l} H_l(x, y)\ s^{l-k}, \quad k = 0, 1, 2, ..., \tag{4.5}$$

$$exp\big(xsb+ys^2b^2-k\tau\big)H_k\Bigg(x\Bigg(1+\frac{2ysb}{x}\Bigg)\ exp(\tau),y\ exp(2\tau)\Bigg)$$
$$=\sum_{l=0}^{\infty}\frac{b^{l-k}}{(l-k)!}H_l(x,y)\ s^{l-k},\quad k=0,1,2,..., \tag{4.6}$$
$$\gamma^{-\frac{(k+1)}{2}}exp\big(\gamma^{-\frac{1}{2}}\varphi+a\big)H_k\Bigg(x\gamma^{-\frac{1}{2}}\Bigg(1+\frac{2ysb}{x}+\frac{c\gamma}{sx}\Bigg)\ exp(\tau),y\ exp(2\tau)\Bigg)$$
$$=\sum_{l=0}^{\infty}\mathcal{C}_{lk}(\mathrm{g})H_l(x,y)\ s^{l-k},\quad k=0,1,2,..., \tag{4.7}$$
$$\gamma^{\frac{-(k+1)}{2}}\ exp\Bigg(\frac{x^2s^2q+xsb+ys^2b^2}{\gamma}\Bigg)H_k\Bigg(x\gamma^{-\frac{1}{2}}\Bigg(1+\frac{2ysb}{x}\Bigg),y\Bigg)$$
$$=\sum_{l=0}^{\infty}\frac{(-q)^{\left(\frac{l-k}{2}\right)}}{(l-k)!}H_{l-k}\Bigg(\frac{b}{2(-q)^{\frac{1}{2}}}\Bigg)H_l(x,y)\ s^{l-k},l\geq k\geq 0;k=0,1,2,.... \tag{4.8}$$

Further, replacing x by $2x$ and y by $-y$ in equations (4.4) and (4.8), reduce to known result [14, p. 133(3.3,3.4)].

Example 3. Taking $z=u=\rho=0; y=-1$ and replacing x by $2x$, in equations (2.13), (2.18), (2.20), (3.14) and (3.18) and using relation (1.9) in the resultant equations, the following implicit summation formulae involving 2VHP $H_m(x,y)$ are obtained

$$exp\big(2xsb-s^2b^2-k\tau\big)H_k\Bigg(2x\Bigg(1-\frac{ysb}{x}+\frac{c}{2sx}\Bigg)\ exp(\tau)\Bigg)$$
$$=\sum_{l=0}^{\infty}c^{k-l}L_l^{(k-l)}(-bc)H_l(x)\ s^{l-k},\quad k,=0,1,2,..., \tag{4.9}$$
$$exp\big(-k\tau\big)H_k\Bigg(2x\Bigg(1+\frac{c}{2sx}\Bigg)\ exp(\tau)\Bigg)$$
$$=\sum_{l=0}^{k}(-c)^{k-l}\binom{k}{k-l}H_l(x)\ s^{l-k},\quad k=0,1,2,..., \tag{4.10}$$
$$exp\big(2xsb-s^2b^2-k\tau\big)H_k\Bigg(2x\Bigg(1-\frac{sb}{x}\Bigg)\ exp(\tau)\Bigg)$$
$$=\sum_{l=0}^{\infty}\frac{b^{l-k}}{(l-k)!}H_l(x)\ s^{l-k},\quad k=0,1,2,..., \tag{4.11}$$

$$\big(1+4qs^2\big)^{-\frac{(k+1)}{2}}exp\big(\big(1+4qs^2\big)^{-\frac{1}{2}}\varphi+a\big)$$

$$H_k\left(2x\left(1+4qs^2\right)^{-\frac{1}{2}}\left(1-\frac{sb}{x}+\frac{c\left(1+4qs^2\right)}{2sx}\right)\,exp(\tau)\right)$$
$$=\sum_{l=0}^{\infty}\mathcal{C}_{lk}(\mathtt{g})H_l(x)\;s^{l-k},\;\;k=0,1,2,...,\tag{4.12}$$

$$\left(1+4qs^2\right)^{\frac{-(k+1)}{2}}\;exp\left(\frac{4x^2s^2q+2xsb-s^2b^2}{1+4qs^2}\right)H_k\left(2x\left(1+4qs^2\right)^{-\frac{1}{2}}\left(1-\frac{sb}{x}\right)\right)$$
$$=\sum_{l=0}^{\infty}\frac{(-q)^{\left(\frac{l-k}{2}\right)}}{(l-k)!}H_{l-k}\left(\frac{b}{2(-q)^{\frac{1}{2}}}\right)H_l(x)\;s^{l-k},l\geq k\geq 0;\quad k=0,1,2,....\tag{4.13}$$

In the next section, we use another approach to derive some properties of the 2I4V1PHP $H_{m,n}(x,y,z,u;\rho)$.

5 Concluding remarks

In the previous sections, we have established certain results for the 2I4V1PHP $H_{m,n}(x,y,z,u;\rho)$ using Lie algebraic technique. The Lie algebraic aspect of special functions is limited to the Lie algebras generated by raising, lowering and maintaining operators.

Radulescu [16] established some other important properties of special functions using operators defined on Lie algebras. Recently, this formalism has been extended to double index sequences [9]. In this concluding section, we show how the Lie algebraic technique can be used to derive the differential and pure recurrence relations and differential equations for the 2I4V1PHP $H_{m,n}(x,y,z,u;\rho)$ through a formalism of double index sequences. In this section, we establish some properties of the 2I4V1PHP $H_{m,n}(x,y,z,u;\rho)$ by using a quadratic combination of four operators defined on a Lie algebra of endomorphisms of a vector space.

Let End $\mathcal{V}$ be the Lie algebra of endomorphism of a vector space $\mathcal{V}$. Let $\mathcal{V}=\mathtt{C}^{\infty}(\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R})$ and $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}\in$ End $\mathcal{V}$ such that

$$\mathcal{A}f(x,y,z,u,s,t;\rho)=\frac{1}{s}\frac{\partial f}{\partial x},\tag{5.1}$$
$$\mathcal{B}f(x,y,z,u,s,t;\rho)=2ys\frac{\partial f}{\partial x}+\rho s\frac{\partial f}{\partial z}+xsf,\tag{5.2}$$
$$\mathcal{C}f(x,y,z,u,s,t;\rho)=\frac{1}{t}\frac{\partial f}{\partial z},\tag{5.3}$$
$$\mathcal{D}f(x,y,z,u,s,t;\rho)=2ut\frac{\partial f}{\partial z}+\rho t\frac{\partial f}{\partial x}+ztf,\tag{5.4}$$

for every $(x, y, z, u, s, t; \rho) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

Now, according to the result [9, Corollary 2.3] and in view of equation (2.3), we have

$$\mathcal{B}\{\Psi_{m,n}(x, y, z, u; \rho)s^m t^n\} = \Psi_{m+1,n}(x, y, z, u; \rho)s^{m+1}t^n \tag{5.5}$$

and

$$\mathcal{D}\{\Psi_{m,n}(x, y, z, u; \rho)s^m t^n\} = \Psi_{m,n+1}(x, y, z, u; \rho)s^m t^{n+1}. \tag{5.6}$$

Consequently, we get

$$\mathcal{A}\{\Psi_{m,n}(x, y, z, u; \rho)s^m t^n\} = m\Psi_{m-1,n}(x, y, z, u; \rho)s^{m-1}t^n \tag{5.7}$$

and

$$\mathcal{C}\{\Psi_{m,n}(x, y, z, u; \rho)s^m t^n\} = n\Psi_{m,n-1}(x, y, z, u; \rho)s^m t^{n-1}, \tag{5.8}$$

respectively.

Next, on using equations (5.2) and (5.4) in equations (5.5) and (5.6), respectively, we get the differential recurrence relations

$$\left(2y\frac{\partial}{\partial x} + \rho\frac{\partial}{\partial z} + x\right)\Psi_{m,n}(x, y, z, u; \rho) = \Psi_{m+1,n}(x, y, z, u; \rho) \tag{5.9}$$

and

$$\left(2u\frac{\partial}{\partial z} + \rho\frac{\partial}{\partial x} + z\right)\Psi_{m,n}(x, y, z, u; \rho) = \Psi_{m,n+1}(x, y, z, u; \rho), \tag{5.10}$$

respectively.

Again, on using equations (5.1) and (5.3) in equations (5.7) and (5.8), respectively, we get the differential recurrence relations

$$\frac{\partial}{\partial x}\Psi_{m,n}(x, y, z, u; \rho) = m\Psi_{m-1,n}(x, y, z, u; \rho) \tag{5.11}$$

and

$$\frac{\partial}{\partial z}\Psi_{m,n}(x, y, z, u; \rho) = n\Psi_{m,n-1}(x, y, z, u; \rho), \tag{5.12}$$

respectively.

Further, using equations (5.11) and (5.12) in equations (5.9) and (5.10), respectively, we get the following differential recurrence relations

$$\begin{aligned}&\frac{\partial}{\partial x}\Psi_{m,n}(x, y, z, u; \rho)\\ &\quad = \frac{1}{\rho}\{-2my\Psi_{m-1,n}(x, y, z, u; \rho) - x\Psi_{m,n}(x, y, z, u; \rho) + \Psi_{m+1,n}(x, y, z, u; \rho)\}\end{aligned} \tag{5.13}$$

and

$$\frac{\partial}{\partial z}\Psi_{m,n}(x,y,z,u;\rho) = \frac{1}{\rho}\{-2nu\Psi_{m,n-1}(x,y,z,u;\rho) - z\Psi_{m,n}(x,y,z,u;\rho) + \Psi_{m,n+1}(x,y,z,u;\rho)\}, \quad (5.14)$$

respectively.

Also, in view of equations (5.11) and (5.12), equations (5.9) and (5.10), become

$$2my\Psi_{m-1,n}(x,y,z,u;\rho) + n\rho\Psi_{m,n-1}(x,y,z,u;\rho) + x\Psi_{m,n}(x,y,z,u;\rho) = \Psi_{m+1,n}(x,y,z,u;\rho) \quad (5.15)$$

and

$$2nu\Psi_{m,n-1}(x,y,z,u;\rho) + m\rho\Psi_{m-1,n}(x,y,z,u;\rho) + z\Psi_{m,n}(x,y,z,u;\rho) = \Psi_{m,n+1}(x,y,z,u;\rho), \quad (5.16)$$

respectively.

Now, in view the identity [9]:

$$(\mathcal{BA})Y_{m,n} = mY_{m,n} \quad (5.17)$$

and equations (5.1) and (5.2), we get the following differential equation

$$\left(2y\frac{\partial^2}{\partial x^2} + \rho\frac{\partial^2}{\partial x\partial z} + x\frac{\partial}{\partial x} - m\right)\Psi_{m,n}(x,y,z,u;\rho) = 0. \quad (5.18)$$

Similarly, using equations (5.3) and (5.4) and in view of the identity [9]:

$$(\mathcal{DC})Y_{m,n} = nY_{m,n}, \quad (5.19)$$

we get

$$\left(2u\frac{\partial^2}{\partial z^2} + \rho\frac{\partial^2}{\partial x\partial z} + z\frac{\partial}{\partial z} - n\right)\Psi_{m,n}(x,y,z,u;\rho) = 0. \quad (5.20)$$

Finally, it is observed that equations (5.18) and (5.20) are the differential equations satisfied by the 2I4V1PHP $H_{m,n}(x,y,z,u;\rho)$. Also, note that equations (5.11)-(5.16) are differential and pure recurrence relations satisfied by the 2I4V1PHP $H_{m,n}(x,y,z,u;\rho)$.

The established results in this paper show that the use of multi-variable polynomials with their associated formalism offers wide possibilities in the applications of pure and applied mathematics. This approach can be further extended to derive the properties of other generalized special functions of mathematical physics and is an interesting problem for further research.

References

[1] L. C. Andrews: Special functions for engineers and applied mathematicians, Macmillan, Co. New York, 1985.

[2] P. Appell and J. Kampé de Fériet: Fonctions Hypergéométriques et Hypersphériques; Polynômes d'Hermite, Gauthier-Villars, Paris, 1926.

[3] G. Dattoli: *Hermite–Bessel and Laguerre–Bessel functions: A by-product of the monomiality principle. Advanced special functions and applications,* in Proc. Melfi Sch. Adv. Top. Math. Phys, Melfi, **1** (1999), 147–164.

[4] G. Dattoli: *Incomplete 2d Hermite polynomials: properties and applications,* J. Math. Anal. Appl., **284** (2003), no. 2, 447–454.

[5] G. Dattoli, C. Chiccoli, S. Lorenzutta, G. Maino and A. Torre: *Generalized Bessel functions and generalized Hermite polynomials*, J. Math. Anal. Appl., **178** (1993), no. 2, 509–516.

[6] G. Dattoli, H. Srivastava and D. Sacchetti: *The Hermite polynomials and the Bessel functions from a general point of view*, Int. J. Math. Math. Sci., **2003** (2003), no. 57, 3633–3642.

[7] G. Dattoli, H. Srivastava and K. Zhukovsky: *Orthogonality properties of the Hermite and related polynomials,* J. Comput. Appl. Math., **182** (2005), no. 1, 165–172.

[8] G. Dattoli, H. M. Srivastava and S. Khan: *Operational versus Lie-algebraic methods and the theory of multi-variable Hermite polynomials,* Integral Transforms Spec. Funct., **16** (2005), no.1, 81–91.

[9] S. Khan and M. Ali: *Certain results concerning operators on a Lie algebra and double index sequences,* Georgian Math. J., **24** (2017), no. 4, 535–544.

[10] S. Khan, M. Pathan, N. A. M. Hassan and G. Yasmin: *Implicit summation formulae for Hermite and related polynomials,* J. Math. Anal. Appl., **344** (2008), no. 1, 408–416.

[11] N. N. Lebedev, R. A. Silverman and D. Livhtenberg: Special functions and their applications, (Translated from the Russian) Prentice-Hall, Englewood Cliffs, New Jersey, 1965.

[12] W. Miller: Lie theory and special functions, Academic Press, New York and London, 1968.

[13] M. Pathan, S. Khan and G. Yasmin: *Representation of a Lie algebra* $\mathcal{G}(0,1)$ *and three variable generalized Hermite polynomials* $H_n(x,y,z)$, Integral Transforms Spec., **13** (2002), no. 1, 59–64.

[14] M. Pathan, S. Khan and G. Yasmin: *Lie theory and two variables generalized Hermite polynomials*, Italian J. Pure Appl. Math., **13** (2003), 129–136.

[15] L. S. Pontrjagin: Topological groups, Princeton Univ. Press, Princeton, New Jersey, 1958.

[16] V. Radulescu; *A study of some special functions with Lie theory*, Stud. Cerc. Mat., **43** (1991), no. 1–2, 67–71.

[17] H. Srivastava and G. B. Djordjević: *Some generating functions and other properties associated with the generalized Hermite and related polynomials,* Integral Transforms Spec. Funct., **22** (2011), no. 12, 895–906.

[18] H. M. Srivastava, M. Özarslan and B. Yilmaz: *Some families of differential equations associated with the Hermite-based Appell polynomials and other classes of Hermite-based polynomials,* Filomat, **28** (2014), no. 4, 695–708.

[19] N. I. Vilenkin: Special functions and the theory of group representations, American Math. Soc., Providence, 1968.

[20] L. Weisner: *Group-theoretic origin of certain generating function,* Pacific J. Math., **5** (1955), no. 6, 1033–1039.

[21] G. Yasmin and A. Muhyi: *Lie algebra $\mathcal{G}(0,1)$ and 3-variable 2-parameter Hermite polynomials*, A. Afr. Mat., **30** (2019), no. 1-2, 231–246.

[22] G. Yasmin and S. Khan: *Hermite matrix based polynomials of two variables and Lie algebraic techniques*, South East Asian Bull. Math., **38** (2014), 603–618.