Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **38** (2018) no. 2, 55–92. doi:10.1285/i15900932v38n2p55

On the ranks of homogeneous polynomials of degree at least 9 and border rank 5

Edoardo Ballico[i]
Department of Mathematics, University of Trento
ballico@science.unitn.it

Received: 10.4.2018; accepted: 9.8.2018.

Abstract. Let f be a degree $d \ge 9$ homogenous polynomial with border rank 5. We prove that it has rank at most $4d-2$ and give better results when f essentially depends on at most 3 variables or there are other conditions on the scheme evincing the cactus and border rank of f. We always assume that f essentially depends on at most 4 variables, because the other case was done by myself in Acta Math. Vietnam. 42 (2017), 509–531.

Keywords: symmetric tensor rank, border rank, cactus rank

MSC 2000 classification: primary 14N05, secondary 15A69

Introduction

A more descriptive title would be " Geometry of low degree zero-dimensional curvilinear schemes and an application to the ranks of homogeneous polynomials of degree at least 9 and border rank 5 ". Let $\nu_{d,m} : \mathbb{P}^m \to \mathbb{P}^r$, $r := \binom{m+d}{m} - 1$, denote the Veronese embedding of $\mathbb{P}^m$, i.e. the embedding of $\mathbb{P}^m$ induced by the complete linear system $|\mathcal{O}_{\mathbb{P}^m}(d)|$. If M is a k-dimensional linear subspace of $\mathbb{P}^m$ and $a \in M$, then $\nu_{d,k}(a) = \nu_{d,m}(a)$ (here we use the image of $|\mathcal{O}_{\mathbb{P}^m}(d)|$ in $|\mathcal{O}_M(d)|$ to get the Veronese embedding of M). Thus we usually write ν_d instead of $\nu_{d,m}$ (this paper is a continuation of [2] and we used ν_d in [2]). For any $q \in \mathbb{P}^r$ the *rank* $r_{m,d}(q)$ of q is the minimal cardinality of a finite set $S \subset \mathbb{P}^m$ such that $q \in \langle \nu_{m,d}(S) \rangle$, where $\langle\ \ \rangle$ denote the linear span. For all integers $a > 0$ the a-secant variety $\sigma_a(\nu_d(\mathbb{P}^m))$ of $\nu_d(\mathbb{P}^m)$ is the closure in $\mathbb{P}^r$ of the union of all $\langle \nu_d(S) \rangle$, where S is a subset of $\mathbb{P}^m$ with cardinality a. For any $q \in \mathbb{P}^r$ the *border rank* $b_{m,d}(q)$ of q is the minimal integer a such that $q \in \sigma_a(\nu_d(\mathbb{P}^m))$. By concision we have $r_{m,d}(q) = r_{k,d}(q)$ and $b_{m,d}(q) = b_{m,k}(q)$ if $q \in \langle \nu_{d,k}(M) \rangle$ with M a k-dimensional linear subspace of $\mathbb{P}^m$ ([10, Exercise 3.2.2.2], [11, §3.2]). Let $Z \subset \mathbb{P}^m$ be a zero-dimensional scheme. Z is said to be *curvilinear* if for each point q of its support $Z_{\rm red}$ the Zariski tangent space of Z at q has dimension at most 1. A zero-dimensional scheme is *curvilinear* if and only if it is contained

[i]This work is partially supported by MIUR and GNSAGA (INDAM)

in a smooth curve (and if and only if it is contained in a reduced curve whose smooth locus contains $Z_{\rm red}$).

Another possible title would be " The stratification by ranks of the homogeneous polynomials with border rank 5 and depending on at most 4 variables ", because the opposite was done in [2]. By concision ([10, Exercise 3.2.2.2]) we are basically working in $\mathbb{P}^3$.

We prove the following result.

Theorem 1. *Assume $d \ge 9$. Let $P \in \mathbb{P}^r$ be a point with border rank 5. Then $r_{m,d}(P) \le 4d-2$.*

We do not have a complete description of all the possible integers $r_{m,d}(P)$ with P of border rank 5. Since $d \ge 4$, each $P \in \mathbb{P}^r$ is contained in the linear span $\langle \nu_d(A)\rangle$ of $\nu_d(A)$, where $A \subset \mathbb{P}^m$ is a degree 5 zero-dimensional smoothable scheme A ([7, Lemma 2.6], [6, Proposition 2.5]). A is Gorenstein ([6, Lemma 2.3]). Since $\deg(A)$ is so low, we get very strong restrictions on the possible schemes A, both as abstract schemes and as embedded subschemes of $\mathbb{P}^m$. The structure of A gives very strong restrictions on the rank of P. The main result is not Theorem 1, but a long list of cases in which we compute the value $r_{m,d}(P)$. A main step in the proofs of all intermediate results is the use of certain zero-dimensional schemes with low degree. For each of these schemes A we give an upper bound for the ranks of the points associated to A. For some A we give the precise value of the ranks. In most cases we only need curvilinear subschemes ([2, Remark 1]) and that each zero-dimensional curvilinear scheme has only finitely many subschemes. However, even for these easy schemes there is a positive-dimensional family Γ of associated polynomials (a projective space minus finitely many of its hyperplanes) and often it is not easy to check the exact value of the rank for all these polynomials, not only for the general element of Γ.

We summarize parts of Propositions 3, 4 and 5 in the following way.

Proposition 1. *Assume $m \ge 3$ and $d \ge 9$. Fix a 3-dimensional linear space $\mathbb{H} \subseteq \mathbb{P}^m$. Let $A_1 \subset \mathbb{H}$ be a degree 4 connected and curvilinear scheme with $\langle A_1\rangle = \mathbb{H}$. Let A_1 (resp. A_2) be the degree 2 (resp. 3) subscheme of A. Fix $O_2 \in \langle A_1\rangle \setminus (A_1)_{\rm red}$ and set $A := A_1 \cup \{O_2\}$. Fix $P \in \langle \nu_d(A)\rangle$ such that $P \notin \langle \nu_d(E)\rangle$ for any $E \subsetneq A$.*

(i) If $O_2 \notin \langle A''\rangle$, then $r_{m,d}(P) = 3d-3$.

(ii) If $O_2 \in \langle \nu_d(A'')\rangle \setminus \langle \nu_d(A')\rangle$, then $3d-3 \le r_{m,d}(P) \le 3d-2$.

(iii) If $O_2 \in \langle \nu_d(A')\rangle$, then $r_{m,d}(P) = 3d-1$.

Part (iii), i.e. Proposition 4, is the part of the paper with the longer proof.

We summarize Propositions 8, 10, 11, 12 in the following way.

Proposition 2. *Assume $d \ge 9$. Let $A_1, A_2 \subset \mathbb{P}^m$, $m \ge 3$, be disjoint curvi-*

linear schemes such that $\deg(A_1) = 3$ *and* $\deg(A_2) = 2$. *Set* $A := A_1 \cup A_2$. *Assume* $\dim(\langle A\rangle) = 2$ *and take* $P \in \langle\nu_d(A)\rangle$ *such that* $P \notin \langle\nu_d(E)\rangle$ *for any* $E \subsetneq A$.

(a) *If* A *is in linearly general position in* $\langle A\rangle$, *then* $r_{m,d}(P) = 3d - 3$.

(b) *If* A *is not in linearly general position in* $\langle A\rangle$, *then* $r_{m,d}(P) = 3d - 2$.

In this paper we also prove the following results:

(1) If A is not connected and $d \geq 9$, then $r_{m,d}(P) \leq 3d - 1$ (Lemma 9).

(2) If $m = 3$, $d \geq 9$, A is connected and A is in linearly general position in $\mathbb{P}^3$, then $r_{m,d}(P) = 3d - 3$ (Proposition 3).

Take $d \geq 9$ and A as in Theorem 1. We recall ([2, Proposition 5]) that if $\dim\langle A\rangle = 2$ (and in particular if $m = 2$), then $r_{m,d}(P) \leq 3d$.

Many thanks are due to the referee for many comments and relevant remarks.

1 Preliminaries

For any $P \in \mathbb{P}^r$ let $r_{m,d}(P)$ (the rank of P) denote the minimal cardinality of a finite set $B \subset \mathbb{P}^m$ such that $P \in \langle\nu_d(S)\rangle$ and let $\mathcal{S}(P)$ denote the set of all subsets of $\mathbb{P}^m$ evincing the rank of P, i.e. the set of all subset $B \in \mathbb{P}^m$ such that $P \in \langle\nu_d(B)\rangle$ and $\sharp(B) = r_{m,d}(P)$. For any $P \in \mathbb{P}^r$ let $b_{m,d}(P)$ denote the border rank of P. For any $P \in \mathbb{P}^r$ the *cactus rank* of P is the minimal degree of a zero-dimensional scheme $Z \subset \mathbb{P}^m$ such that $P \in \langle\nu_d(Z)\rangle$.

Remark 1. Fix $q \in \mathbb{P}^r$ such that there is a line $L \subseteq \mathbb{P}^m$ with $q \in \langle\nu_d(L)\rangle$. Sylvester's theorem says that $b_{1,d}(q) \leq \lfloor (d+2)/2\rfloor$, either $b_{1,d}(q) = r_{1,d}$ or $r_{1,d}(q) = d+2-b_{1,d}(q)$, that each integer y with $1 \leq y \leq \lfloor (d+2)/2\rfloor$ is the border rank of some $P \in L$ and that each integer x such that $1 \leq x \leq d$ occurs as a rank for some $q \in \langle\nu_d(L)\rangle$. Moreover, each $q \in \langle\nu_d(L)\rangle$ has cactus rank equal to its border rank $b_{1,d}(q)$. There is a unique zero-dimensional scheme $Z \subset L$ evincing the cactus rank of q (and hence $\deg(Z) = b_{1,d}(q)$). We have $b_{1,d}(q) = r_{1,d}(q)$ if and only if Z is reduced. If $b_{1,d}(q) \neq r_{1,d}(q)$ and $B \in \mathcal{S}(q)$, then $B \cap Z = \emptyset$ ([8], [11, 4.1]). The fact that $b_{1,d}(q)$ is at least the cactus rank of q follows from general statements ([7, Lemma 2.6], [6, Proposition 2.5]) and easily implied that for q border and cactus ranks coincides. Granted this fact the uniqueness of Z and the fact that if $b_{1,d}(q) \neq r_{1,d}(q)$, then $b_{1,d}(q) + r_{1,d}(q) \geq d + 2$ follows from [3, Lemma 1] and the fact that $h^1(\mathbb{P}^1, \mathcal{I}_W(d)) = 0$ for every zero-dimensional scheme $W \subset \mathbb{P}^1$ with $\deg(W) \leq d + 1$.

Let X be any projective scheme and let $D \subset X$ be an effective Cartier divisor of X. For any zero-dimensional scheme $Z \subset X$ let $\mathrm{Res}_D(Z)$ denote the closed subscheme of X with $\mathcal{I}_Z : \mathcal{I}_D$ as its ideal sheaf. We have $\deg(Z) =$

$\deg(Z \cap D) + \deg(\mathrm{Res}_D(Z))$. For any line bundle $\mathcal{L}$ on X we have the exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_{D(Z)}} \otimes \mathcal{L}(-D) \to \mathcal{I}_Z \otimes \mathcal{L} \to \mathcal{I}_{Z \cap D, D} \otimes \mathcal{L}_{|D} \to 0 \tag{1}$$

We say that (1) is the residual exact sequence of the inclusion $D \subset X$.

The next 4 easy statements are contained in [2]. They are easily proved and they are the only parts of [2] that we use (except of course the statement of the main theorem of [2], which basically reduce the proof of Theorem 1 to prove it when $m = 3$; the case $m = 3$ is exactly the content of this paper).

Remark 2. Let $T \subset \mathbb{P}^2$ be a reduced curve of degree $t < d$. It is connected and the projective space $\langle \nu_d(T) \rangle$ has dimension $x := \binom{d+2}{2} - \binom{d-t+2}{2} - 1$. Every point of $\langle \nu_d(T) \rangle$ has rank at most x with respect to the curve $\nu_d(T)$ (the proof of [11, Proposition 5.1] works verbatim for reduced and connected curves). Hence if $B \in \mathcal{S}(P)$, then $\sharp(B \cap T) \le x$. If $t = 1$ (resp. $t = 2$, resp $t = 3$) then $x = d$ (resp. $2d$, resp. $3d - 1$).

Lemma 1. *Fix an integer $d \ge 6$. Let $Z \subset \mathbb{P}^m$, $m \ge 2$, be a zero-dimensional scheme with $\deg(Z) \le 3d + 1$ and $h^1(\mathcal{I}_Z(d)) > 0$. Then either there is a line $L \subset \mathbb{P}^m$ with $\deg(L \cap Z) \ge d + 2$ or there is a conic $T \subset \mathbb{P}^m$ with $\deg(T \cap Z) \ge 2d + 2$ or there is a plane cubic F with $\deg(F \cap Z) \ge 3d$.*

Remark 3. Take the set-up of Lemma 1 and assume the existence of a plane conic T with $\deg(T \cap Z) \ge 2d + 2$, but that there is no line $L \subset \mathbb{P}^m$ with $\deg(L \cap Z) \ge d + 2$. In many cases (e.g. when Z has many reduced connected components), it is obvious that T must be reduced. Assume that T is reduced and reducible, say $T = D \cup R$, with D and R lines and $D \ne R$. Set $\{o\} := D \cap R$. Since $\deg(D \cap Z) \le d + 1$ and $\deg(R \cap Z) \le d + 1$. We get $\deg(D \cap Z) = \deg(R \cap Z) = d + 1$ and that either $o \notin Z_{\mathrm{red}}$ or that Z is a Cartier divisor of the nodal curve T (it is a general property of nodal curves). Now assume the existence of a plane cubic F with $\deg(F \cap Z) \ge 3d$. F is not reduced if and only if there is a line $L \subset F$ appearing in F with multiplicity at least two. To get that F is reduced it is sufficient to assume that $\deg(R \cap Z) \le d + 1$ for each line R and that Z has at least $2d + 2$ reduced connected components.

Lemma 2. *([2, Proposition 5]) In the set-up of Theorem 1 if $\dim(\langle A \rangle) \le 2$, then $r_{m,d}(P) \le 3d$.*

2 A few lemmas

A connected zero-dimensional scheme $A \subset \mathbb{P}^n$ is called *curvilinear* if it has embedding dimension ≤ 1, i.e. if and only if either it is a point with its reduced structure or $\dim(\mu/\mu^2) = 1$, where μ is the maximal ideal of the local ring $\mathcal{O}_{A,O}$,

$\{O\} := A_{\mathrm{red}}$. A zero-dimensional scheme $A \subset \mathbb{P}^n$ is called *curvilinear* if its connected components are curvilinear. If A is connected and curvilinear, then for each integer z with $1 \le z \le \deg(A)$ there is a unique degree z subscheme of A. Hence a curvilinear zero-dimensional scheme has only finitely many subschemes. Usually for a projective scheme X and a coherent sheaf $\mathcal{F}$ on X we write $H^i(X, \mathcal{F})$, $i \in \mathbb{N}$, for its cohomology group and set $h^i(X, \mathcal{F}) := \dim H^i(X, \mathcal{F})$, but we often write $H^i(\mathcal{F})$ and $h^i(\mathcal{F})$ if X is a projective space obvious from the context.

Let $Q \subset \mathbb{P}^3$ be any smooth quadric surface. We have $\mathrm{Pic}(Q) \cong \mathbb{Z}^2$ and we take as a free basis of it the line bundles $\mathcal{O}_Q(1,0)$ and $\mathcal{O}_Q(0,1)$ whose complete linear systems induce the two projections $Q \to \mathbb{P}^1$. Both $\mathcal{O}_Q(1,0)$ and $\mathcal{O}_Q(0,1)$ are base-point free, $h^0(\mathcal{O}_Q(1,0)) = h^0(\mathcal{O}_Q(0,1)) = 2$ and $\mathcal{O}_Q(1) \cong \mathcal{O}_Q(1,1)$. The integers $h^i(Q, \mathcal{O}_Q(a,b))$, $i = 0,1,2$, $(a,b) \in \mathbb{Z}^2$, are computed using the Künneth's formula and the cohomology of line bundles on $\mathbb{P}^1$. In particular we have $h^1(Q, \mathcal{O}_Q(a,b)) = 0$ and $h^0(Q, \mathcal{O}_Q(a,b)) = (a+1)(b+1)$ if $a \ge -1$ and $b \ge -1$.

Lemma 3. *Fix an integer $d \ge 8$. Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface and let $Z \subset Q$ be a zero-dimensional scheme with $\deg(Z) \le 3d+3$ and $h^1(\mathcal{I}_Z(d)) > 0$. If $\deg(Z) > 3d$, then assume that the union of the non-reduced connected components of Z has degree ≤ 5. Then one of the following cases occurs:*

(1) there is $L \in (|\mathcal{O}_Q(1,0)| \cup |\mathcal{O}_Q(0,1)|)$ with $\deg(L \cap Z) \ge d+2$;

(2) there is $T \in |\mathcal{O}_Q(1,1)|$ with $\deg(T \cap D) \ge 2d+2$;

(3) there is $F \in (|\mathcal{O}_Q(2,1)| \cup |\mathcal{O}_Q(1,2)|)$ with $\deg(F \cap Z) \ge 3d+2$;

Proof. Taking a minimal $Z' \subseteq Z$ with $h^1(\mathcal{I}_{Z'}(d)) > 0$, we reduce to the case in which $h^1(\mathcal{I}_E(d)) = 0$ for all $E \subsetneq Z$. If $\deg(Z) \le 3d$, then use Lemma 1. In particular we may assume $3d < \deg(Z) \le 3d+3$ and that the lemma is true for all integers $d' < d$. Fix $D \in |\mathcal{O}_Q(2,2)|$ with $x := \deg(D \cap Z)$ maximal.

(a) Assume that $h^1(D, \mathcal{I}_{D\cap Z}(d)) > 0$. Since $h^1(Q, \mathcal{I}_D(d)) = 0$, we get $h^1(\mathcal{I}_{Z\cap D}(d)) > 0$. Since $h^1(\mathcal{I}_E(d)) = 0$ for all $E \subsetneq Z$, we get $Z \subset D$. Let $G \subseteq D$ be a minimal subcurve such that $Z \subset G$. Assume for the moment that G is not reduced, i.e. it has a multiple component. If G has no line counted with multiplicity ≥ 2, then see the case $C_1 = C_2$ of step (a2) below. Assume that G has a line with multiplicity 2, say $G = 2L \cup J$ with $L \in |\mathcal{O}_Q(1,0)|$ and either $J = \emptyset$ or $J \in |\mathcal{O}_Q(0,e)|$ with $e \in \{1,2\}$; set $e := 0$ if $J = \emptyset$. Since the union of the non-reduced connected components of Z has degree ≤ 5 and $Z \subset 2L \cup J$, we get $\deg(\mathrm{Res}_{L\cup J}(Z)) \le 2$ and hence $h^1(Q, \mathcal{I}_{\mathrm{Res}_{L\cup J}(Z)}(d-1, d-e)) = 0$. The residual exact sequence of the inclusion $L \cup J \subset Q$ gives $h^1(L\cup J, \mathcal{I}_{Z\cap(L\cup J)}(d)) > 0$. Since

$h^1(Q, \mathcal{O}_Q(d-e, d-1)) = 0$, we get $h^1(Q, \mathcal{I}_{Z\cap(L\cup J)}(d)) > 0$, contradicting the definition of G. Therefore except in step (a2) we freely use that G is reduced. G is a union of lines if and only if C is a union of 4 lines. Since $h^1(Q, \mathcal{O}_Q(d)(-G)) = 0$, we have $h^1(G, \mathcal{I}_Z(d)) > 0$. If $G \subsetneq D$, then we are in one of the cases listed in the statement of the lemma.

(a1) Assume that D is irreducible. Since D is a complete intersection of two quadric surfaces, $\omega_D \cong \mathcal{O}_D$. Therefore Riemann-Roch gives $\deg(Z) \geq 4d$, a contradiction.

(a2) Assume $D = C_1 \cup C_2$ with C_i irreducible conics (we allow the case $C_1 = C_2$). Note that $\mathrm{Res}_{C_1}(Z) \subset C_1$. If $C_1 \neq C_2$ up to a change of the labels we may assume $\deg(C_1 \cap Z) \geq \deg(C_2 \cap Z)$ and hence $\deg(Z \cap C_1) \geq \deg(Z)/2$. If $C_1 = C_2$, then note that $\mathrm{Res}_{C_1}(Z) \subseteq Z$ and so $\deg(C_1 \cap Z) \geq \deg(\mathrm{Res}_{C_1}(Z) \cap C_1) = \deg(\mathrm{Res}_{C_1}(Z))$, i.e. $\deg(C_1 \cap Z) \geq \deg(Z)/2$. Since C_1 is a Cartier divisor of Q, we have $\deg(\mathrm{Res}_{C_1}(Z)) = \deg(Z) - \deg(C_1 \cap Z) \leq (3d+3)/2 < 2d$. Since $\mathrm{Res}_{C_1}(Z) \subset C_2$ and C_2 is irreducible, we get $h^1(\mathcal{I}_{\mathrm{Res}_{C_1}(Z)}(d-1)) = 0$. The residual exact sequence of the inclusion $C_1 \subset Q$ gives $h^1(C_1, \mathcal{I}_{C_1\cap Z}(d)) > 0$ and hence $\deg(C_1 \cap Z) \geq 2d+2$.

(a3) By steps (a1) and (a2) we may assume that $G = D$ is reduced and that it contains a line L, say of type $(1,0)$. Take $F \in |\mathcal{O}_Q(1,2)|$ with $D = L+F$. Since $G = D$, we have $Z \cap F \subsetneq Z$ and hence $h^1(\mathcal{I}_{F\cap Z}(d)) = 0$. Thus the residual exact sequence of the inclusion $F \subset Q$ gives $h^1(\mathcal{I}_{\mathrm{Res}_F(Z)}(d-1, d-2)) > 0$ and hence $\deg(L \cap \mathrm{Res}_F(Z)) \geq d$. Hence $\deg(F \cap Z) \leq 2d+3$. First assume that F is irreducible. We get $h^1(F, \mathcal{I}_{F\cap Z}(d-1)) = 0$. Hence $h^1(F, \mathcal{I}_{\mathrm{Res}_L(Z)}(d-1, d)) = 0$. The residual exact sequence of the inclusion $L \subset Q$ gives $h^1(L, \mathcal{I}_{Z\cap L}(d)) > 0$ and hence $Z \cap L = Z$ and $G = L$, a contradiction. Assume that F is reducible and take a curve $C' \subset F$ of type $(1,1)$ (it may be reducible). Write $F = C' + R$. Since $h^1(\mathcal{I}_{\mathrm{Res}_{C'+L}(Z)}(d-2, d-1)) > 0$, we get $\deg(\mathrm{Res}_{C'+L}(Z)) \geq d$ and hence $\deg((L \cup R) \cap Z)) \geq 2d$. Therefore $\deg(\mathrm{Res}_{R+L}(Z)) \leq d+3$. Since $h^1(C', \mathcal{I}_{\mathrm{Res}_{R+L}(Z)}(d-1)) > 0$, we get that C' is reducible and there is a component J of C' with $\deg(J \cap \mathrm{Res}_{R+L}(Z)) \geq d+1$. Since $\deg(\mathrm{Res}_{J+R+L}(Z)) \leq 2$, we have $h^1(Q, \mathcal{I}_{\mathrm{Res}_{J+R+L}(Z)}(d-1)(-J)) = 0$. Hence $G \subseteq J \cup R \cup L$, a contradiction.

(b) Assume $h^1(D, \mathcal{I}_{D\cap Z}(d)) = 0$. The residual exact sequence of D in Q gives $h^1(\mathcal{I}_{\mathrm{Res}_D(Z)}(d-2)) > 0$. Set $W := \mathrm{Res}_D(Z)$. Since $h^0(\mathcal{O}_Q(2)) = 9$, we have $x \geq 8$ and hence $\deg(W) \leq 3d-5 = 3(d-2)+1$. The inductive assumption gives that either there is a line $L \subset Q$ with $\deg(L \cap W) \geq d$ or there is $E \in |\mathcal{O}_Q(1,1)|$ with $\deg(W \cap E) \geq 2d-2$ or there a curve $F \in (|\mathcal{O}_Q(2,1)| \cup |\mathcal{O}_Q(1,2)|)$ with $\deg(F \cap Z) \geq 3d-4$.

Assume the existence of E. Note that $\mathcal{O}_Q(2,2)(-E) = \mathcal{O}_Q(1,1)$. We have

$h^0(Q,\mathcal{O}_Q(1,1)) = 4$. Thus there is a curve $N \in |\mathcal{O}_Q(1,1)|$ such that $\deg(\mathrm{Res}_E(Z) \cap N) \ge \min\{\deg(\mathrm{Res}_E(Z),3\}$. Since $\deg(E\cap Z) \ge \deg(E\cap W) \ge 2d-2$, we get $x \ge 2d+1$ and hence $\deg(W) \le d+2$, a contradiction. In the same way we exclude F. Therefore there is a line $L \subset Q$ with $\deg(L\cap W) \ge d$. Set $Z_0 := Z$. Fix $N_1 \in |\mathcal{I}_L(1,1)|$ such that $f_1 := \deg(Z_0 \cap N_1)$ is maximal. Since $h^0(Q,\mathcal{I}_L(1,1)) = 2$, we have $f_1 \ge 1+\deg(Z\cap L) \ge d+1$. Set $Z_1 := \mathrm{Res}_{N_1}(Z_0)$. Take $N_2 \in |\mathcal{O}_Q(1,1)|$ such that $f_2 := \deg(N_2\cap Z_1)$ is maximal and set $Z_2 := \mathrm{Res}_{N_2}(Z_1)$. Fix an integer $h > 2$ and assume defined f_i, N_i and Z_i for all $i < h$. Take any $N_h \in |\mathcal{O}_Q(1,1)|$ such that $f_h := \deg(N_h \cap Z_{h-1})$ is maximal. We have just defined N_i, f_i, Z_i for all $i \ge 1$. We have $f_i \ge f_{i+1}$ for all $i \ge 2$ and if $f_i \le 2$, then $f_{i+1} = 0$ and $Z_i = \emptyset$. Since $f_1 \ge d+1$, we have $\sum_{i\ge 2} f_i \le 2d+2$. Recall that $h^1(Q,\mathcal{I}_{Z_0,Q}(d)) > 0$. Fix an integer $h \ge 2$ and assume $h^1(N_i,\mathcal{I}_{Z_{i-1}\cap N_i,N_i}(d+1-i)) = 0$ for all $i < h$. The residual exact sequence of $N_h \subset Q$ gives $h^1(Q,\mathcal{I}_{Z_h,Q}(d+1-h)) > 0$. Since $h^1(\mathcal{O}_Q(t)) = 0$ for all integers t, there is a minimal integer g' such that $h^1(N_{g'},\mathcal{I}_{Z_{g'-1}\cap N_{g'},N_{g'}}(d+1-g')) > 0$. Note that $f_{g'} > 0$. Since $f_i \ge 3$ if $f_{i+1} > 0$ and $\sum_{i\ge 2} f_i \le 2d+2$, we have $g' < d$. Since $h^1(\mathbb{P}^3,\mathcal{I}_{N_{g'}\cap Z_{g'-1}}(d+1-g')) = h^1(N_{g'},\mathcal{I}_{N_{g'}\cap Z_{g'-1}}(d+1-g')) > 0$, there is a line $R \subset \mathbb{P}^3$, such that $\deg(R\cap Z_{g'-1}) \ge d+3-g'$ ([5, Lemma 34]). Since $\deg(R\cap Z_{g'-1}) \ge 3$, we have $R \subset Q$. Note that $R \ne L$, because $\deg(R \cap \mathrm{Res}_{L\cup R}(Z)) \ge d \ge 8$ and the sum of the degrees of the unreduced connected components of Z is at most 5. Assume for the moment $g' \ge 2$. Since $g' < d$ and $(g'-1)(d+3-g') \le 2d+2$, we get $g' \le 3$ and hence $\deg(R\cap Z_1) \ge d$. If $h^1(Q,\mathcal{I}_{Z\cap(L\cup R}(d)) > 0$, then the minimality of Z gives $Z \subset L\cup R$. $L\cup R$ is either a plane conic or the union of 2 disjoint lines and in both cases we conclude, because we assumed at the beginning of the proof $\deg(Z) > 3d$. Now assume $h^1(Q,\mathcal{I}_{Z\cap(L\cup R)}(d)) = 0$. The residual exact sequence of $L\cup R \subset Q$ gives $h^1(Q,\mathcal{I}_{\mathrm{Res}_{L\cup R}(Z)}(d)(-L-R)) > 0$. Hence $h^1(Q,\mathcal{I}_{\mathrm{Res}_{L\cup R}(Z)}(d-2,d-2) > 0$. Since $\deg(\mathrm{Res}_{L\cup R}(Z)) \le d+2$, [5, Lemma 34] gives the existence of a line $J \subset \mathbb{P}^3$ such that $\deg(J\cap \mathrm{Res}_{L\cup R}(Z)) \ge d$. Bezout's theorem gives $J \subset Q$. Note that $J \ne R$ and $J \ne L$, because $\deg(J\cap \mathrm{Res}_{L\cup R}(Z)) \ge d \ge 8$ and the sum of the degrees of the unreduced connected components of Z is at most 5. Since $\deg(\mathrm{Res}_{L\cup R\cup J}(Z)) \le 3$, we get $h^1(Q,\mathcal{I}_{\mathrm{Res}_{L\cup R\cup J}(Z)}(d)(-L-R-J)) = 0$ (even if the lines L, R, J are in the same ruling of Q, because $d \ge 8$. The minimality of Z gives $Z \subset L\cup J\cup R$. We start with one of the lines L, R, J (call it L_1) with $\deg(L_1\cap Z)$ maximal. If $\deg(L_1\cap Z) \ge d+2$, the proof is over. If $\deg(L_1\cap Z) \le d+1$ and all lines L, R and J are in the same ruling of Q, then taking the residual first with respect to L, then to R and then to J we get $h^1(Q,\mathcal{I}_Z(d)) = 0$, a contradiction. Thus at least one of the other lines meets L_1 and we call L_2 a line among $\{L,R,J\}\setminus L_1$ meeting L_1 and with $\deg(L_2\cap \mathrm{Res}_{L_1}(Z))$ maximal. Note that $\deg(Z\cap(L_1\cup L_2)) = \deg(L_1\cap Z) + \deg(L_2\cap \mathrm{Res}_{L_1}(Z))$. If

$\deg(Z\cap(L_1\cup L_2)) \ge 2d+2$, then we are in case (2). If $\deg(Z\cap(L_1\cup L_2)) \le 2d+1$, then $h^1(Q,\mathcal{I}_{Z\cap(L_1\cup L_2)}(d)) = 0$. Since $L_1 \cup L_2 \in |\mathcal{O}_Q(1,1)|$, the residual exact sequence of $L_1 \cup L_2 \subset Q$ gives $h^1(Q,\mathcal{I}_{\mathrm{Res}_{L_1\cup L_2}(Z)}(d-1,d-1)) > 0$. Hence [5, Lemma 34] gives $\deg(\mathrm{Res}_{L_1\cup L_2}(Z)) \ge d+1$. Call L_3 the line in $\{L,R,J\}\setminus\{L_1,L_2\}$. Since $d+1 \ge \deg(Z\cap L_1) \ge \deg(Z\cap L_3)$, we get $\deg(L_1\cap Z) = \deg(L_3\cap Z) = \deg(L_3\cap \mathrm{Res}_{L_1\cup L_2}(Z)) = d+1$. We see that L_1 and L_3 are in the same ruling of Q, say $|\mathcal{O}_Q(1,0)|$ and that $h^1(Q,\mathcal{I}_{Z\cap(L_1\cup L_3)}(d)) = 0$. The residual exact sequence of $L_1\cup L_3 \subset Q$ gives $h^1(Q,\mathcal{I}_{\mathrm{Res}_{L_1\cup L_3}(Z)}(d-2,d)) > 0$ and hence $\deg(\mathrm{Res}_{L_1\cup L_3}(Z)) \ge d$. Since $\mathrm{Res}_{L_1\cup L_3}(Z)) \subset L_3 \in |\mathcal{O}_Q(0,1)|$, we are in case (3) with F union of 3 lines. QED

Lemma 4. *Fix an integer $d \ge 8$. Fix $o \in \mathbb{P}^3$ and 3 distinct lines L_1, L_2, L_3 of $\mathbb{P}^3$ with $\{o\} = L_1\cap L_2\cap L_3$ and $\langle L_1\cup L_2\cup L_3\rangle = \mathbb{P}^3$. Let $Z \subset L_1\cup L_2\cup L_3$ be a zero-dimensional scheme such that $\deg(Z\cap L_1) \ge \deg(Z\cap L_i)$ for all i and $\deg(Z\cap(L_1\cup L_2)) \ge \deg(Z\cap(L_1\cup L_3))$. We have $h^1(\mathbb{P}^3,\mathcal{I}_Z(d)) = h^1(L_1\cup L_2\cup L_3,\mathcal{I}_Z(d))$. We have $h^1(\mathbb{P}^3,\mathcal{I}_Z(d)) > 0$ if and only if either $\deg(L_1\cap Z) \ge d+2$ or $\deg(Z\cap(L_1\cup L_2)) \ge 2d+2$ or $\deg(Z) \ge 3d+2$.*

Proof. We have $h^1(\mathbb{P}^3,\mathcal{I}_Z(d)) = h^1(L_1\cup L_2\cup L_3,\mathcal{I}_Z(d))$, because $L_1\cup L_2\cup L_3$ is arithmetically Cohen-Macaulay. Since $h^0(\mathcal{O}_{L_1\cup L_2\cup L_3}(d)) = 3d+1$ and $h^0(\mathcal{O}_{L_1\cup L_2}(d)) = 2d+1$, the " if " part is obvious. Assume $\deg(Z\cap L_3) \le d$. Set $H := \langle Z\cap(L_1\cup L_2)\rangle$. We have $\dim(H) \le 2$ and we may assume $\dim(H) = 2$, because the lemma is true if $Z \subset L_i$ for some i. We may apply [5, Lemma 34] to $Z\cap H$, because $H\cap Z = Z\cap(L_1\cup L_2)$ and if $\deg(Z\cap H) \ge 2d+2$, then we are done. Therefore we may assume $h^1(\mathcal{I}_{Z\cap H}(d)) = 0$. Hence a residual exact sequence gives $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(d-1)) > 0$. Since $\mathrm{Res}_H(Z) \subseteq Z\cap L_3$, we get $\deg(Z\cap L_3) \ge d+1$. By the proof just given we may also assume $\deg(Z\cap L_2) = \deg(Z\cap L_1) = d+1$. Since $L_1\cap L_2 = \{o\}$ (scheme-theoretically) we have $\deg(Z\cap(L_1\cup L_2)) \ge 2d+1$. Since $(L_1\cup L_2)\cap L_3 = \{o\}$ (scheme-theoretically), we get $\deg(Z) \ge \deg(Z\cap H) + \deg(Z\cap L_3) - 1 \ge 3d+2$. QED

Lemma 5. *Fix an integer $d \ge 3$. Let $Q \subset \mathbb{P}^3$ be an irreducible quadric cone with vertex o and $Z \subset Q$ a zero-dimensional scheme with $\deg(Z) \le 3d+3$ and $h^1(\mathcal{I}_Z(d)) > 0$. If $\deg(Z) \ge 3d$, then assume $d \ge 8$, that the union of the unreduced connected components has degree ≤ 5 and that each of them is curvilinear and linearly independent. Then one of the following cases occurs:*

(i) there is a line $L \subset Q$ with $\deg(L\cap Z) \ge d+2$;

(ii) there is a plane section $D \subset Q$ with $\deg(D\cap Z) \ge 2d+2$; if we are not in case (i) either D is smooth or $D = L_1\cup L_2$ with L_1, L_2 distinct lines and $\deg(Z\cap L_1) = \deg(Z\cap L_2) = d+1$;

(iii) there is a curve $F \subset Q$ with $\deg(F \cap Z) \geq 3d+2$ and either F is the union of a plane section of Q and a line of Q or it is a rational normal curve; if we are not in cases (i) or (ii) then either F is a smooth rational normal curve or $F = D \cup L$ with D a smooth conic, L a line, $\deg(D \cap Z) = 2d+1$ and $\deg(L \cap Z) = d+1$.

Proof. By Lemma 1 we may assume $\deg(Z) \geq 3d+1$. Therefore we may assume $d \geq 8$. We immediately reduce to the case $h^1(\mathcal{I}_W(d)) = 0$ for all $W \subsetneq Z$. Since the case in which Z is reduced is known ([1]), we may assume $Z \neq Z_{\mathrm{red}}$. We may assume that $\deg(Z) \geq 3d+1$ even after these reductions, because any subscheme of a curvilinear scheme is curvilinear.

Fix $D \in |\mathcal{O}_Q(2)|$ such that $x := \deg(D \cap Z)$ is maximal. Since $h^0(\mathcal{O}_Q(2)) = 9$, we have $m \geq 8$.

(a) Assume $h^1(D, \mathcal{I}_{D\cap Z}(d)) > 0$. Hence $D \cap Z = Z$, i.e. $Z \subset D$. We have $h^0(\mathbb{P}^3, \mathcal{I}_D(2)) = 2$, because $h^0(Q, \mathcal{I}_{D,Q}(2)) = h^0(Q, \mathcal{O}_Q) = 1$ and $h^0(\mathbb{P}^3, \mathcal{I}_Q(2)) = 1$; equivalently, we use that D is a complete intersection of 2 quadric surfaces. Take a general quadric $Q' \subset \mathbb{P}^3$ containing D. Since Q is irreducible, Q' is irreducible. If Q' is smooth, then we apply Lemma 3 and get the existence of a certain curve L or T or F inside Q' (call T' this curve) with $h^1(\mathcal{I}_{T'\cap Z}(d)) > 0$, $\deg(T' \cap Z)$ large and $h^1(\mathcal{I}_{T''\cap Z}(d)) = 0$ for every proper subcurve T'' of T. Bezout's theorem gives $T' \subset Q$ (in all cases, even if T' is reducible). Now assume that all quadrics Q' are singular. Since Q is irreducible and Q' is a general element of $|\mathcal{I}_D(2)|$, Q' is irreducible. Bertini's theorem gives that a general Q' has singular point contained in D. Each complete intersection curve of a surface singular at some $o' \in \mathbb{P}^3$ with a surface containing o' is singular at o'. We get that either D has a multiple component or that D is the complete intersection of two quadric cones with the same vertex, o. In the latter case D is the union of 4 lines of Q through o (if D has no multiple component).

(a1) Assume that D has no multiple component. In this case case $D = L_1 \cup L_2 \cup L_3 \cup L_4$ with each L_i a line. Set $m_i = \deg(L_i \cap Z)$. We order the lines L_1, L_2, L_3, L_4 of D so that $m_4 \leq m_i$ for all i.

(a1.1) First assume $m_4 \leq d-1$. If $h^1(L_1 \cup L_2 \cup L_3, \mathcal{I}_{Z\cap(L_1\cup L_2\cup L_3)}(d)) > 0$, then we use Lemma 4. Now assume $h^1(L_1 \cup L_2 \cup L_3, \mathcal{I}_{Z\cap(L_1\cup L_2\cup L_3)}(d)) = 0$. Since Z is curvilinear, it has only finitely many subschemes. Since $L_1 \cup L_2 \cup L_3$ is scheme-theoretically cut out by quadrics, we have $Q_1 \cap Z = Z \cap (L_1 \cup L_2 \cup L_3)$ for a general quadric surface $Q_1 \supset L_1 \cup L_2 \cup L_3$. Since $h^1(L_1 \cup L_2 \cup L_3, \mathcal{I}_{Z\cap(L_1\cup L_2\cup L_3)}(d)) = 0$ and $Q_1 \cap Z = Z \cap (L_1 \cup L_2 \cup L_3)$, the residual exact sequence of the inclusion $Q_1 \subset \mathbb{P}^3$ gives $h^1(\mathcal{I}_{\mathrm{Res}_{Q_1}(Z)}(d-2)) > 0$. Since $\mathrm{Res}_{Q_1}(Z) \subseteq Z \cap L_4$, we get $m_4 \geq d$, a contradiction.

(a1.2) Now assume $m_4 \geq d$ and hence $m_i \geq d$ for all i. As in the last part of the proof of Lemma 4 we get $\deg(Z \cap (L_1 \cup L_2)) \geq 2d-1$ and $\deg(Z \cap$

$(L_1 \cup L_2 \cup L_3)) \geq 3d-2$. Since $\deg(L_3 \cap (L_1 \cup L_2 \cup L_3)) = 2$, we also get $\deg(Z) \geq 4d-4 > 3d+3$, a contradiction.

(a2) Assume that D has at least one multiple component. Set $f := \deg(D) - \deg(D_{\mathrm{red}})$. Since $Z_{\mathrm{red}} \subset D_{\mathrm{red}}$ and $\deg(Z_{\mathrm{red}}) \geq \deg(Z) - 4 \geq 3d-3$, we may assume $f = 1$, i.e. that D is the union of the double $2L$ of a line L (i.e. the scheme-theoretic intersection of Q with a plane tangent to Q at a point of $L \backslash \{0\}$) and a conic C (a smooth plane section of Q or the union of two lines through o). Since Z is curvilinear, it has finitely many subschemes. Since $C \cup L$ is the scheme-theoretic base locus of $|\mathcal{I}_{C\cup L}(2)|$ (take a general plane $H \subset \mathbb{P}^3$ and use that 3 non-collinear points of H are cut out by conics), we have $Z \cap Q' = Z \cap (C \cup L)$ for a general $Q' \in |\mathcal{I}_{C\cup L}(2)|$. Since $Z \neq Z_{\mathrm{red}}$, we have $h^1(Q, \mathcal{I}_{Z_{\mathrm{red}}}(d)) = 0$ and hence $h^1(\mathbb{P}^3, \mathcal{I}_{Z_{\mathrm{red}}}(d)) = 0$ and hence $h^1(Q', \mathcal{I}_{Z_{\mathrm{red}}}(d)) = 0$. The residual exact sequence of the inclusion $Q' \subset \mathbb{P}^3$ gives $h^1(\mathcal{I}_{\mathrm{Res}_{Q'}(Z)}(d-2)) > 0$. Hence $\deg(\mathrm{Res}_{Q'}(Z)) \geq d$. Since $\deg(\mathrm{Res}_{Q'}(Z)) = \deg(Z) - \deg(Q' \cap Z) \leq \deg(Z) - \deg(Z_{\mathrm{red}}) \leq 4$, we get a contradiction.

(b) In this step we assume $h^1(C, \mathcal{I}_{D\cap Z}(d)) = 0$. A residual exact sequence gives $h^1(\mathcal{I}_{\mathrm{Res}_{D}(Z))}(d-2)) > 0$. As in step (b) of the proof of Lemma 3 we first get the existence of a line $L \subset Q$ with $\deg(L \cap Z) \geq d$, then define N_i, f_i, Z_i, g', get $g' \leq 3$ and we land in one of the cases (i), (ii) or (iii) with curves unions of lines. Note that for two different lines of T, say L are R, the divisor $L \cup R$ is a Cartier divisor of Q (it is a plane section of Q) and hence we may define the residual sequence with respect to $L \cup R$, but not with respect to L. We call L_1 an element of $\{L, R, J\}$ with $\deg(L_1 \cap Z)$ maximal. Then we call L_2 one of the other 2 lines with $\deg(Z \cap (L_1 \cup L_2))$ maximal. QED

Lemma 6. *Let $A \subset \mathbb{P}^m$, $m \geq 2$, be a connected curvilinear scheme such that $\deg(A) = 3$ and $\dim(\langle A \rangle) = 2$. Set $\{o\} := A_{\mathrm{red}}$. Let L, R be lines of $\mathbb{P}^m$ such that $L \neq R$. We have $A \subset L \cup R$ if and only if $L \cup R \subset \langle A \rangle$, $o \in L \cap R$ and one of the lines L, R contains the degree two subscheme of A.*

Proof. If either $o \notin L$ or $o \notin R$, then $A \nsubseteq L \cup R$, because $A_{\mathrm{red}} = \{o\}$ and $\dim(\langle A \rangle) = 2$. Now assume $\{o\} = L \cap R$. Therefore $M := \langle L \cup R \rangle$ is a plane. If $A \subset L \cup R$, then $\langle A \rangle \subseteq \langle L \cup R \rangle$. Therefore we may assume $L \cup R \subset \langle A \rangle$. We use that L, R are Cartier divisors of the plane $\langle A \rangle$ and hence $\mathrm{Res}_L(\mathrm{Res}_R(A)) = \mathrm{Res}_{L+R}(A) = \mathrm{Res}_{R+L}(A) = \mathrm{Res}_R(\mathrm{Res}_L(A))$. Since $\langle A \rangle = \mathbb{P}^2$, for any line D we have $\deg(A \cap D) \leq 2$ and equality holds if and only if D is the line spanned by the the degree two zero-dimensional scheme A' of A. Therefore $A \nsubseteq L \cup R$ if $o \notin L \cup R$. Assume $o \in L \cup R$ and take one of the lines L, R, say R, which doesn't contain A'. We have $\mathrm{Res}_R(A) = A'$. Therefore $A \subset L \cup R$ if and only if $A' \subset L$. QED

Lemma 7. *Fix $o \in \mathbb{P}^3$ and let L, R, D 3 distinct lines of $\mathbb{P}^3$ such that $o \in L \cap R \cap D$ and $\langle L \cup D \cup R \rangle = \mathbb{P}^3$. Let $A \subset \mathbb{P}^3$ be a connected curvilinear scheme such that $\deg(A) = 4$ and $\langle A \rangle = \mathbb{P}^3$. Then $A \nsubseteq L \cup D \cup R$.*

Proof. If $\{o\} \neq (A)_{\mathrm{red}}$, then the lemma is obvious, because A is linearly independent and in particular it is not contained in a line. Therefore we may assume $\{o\} = A_{\mathrm{red}}$. Let A' (resp. A'') be the degree two (resp. 3) subscheme of A. At most one of the lines L, R, D contains A', i.e. it is the line $\langle A' \rangle$. Take lines L, R which do not contain A'.

First assume $A' \nsubseteq \langle L \cup R \rangle$. In this case the plane $\langle L \cup R \rangle$ is transversal to $\langle A' \rangle$ and hence $\deg(A \cap \langle L \cup R \rangle) = 1$. Therefore $\mathrm{Res}_{\langle L \cup R \rangle}(A) = A''$. Since A is linearly independent, then $A'' \nsubseteq D$ and hence $A \nsubseteq (\langle L \cup R \rangle) \cup D$.

Now assume $\langle A' \rangle \subset \langle L \cup R \rangle$. Since $L \neq \langle A' \rangle$ and $R \neq \langle A' \rangle$, Lemma 6 gives $\deg(A \cap \langle L \cup R \rangle) = 2$ and hence $\mathrm{Res}_{\langle L \cup R \rangle}(A) = A'$. Since $D \nsubseteq \langle L \cup R \rangle$, we get $A' \nsubseteq D$ and hence $A \nsubseteq (\langle L \cup R \rangle \cup D)$. QED

Lemma 8. *Let $A \subset \mathbb{P}^3$ be a connected curvilinear degree 4 scheme such that $\deg(A) = 4$ and $\langle A \rangle = \mathbb{P}^3$. Set $\{o\} := A_{\mathrm{red}}$ and let A' be the degree 3 closed subscheme of A. Let $C \subset \langle A' \rangle$ be any smooth conic containing A'. There is a line $L \subset \mathbb{P}^3$ such that $A \subset C \cup L$ and $o \in L$ for any such a line L.*

Proof. If L exists, then obviously $o \in L \nsubseteq \langle A' \rangle$. Since $A \nsubseteq \langle A' \rangle$, we have $(A \cup C) \cap \langle A' \rangle = C$. We get the existence of a smooth quadric $Q \supset C \cup A$. Call $\mathcal{O}_C(1, 0)$ any ruling of Q and let L be the line of $|\mathcal{O}_Q(1, 0)|$ containing o. We have $\mathrm{Res}_C(A) = \{o\} \in L$. Since C, L are Cartier divisors of Q, we get $A \subset C \cup L$. QED

3 The main results

Let $A \subset \mathbb{P}^n$, $n \geq 2$, be a zero-dimensional scheme. We recall that A is said to be *in linearly general position in* $\mathbb{P}^n$ if $\deg(V \cap A) \leq \dim V + 1$ for every linear subspace $V \subsetneq \mathbb{P}^n$. If $\deg(A) > n$ (and in particular if $\langle A \rangle = \mathbb{P}^n$), A is in linearly general position if and only if $\deg(H \cap A) \leq n$ for all hyperplanes $H \subset \mathbb{P}^n$. If A is in linearly general position in $\mathbb{P}^n$, then each subscheme of A is in linearly general position in $\mathbb{P}^n$. If $\deg(A) \leq n+1$, A is in linearly general position in $\mathbb{P}^n$ if and only if it is linearly independent, i.e. if and only if $\dim(\langle A \rangle) = \deg(A) - 1$.

Proposition 3. *Assume $d \geq 7$ and $m \geq 3$ and take $P \in \mathbb{P}^r$ with $b_{m,d}(P) = 5$ and $A \subset \mathbb{P}^m$ evincing the cactus rank of P with $A = A_1 \sqcup \{O_2\}$, $\deg(A_1) = 4$ and A_1 connected. Assume the existence of a 3-dimensional linear subspace $\mathbb{H} \subseteq \mathbb{P}^m$ such that $\mathbb{H} \supset A$ and A is in linearly general position in $\mathbb{H}$. Then $r_{m,d}(P) = 3d - 3$.*

Proof. By concision ([10, Exercise 3.2.2.2]) we may assume $m = 3$. Since A_1 is Gorenstein ([6, part (ii) of Proposition 2.2]) and $\dim\langle A_1\rangle = \deg(A_1) - 1$, A_1 is unramified and curvilinear ([9, Theorem 1.3]).

Claim: A is contained in a rational normal curve C.

Proof of the Claim: Since A is curvilinear, it has only finitely many subschemes. This property and a dimensional count give that the scheme $A \cup \{Q\}$ is in linearly general position for a general $Q \in \mathbb{P}^3$. By [9, part (b) of Theorem 1] $A \cup \{Q\}$ is contained in a unique rational normal curve. Hence A is contained in a rational normal curve.

Since $\nu_d(C)$ is a degree $3d$ rational normal curve in its linear span and $A \subset C$, Sylvester's theorem (Remark 1) says that P has at most rank $3d-3$ with respect to $\nu_d(C)$. Hence $r_{m,d}(P) = r_{3,d}(P) \le 3d - 3$. Assume $r_{3,d}(P) \le 3d - 4$ and take any $B \subset \mathbb{P}^3$ evincing the rank of P. We have $\deg(A \cup B) \le 3d + 1$. The proof of [4, Proposition 5.19] gives a contradiction (see the proofs of Propositions 4 and Proposition 5 for similar, but harder proofs). Alternatively, take $P_1 \in \langle\nu_d(A_1)\rangle$ such that $P \in \langle\{P_1, \nu_d(A)\}\rangle$; it is easy to check that $P_1 \notin \langle\nu_d(E)\rangle$ for any $E \subsetneq A_1$; since $d \ge 7$, A_1 is the unique zero-dimensional scheme $F \subset \mathbb{P}^m$ with $\deg(F) \le 4$ and $P_1 \in \langle\nu_d(F)\rangle$; therefore A_1 evinces the cactus rank of P and hence $r_{m,d}(P_1) = 3d - 2$ ([4, Proposition 5.19]); since $P_1 \in \langle\{\nu_d(O_2), P\}\rangle$, we get $r_{m,d}(P) \ge 3d - 3$. QED

Proposition 4. *Assume $d \ge 9$. Let $A_1 \subset \mathbb{P}^m$, $m \ge 3$, be a connected and curvilinear zero-dimensional scheme such that* $\deg(A_1) = 4$ *and* $\dim(\langle A_1\rangle) = 3$. *Set $\{O_1\} := (A_1)_{\mathrm{red}}$. Let A' be the degree 2 subscheme of A_1. Fix $O_2 \in \langle A'\rangle\setminus\{O_1\}$ and set $A := A_1 \cup \{O_2\}$. Take any $P \in \langle A\rangle$ such that $P \notin \langle E\rangle$ for any scheme $E \subsetneq A$. Then $r_{m,d}(P) = 3d-1$, $b_{m,d}(P) = 5$ and A is the only scheme evincing the cactus rank of P.*

Proof. By concision ([10, Exercise 3.2.2.2]) we may assume $m = 3$. There is a unique $P_1 \in \langle\nu_d(A_1)\rangle$ such that $P \in \langle\{P_1, \nu_d(O_2)\}\rangle$. Since $P \in \langle A\rangle$ and $P \notin \langle E\rangle$ for any $E \subsetneq A$, then $P_1 \in \langle A\rangle$ and $P_1 \notin \langle E\rangle$ for any $E \subsetneq A_1$. Hence $r_{m,d}(P_1) = 3d - 2$ ([4, Proposition 5.19]). Since $P \in \langle\{P_1, \nu_d(O_2)\}\rangle$ and $P_1 \in \langle\{P, \nu_d(O_2)\}\rangle$, then $3d - 3 \le r_{m,d}(P) \le 3d - 1$. Assume $r_{m,d}(P) \le 3d - 2$ and take $B \in \mathcal{S}(P)$. Set $W_0 := A \cup B$. We have $\deg(W_0) \le 3d + 3$. We have $h^1(\mathcal{I}_{W_0}(d)) > 0$ ([3, Lemma 1]).

Claim 1: A_1 is not contained in a union of 3 distinct lines.

Proof of Claim 1: Assume $A_1 \subset L \cup D \cup R$ with L, D, R distinct lines. Since A_1 is connected and $\langle A_1\rangle = \mathbb{P}^3$, we have $\langle L \cup R \cup D\rangle = \mathbb{P}^3$ and $O_1 \in L \cap D \cap R$, contradicting Lemma 7.

Claim 2: A is not contained in a union of a reduced conic C and a line L.

Proof of Claim 2: Assume $A \subset C \cup L$. Claim 1 gives that C is a smooth conic. Since A_1 is connected and $\langle A_1 \rangle = \mathbb{P}^3$, we have $L \nsubseteq \langle C \rangle$ and $\{O_1\} = C \cap L$. Since $\deg(C \cap D) \le 2$ for each line D, while $\deg(\langle \{O_1, O_2\} \rangle \cap A) = 3$, we get $L = \langle \{O_1, O_2\} \rangle$. Since $L \nsubseteq \langle C \rangle$, we have $\deg(A_1 \cap \langle C \rangle) = 1$ and $O_2 \notin \langle C \rangle$. Therefore $\mathrm{Res}_{\langle C \rangle}(A_1) = A'' \nsubseteq L$. Therefore $A_1 \nsubseteq \langle C \rangle \cup L$, a contradiction.

Claim 3: We have $O_2 \notin B$.

Proof of Claim 3: Assume $O_2 \in B$ and set $B' := B \setminus \{O_2\}$. The curvilinear scheme A_1 is contained in a rational normal curve of $\mathbb{P}^3$. Using this curve we see that A_1 is cut out by quadrics and hence by surfaces of degree d. Since $O_2 \ne O_1$, we get $h^1(\mathbb{P}^3, \mathcal{I}_A(d)) = 0$ and hence $\nu_d(O_2) \notin \langle \nu_d(A_1) \rangle$. Since $O_2 \in B$, $P \notin \langle \nu_d(A_1) \rangle$ and $P \in \langle \nu_d(A) \rangle$, the line $\langle \{\nu_d(O_2), P \rangle$ contains at least one point, P_2, of $\langle \nu_d(A_1) \rangle$. If $P_2 \in \langle \nu_d(E) \rangle$ for some $E \nsubseteq A_1$, we have $r_{m,d}(P_2) \le 2d - 1$ by [5] and so $r_{m,d}(P) \le 2d$, contradicting the inequality $r_{m,d}(P) \ge 3d - 2$. If $P_2 \notin \langle \nu_d(E) \rangle$ for any $E \nsubseteq A_1$, we get $r_{m,d}(P_2) = 3d - 1$ ([4, Proposition 5.19]). Hence $\sharp(B') \ge 3d - 1$, contradicting the assumption $\sharp(B) \le 3d - 1$.

Let $H_1 \subset \mathbb{P}^3$ be a plane such that $e_1 := \deg(W_0 \cap H_1)$ is maximal. Set $W_1 := \mathrm{Res}_{H_1}(W_0)$. Fix an integer $i \ge 2$ and assume to have defined the integers e_j, the planes H_j and the scheme W_j, $1 \le j < i$. Let $H_i \subset \mathbb{P}^3$ be any plane such that $e_i := \deg(H_i \cap W_{i-1})$ is maximal. Set $W_i := \mathrm{Res}_{H_i}(W_{i-1})$. We have $e_i \ge e_{i+1}$ for all i. For each integer $i > 0$ we have the residual exact sequence

$$0 \to \mathcal{I}_{W_i}(d-i) \to \mathcal{I}_{W_{i-1}}(d+1-i) \to \mathcal{I}_{W_{i-1} \cap H_i, H_i}(d+1-i) \to 0 \qquad (2)$$

Since $h^1(\mathcal{I}_{W_0}(d)) > 0$, there is an integer $i > 0$ such that $h^1(H_i, \mathcal{I}_{W_{i-1} \cap H_i, H_i}(d+1-i)) > 0$. We call g the minimum such an integer. Since $h^1(\mathcal{O}_{\mathbb{P}^3}(t)) = 0$ for every integer t, we have $e_g > 0$. Since any zero-dimensional scheme with degree 3 of $\mathbb{P}^3$ is contained in a plane, if $e_i \le 2$, then $W_i = \emptyset$ and $e_j = 0$ if $j > i$. We have $\sum_i e_i = \deg(W_0) \le 3d + 3$. Since A is not in linearly general position, we have $e_1 \ge 4$.

(a) Assume $g \ge d + 2$. In particular $e_{d+2} > 0$. Therefore $e_i \ge 3$ for $1 \le i \le d + 1$. We get $\deg(W_0) > 3d + 3$, a contradiction.

(b) Assume $g = d + 1$, i.e. assume $h^1(\mathcal{I}_{H_{d+1} \cap W_d}) > 0$. We get $e_{d+1} \ge 2$. Since $e_1 \ge 4$, we get $e_1 = 4$, $e_i = 3$ for $1 \le i \le d$, $e_{d+1} = 2$, $W_{d+1} = \emptyset$ and $\deg(W_0) = 3d + 3$. In particular we have $A \cap B = \emptyset$ and hence $O_2 \notin B$. Let $Q \subset \mathbb{P}^3$ be a quadric surface such that $\gamma := \deg(Q \cap W_0)$ is maximal. Since $h^0(\mathcal{O}_{\mathbb{P}^3}(2)) = 10$, we have $\gamma \ge 9$. Set $E_2 := \mathrm{Res}_Q(W_0)$. Since $\gamma \ge 9$, then $\deg(E_2) \le 3d - 6$. Let $M_3 \subset \mathbb{P}^3$ be a plane such that $h_3 := \deg(M_3 \cap E_2)$ is maximal. Set $E_3 := \mathrm{Res}_{M_3}(E_2)$. Fix an integer $i \ge 4$ and assume to have defined the plane M_j, the scheme E_j and the integer h_j for all $j \in \{2, \dots, i-1\}$. Let M_i be a plane such that $h_i := \deg(M_i \cap E_{i-1})$ is maximal. Set $E_i := \mathrm{Res}_{M_i}(E_{i-1})$. Since any zero-dimensional scheme of degree ≤ 3 of a projective space is contained in a plane, if $h_i \le 2$, then $h_{i+1} = 0$ and $E_i = \emptyset$. Since $\deg(E_2) \le 3d-6$, then $\sum_{i\ge 3} h_i \le 3d-6$.

(b1) Assume $h^1(\mathcal{I}_{W_0\cap Q}(d)) = 0$. The residual exact sequence of the inclusion $Q \subset \mathbb{P}^3$ gives $h^1(\mathcal{I}_{E_2}(d-2)) > 0$. A residual exact sequence like (2) with E_i instead of W_i and M_i instead of H_i gives the existence of an integer $i \ge 3$ such that $h^1(M_i, \mathcal{I}_{M_i\cap E_{i-1}}(d+1-i)) > 0$. We call c the first such an integer. We obviously have $h_c > 0$. Since $h_i \ge 3$ for all $i \in \{3,\dots,c-1\}$ and $\sum_{i\ge 3} h_i \le 3d-6$, we get $c \le d$. By [5, Lemma 34], either $h_c \ge 2(d+1-c)+2$ or there is a line $L \subset H_c$ such that $\deg(L\cap E_{c-1}) \ge d+3-c$. Assume for the moment $c \ge 4$ and the existence of a line $L \subset H_c$ such that $\deg(L\cap E_{c-1}) \ge d+3-c$. Since $h_c > 0$, we get $h_{c-1} \ge d+4-c$. Therefore $\sum_{i\ge 3} h_i \ge (c-3)(d+4-c)+d+3-c$. We obviously get this inequality even if $c = 3$ and L exists. Since $d \ge 9$, in this case we get $3d-6 \ge (c-3)(d+4-c)+d+3-c$ and hence $3 \le c \le 4$. Now assume $h_c \ge 2(d+1-c)+2$. Since $h_i \ge h_{i+1}$ for all $i \ge 3$, we get $3d-6 \ge 2(c-2)(d+2-c)$ and hence $c = 3$. By [5, Lemma 34] we have $h_3 \ge d$. Hence $e_1 \ge d$, a contradiction.

(b2) Assume $h^1(\mathcal{I}_{W_0\cap Q}(d)) > 0$.

(b2.1) Assume $h^1(\mathcal{I}_{E_2}(d-2)) > 0$. As in step (b1) we first see the existence of an integer $i \ge 3$ such that $h^1(M_i, \mathcal{I}_{M_i\cap E_{i-1}}(d+1-i)) > 0$ and then we get $e_1 \ge d-2$, a contradiction.

(b2.2) Assume $h^1(\mathcal{I}_{E_2}(d-2)) = 0$. Since A_1 is connected and it spans $\mathbb{P}^3$, [4, Lemma 5.1] gives that either $W_0 \subset Q$ or $O_2 \in B$, $O_2 \nsubseteq Q$ and $A_1 \cup (B \setminus \{O_2\}) \subset Q$. By Claim 3 we may assume $W_0 \subset Q$.

(b2.2.1) Assume that Q is smooth. Lemma 3 gives that either there is a line $L \subset Q$ such that $\deg(L\cap W_0) \ge d+2$ or there is a conic $T \subset Q$ with $\deg(T\cap W_0) \ge 2d+2$ or there is a degree 3 curve $F \subset Q$ of type $(2,1)$ or of type $(1,2)$ with $\deg(F\cap W_0) \ge 3d+2$. L does not exist, because its existence would imply $e_1 \ge d+3$. T does not exist, because its existence would imply $e_1 \ge 2d+2$. Therefore F exists, say with $F \in |\mathcal{O}_Q(2,1)|$. Since $\deg(\mathrm{Res}_F(W_0)) \le 1$, we have $h^1(Q, \mathcal{I}_{\mathrm{Res}_F(W_0)}(d-2,d-1)) = 0$. Since $O_2 \notin B$ (Claim 3), applying [4, Lemma 5.1] to the inclusion $F \subset Q$ we get $W_0 \subset F$ and in particular $A \subset F$. Since $\deg(\langle A'\rangle \cap A) = 3$, Bezout's theorem gives $\langle A'\rangle \subset Q$. Since F has type $(2,1)$ and $\deg(F\cap\langle A'\rangle\cap A) = 3$, we get that $\langle A'\rangle$ is a component of F. The non-existence of T or L gives $\deg(W_0\cap\langle A'\rangle\cap A) \ge d-1$ and hence $e_1 > 4$, a contradiction.

(b2.2.2) Assume that Q is singular and irreducible. We use Lemma 5 instead of Lemma 3. Cases (i) and (ii) are excluded, because $e_1 = 4$. Therefore there is a curve $F \subset Q$ with $\deg(F\cap Z) \ge 3d+2$ and either F is a smooth rational normal curve or $F = D \cup L$ with D a smooth conic, L a line, $\deg(D\cap W_0) = 2d+1$ and $\deg(L\cap W_0) = d+1$. Since $e_1 = 4$, there is no line $L \subset \mathbb{P}^3$ with $\deg(L\cap W_0) \ge 4$. Now assume that F is a rational normal curve. Since A is not in linearly general position, we get $(A\setminus\{O_2\})\cup B\subset F$ and $O_2\notin F$. Since $\mathcal{I}_F(2)$ is spanned by its global sections (i.e. the evaluation map $H^0(\mathcal{I}_F(2))\otimes\mathcal{O}_{\mathbb{P}^3}\to\mathcal{I}_F(2)$ is surjective), there is a quadric $Q'\supset F$, with $O_2\notin Q'$. Since $\mathrm{Res}_{Q'}(W_0)=\{O_2\}$, [4, Lemma 5.1] gives $O_2\in B$, contradicting Claim 3.

(b2.2.3) Assume that Q is not irreducible. Since $\gamma\geq 9$, there is a plane $M\subset Q$ with $\deg(M\cap W_0)\geq 5$. Hence $e_1\geq 5$, a contradiction.

(c) Assume $g\leq d$. By [5, Lemma 34] either there is a line $L\subset H_g$ such that $\deg(L\cap W_{g-1})\geq d+3-g$ or $e_g\geq 2(d+2-g)+2=2(d+3-g)$. In the latter case we get $3d+3\geq 2g(d+3-g)$ and hence $g=1$. In the former case if $g\geq 2$ we get $e_{g-1}\geq d+4-g$, because A spans $\mathbb{P}^3$. In the former case we also have $e_g\geq\deg(L\cap W_{g-1})\geq d+3-g$. Hence in the former case we get $3d+3\geq e_1+\cdots+e_g\geq g(d+4-g)-1$ and hence $1\leq g\leq 3$.

(c1) Assume $g=3$. We saw that there is a line $L\subset H_3$ such that $\deg(L\cap W_2)\geq d$ and that $e_2\geq d+1$. Therefore $d+1\leq e_1\leq d+2$ and $e_2=d+1$. Let N_1 be a plane containing L and with $f_1:=\deg(N_1\cap W_0)$ maximal among the planes containing L. Set $Z_0:=W_0$ and $Z_1:=\mathrm{Res}_{N_1}(Z_0)$. Let $N_2\subset\mathbb{P}^3$ be a plane such that $f_2:=\deg(Z_1\cap N_0)$ is maximal. Set $Z_2:=\mathrm{Res}_{N_2}(Z_1)$. Fix an integer $i\geq 3$ and assume to have defined f_j,N_j,Z_j for all $j<i$. Let $N_i\subset\mathbb{P}^3$ be any plane such that $f_i:=\deg(N_i\cap Z_{i-1})$ is maximal. Set $Z_i:=\mathrm{Res}_{N_i}(Z_{i-1})$. The residual exact sequences like (2) with N_i instead of H_i and Z_i instead of W_i give the existence of an integer $i>0$ such that $h^1(N_i,\mathcal{I}_{N_i\cap Z_{i-1}}(d+1-i))>0$. Let g' be the minimal such an integer. Since $h^1(\mathcal{O}_{\mathbb{P}^3}(t))=0$ for all integers t, we have $f_{g'}>0$. Since A spans $\mathbb{P}^3$, we have $f_1\geq 1+\deg(L\cap W_2)\geq d+1$. We have $f_i\geq f_{i+1}$ for all $i\geq 2$ and $\sum_{i\geq 2}f_i\leq 3d+3-f_1\leq 2d+2$. Since $f_i\geq 3$ if $f_{i+1}>0$, we get $3(g'-2)+1\leq 2d+2$ and hence (since $d\geq 8$) $g'\leq d$. Hence either $f_{g'}\geq 2(d+1-g')+2$ or there is a line $R\subset N_{g'}$ with $\deg(R\cap Z_{g'-1})\geq d+3-g'$ ([5, Lemma 34]). In the former case (since $f_2\geq\cdots\geq f_{g'-1}\geq f_{g'}$) we get $2(g'-1)(d+2-g')\leq 2d+2$ and hence $1\leq g'\leq 2$. In the latter case if $g'\geq 2$ we have $f_{g'-1}\geq d+4-g'$; hence in the latter case we have $2d+2\geq(g'-1)(d+4-g')-1$ and hence $1\leq g'\leq 3$. Recall that since $g\geq 2$, we have $f_1\leq e_1<3d+2$ and hence $f_2>0$. Thus $f_1\geq 1+\deg(L\cap W_0)\geq d+2$. Since $d+2\geq e_1\geq f_1\geq d+2$, we have $e_1=f_1=d+2$. Hence $f_2+\cdots+f_{g'}\leq 2d$.

(c1.1) Assume $g'=3$. Thus $f_3\geq d$. We saw in (c1) the existence of a line $R\subset N_3$ such that $\deg(Z_2\cap R)\geq d$. Since $\deg(D\cap A)\leq 3$ for each line D and $B\cap Z_1\subset B\setminus B\cap L$, we get $R\cap(B\setminus B\cap L)\neq\emptyset$. Therefore $R\neq L$. We have $R\cap L=\emptyset$ because $e_1<2d-1$. Let $Q\subset\mathbb{P}^3$ be a smooth quadric surface containing $R\cup L$. We have $\delta:=\deg(W_0\cap Q)\geq 2d$ and hence $\deg(\mathrm{Res}_Q(W_0))\leq d+3$.

(c1.1.1) Assume for the moment $h^1(\mathcal{I}_{\mathrm{Res}_Q(W_0)}(d-2))=0$. By [4, Lemma 5.1] either $O_2\in B$ or $W_0\subset Q$. Claim 3 gives $W_0\subset Q$. Since $e_1\leq d+2$, there is no line $D\subset\mathbb{P}^3$ with $\deg(D\cap W_0)\geq d+2$ (use that A_1 spans $\mathbb{P}^3$) and no conic

$T \subset \mathbb{P}^3$ with $\deg(T \cap W_0) \geq 2d+2$. Since $\deg(Z_2 \cap R) \geq d$, $\deg(W_0 \cap L) \geq d$ and $R \cap L = \emptyset$, Lemma 3 gives the existence of $F \subset Q$ of type $(2,1)$ or $(1,2)$ with $\deg(F \cap W_0) \geq 3d+2$. We get $F = L \cup R \cup D$ with D a line. Since $L \cap R = \emptyset$, we have $D \cap L \neq \emptyset$ and hence $e_1 \geq 2d-1$, a contradiction.

(c1.1.2) Now assume $h^1(\mathcal{I}_{\mathrm{Res}_Q(W_0)}(d-2)) > 0$. Since $\deg(\mathrm{Res}_Q(W_0)) \leq d+3 \leq 2(d-2)+1$, there is a line $D \subset \mathbb{P}^3$ such that $\deg(D \cap \mathrm{Res}_Q(W_0)) \geq d$. Since $\deg(D \cap A) \leq 3$, we get $D \cap (B \setminus B \cap (L \cup R)) \neq \emptyset$. Therefore D, R, L are 3 distinct lines. Since $e_1 < 2d-1$, we have $D \cap R = D \cap L = \emptyset$. Let Q' be the only quadric containing $D \cup L \cup R$ (Q' is smooth). Since $\deg(\mathrm{Res}_{Q'}(W_0)) \leq 3d+2-d-d-d$, we have $h^1(\mathcal{I}_{\mathrm{Res}_{Q'}(W_0)}(d-2)) = 0$. Claim 3 and [4, Lemma 5.1] gives $W_0 \subset Q'$. Since $h^1(\mathcal{I}_{W_0}(d)) = h^1(Q', \mathcal{I}_{W_0}(d)) > 0$, $\deg(W_0) \leq \deg(W_0 \cap (L \cup D \cup R)) + 2$ and $e_1 \leq d+2$, Lemma 3 gives a contradiction.

(c1.2) Assume $g' = 2$. Since $f_2 \leq \deg(Z_1) \leq 2d+1$, either there is a line $R \subset N_2$ with $\deg(R \cap Z_1) \geq d+1$ or there is a conic T with $\deg(T \cap Z_1) \geq 2d$ and in particular $f_2 \geq 2d$. The latter case cannot occur, because $e_1 \leq d+2$ for $g = 3$. Hence R exists. We have $R \cap (B \setminus B \cap L) \neq \emptyset$ and hence $R \neq L$. If $R \cap L \neq \emptyset$, then $e_1 \geq 2d-1$, contradicting the inequality $e_1 \leq d+2$. Therefore $R \cap L = \emptyset$. We continue as in step (c1.1.1) and (c1.1.2).

(c1.3) Assume $g' = 1$.

(c1.3.1) Assume $\deg(L \cap W_0) \geq d+2$. Since A spans $\mathbb{P}^3$, we have $e_1 \geq f_1 \geq 1 + \deg(L \cap W_0) \geq d+3$, a contradiction.

(c1.3.2) Assume $\deg(L \cap W_0) \leq d+1$. Since $f_1 \leq e_1 \leq d+2 \leq 2d+1$, Lemma 1 gives the existence of a line $D \subset N_1$ with $\deg(D \cap W_0) \geq d+2$. Since $L \neq D$, $\deg(L \cap W_0) \geq d$ and $L \cup D \subset N_1$, we get $e_1 \geq f_1 \geq 2d+1$, a contradiction.

(c2) Assume $g = 2$. We saw that there is a line $L \subset H_3$ such that $\deg(L \cap W_2) \geq d+1$. Hence $e_1 \leq 2d+2$. Let N_1 be a plane containing L and with $f_1 := \deg(N_1 \cap W_0)$ maximal among the planes containing L. Since A spans $\mathbb{P}^3$, we have $f_1 \geq 1 + deg(L \cap W_2) \geq d+2$. Define N_i, f_i, Z_i, g' as in step (c1). Since $f_i \geq 3$ if $f_{i+1} > 0$, we get $g' \leq d$. Hence either $f_{g'} \geq 2(d+1-g')+2 = 2(d+2-g')$ or there is a line $R \subset N_{g'}$ with $\deg(R \cap Z_{g'-1}) \geq d+3-g'$. In the former case if $g' \geq 2$ we get $2(g'-1)(d+2-g') + d+2 \leq 3d+2$ and hence $1 \leq g' \leq 2$. In the latter case if $g' \geq 2$ we have $f_{g'-1} \geq d+4-g'$; in the latter case we have $3d+2 \geq g'(d+4-g') - 1$, because $f_1 \geq d+2$; thus $1 \leq g' \leq 3$.

(c2.1) Assume $g' = 3$. We saw the existence of a line $R \subset \mathbb{P}^3$ such that $\deg(R \cap Z_2) \geq d$. Since $f_3 > 0$, Z_1 spans $\mathbb{P}^3$. Hence $f_2 \geq \deg(R \cap Z_2) + 1 \geq d+1$. Since $f_1 \geq d+2$ and $\deg(W_0) \leq 3d+3$, we get $\deg(R \cap Z_2) = d$, $Z_2 \subset R$, $f_2 = d+1$ and $f_1 = d+2$. Since $f_1 < 2d$, we have $R \cap L = \emptyset$. Let Q be any smooth quadric containing $R \cup L$. Since $\mathcal{I}_{R \cup L}(2)$ is spanned by its global sections and $A_1 \nsubseteq R \cup L$, [4, Lemma 5.1] gives $h^1(\mathcal{I}_{\mathrm{Res}_{Q(W_0)}}(d-2)) > 0$.

Since $\deg(\mathrm{Res}_Q(W_0)) \le d+2 \le 2(d-2)+1$, there is a line D with $\deg(D \cap \mathrm{Res}_Q(W_0)) \ge d$. We have $D \ne R$ and $D \ne L$. Since $\mathcal{I}_{D\cup R\cup L}(3)$ is spanned, $A_1 \nsubseteq D \cup R \cup L$ (Claim 1) and A is curvilinear, there is $Y \in |\mathcal{I}_{D\cup R\cup L}(3)|$ with $A_1 \nsubseteq Y$. Since $\deg(\mathrm{Res}_Y(W_0)) \le 3d+3-(d+1)-d-d+1 \le d-2$, we have $h^1(\mathcal{I}_{(\mathrm{Res}_Y(W_0)}(d-3)) = 0$, contradicting [4, Lemma 5.1] and the assumption $A_1 \nsubseteq Y$.

(c2.2) Assume $g' = 2$. Since $f_2 \le \deg(Z_1) \le 2d$, either there is a line $R \subset N_2$ with $\deg(R \cap Z_1) \ge d+1$ or $f_2 = \deg(Z_1) = 2d$ and there is a conic $T \supset Z_1$. If R exists, then $R \cap (B \setminus B \cap L) \ne \emptyset$ and hence $R \ne L$.

(c2.2.1) Assume the existence of R and that $R \cap L = \emptyset$. We continue as in step (c2.1).

(c2.2.2) Assume the existence of R and that $R \cap L \ne \emptyset$ and hence $f_1 \ge (d+1)+(d+1)-1$. Since $A_1 \nsubseteq N_1$, [4, Lemma 5.1] gives $h^1(\mathcal{I}_{Z_1}(d-1)) > 0$. Since $\deg(Z_1) \le 3d+2-f_1 \le 2d-1$, [5, Lemma 34] gives the existence of a line $D \subset \mathbb{P}^3$ such that $\deg(D \cap Z_1) \ge d+1$. Using $|\mathcal{I}_{R\cup L\cup D}(3)|$ and [4, Lemma 5.1] as in step (c2.1) we get $A_1 \subset R \cup L \cup D$, contradicting Claim 1.

(c2.2.3) Assume $\deg(Z_1) = 2d$ and the existence of a reduced conic $T \supset Z_1$. The sheaf $\mathcal{I}_{T\cup L}(3)$ is spanned by its global sections. Fix $Y \in |\mathcal{I}_{T\cup L}(3)|$. Since $\deg(\mathrm{Res}_Y(W_0)) \le 4$, we have $h^1(\mathcal{I}_{\mathrm{Res}_Y(W_0)}(d-3)) = 0$. Hence $A_1 \subset Y$ ([4, Lemma 5.1]). Since $\mathcal{I}_{T\cup L}(3)$ is spanned and A is curvilinear, as in step (c2.1) we get $A_1 \subset T \cup L$, contradicting Claim 2.

(c2.3) Assume $g' = 1$.

(c2.3.1) Assume $\deg(L \cap W_0) \ge d+2$. Since A spans $\mathbb{P}^3$, we have $f_1 \ge 1 + \deg(L \cap W_0) \ge d+3$ and hence $\deg(Z_1) \le 2d$. Since $A_1 \nsubseteq N_1$, [4, Lemma 5.1] gives $h^1(\mathcal{I}_{Z_1}(d-1)) > 0$. Lemma 1 gives that either there is a line R with $\deg(R \cap Z_1) \ge d+1$ or $\deg(Z_1) = 2d$ and $Z_1 \subset T$ for some reduced conic. First assume the existence of R. Since $R \cap (B \setminus B \cap L) \ne \emptyset$, Remark 2 gives $A \cap R \ne \emptyset$, i.e. either $O_1 \in R$ or $O_2 \in R$. Hence $R \cap L \ne \emptyset$. Let M be the plane spanned by $R \cup L$. Since $A_1 \nsubseteq M$, [4, Lemma 5.1] gives $h^1(\mathcal{I}_{\mathrm{Res}_M(W_0)}(d-1)) > 0$. Since $\deg(M \cap W_0) \ge (d+2)+(d+1)-1$, there is a line D with $\deg(D \cap \mathrm{Res}_M(W_0)) \ge d+1$. Since $D \cap (B \setminus (B \cap (R \cup L)) \ne \emptyset$, then $D \ne R$ and $D \ne L$. Since $f_1 < 3d$, $\langle D \cup R \cup L\rangle = \mathbb{P}^3$. For all possible configurations of L, D, R we see that $\mathcal{I}_{D\cup R\cup L}(3)$ is spanned by its global sections. We conclude as in the last part of step (c2.1). Now assume $\deg(Z_1) = 2d$ and $Z_1 \subset T$ for some reduced conic T. Since $\mathcal{I}_{T\cup L}(3)$ is spanned and $O_2 \notin B$ (Claim 3), [4, Lemma 5.1] gives $W_0 \subset T \cup L$. Since $\deg(W_0 \cap L) \ge d+2$, Remark 2 gives $L = \langle A'\rangle$. Since $A_1 \subset T \cup L$, we have $L \nsubseteq \langle T\rangle$ and hence $\deg(L \cap T) \le 1$. Since $L = \langle A'\rangle$, we get $T \cap A_1 = \{O_1\}$ as schemes and hence $A_1 \nsubseteq T \cup L$.

(c2.3.2) Assume $\deg(L \cap W_0) \le d+1$. Since $f_1 \le e_1 \le 2d+2$, Lemma 1 gives that either there is a line $D \subset N_1$ with $\deg(D \cap W_0) \ge d+2$ or $f_1 = 2d+2$

and there is a reduced conic $T \supset W_0 \cap N_1$. If D exists, then $D \neq L$ and hence $f_1 \geq (d+1)+(d+2)-1$, i.e. $f_1 = 2d+2$ and $W_0 \cap N_1 \subset D \cup L$. Therefore in both cases we have $\deg(Z_1) \leq 2(d-1)+1$. Since $h^1(\mathcal{I}_{Z_1}(d-1)) > 0$ by [4, Lemma 5.1], there is a line $R \subset \mathbb{P}^3$ with $\deg(R \cap Z_1) \geq d+1$. We have 3 lines R, L, D' with either $D' = D$ or $D' \cup L = T$. In all cases we see that $\mathcal{I}_{D \cup R \cup L}(3)$ is spanned by its global sections and we conclude as in the last part of step (c2.1).

(c3) Assume $g = 1$. Since A_1 is connected and $A_1 \nsubseteq H_1$, [4, Lemma 5.1] gives $h^1(\mathcal{I}_{W_1}(d-1)) > 0$. Since $h^1(H_1, \mathcal{I}_{W_0 \cap H_1}(d)) > 0$, we have $e_1 \geq d+2$ and hence $\deg(W_1) \leq 2d+1$. By Lemma 1 either there is a line $R \subset \mathbb{P}^3$ such that $\deg(R \cap W_1) \geq d+1$ or there is a plane conic T with $\deg(T \cap W_1) \geq 2d$. The latter case does not arise, because it would imply $e_1 \geq 2d$ and hence $\deg(W_1) \leq d+3$. Therefore there is a line $R \subset \mathbb{P}^3$ such that $\deg(R \cap W_1) \geq d+1$. Therefore $e_1 \leq 2d+2$. Since $g = 1$, Lemma 1 gives that either there is line $L \subset H_1$ such that $\deg(L \cap W_0) \geq d+2$ or there is a plane conic $T \subset H_1$ such that $\deg(W_0 \cap T) \geq 2d+2$.

(c3.1) Assume the existence of a plane conic $T \subset H_1$ such that $\deg(W_0 \cap T) \geq 2d+2$. We get $e_1 = 2d+2$, $W_0 \cap H_1 \subset T$ and $W_1 \subset R$. Remark 2 gives $\deg(T \cap B) \leq 2d$ and hence $O_1 \in T$. First assume $R \cap T = \emptyset$. The linear system $|\mathcal{I}_{T \cup R}(2)|$ is formed by the pencil of the reducible quadrics $H_1 \cup M$ with M a plane containing R. By [4, Lemma 5.1] we get $A_1 \subset H_1 \cup M$. Since $R \cap T = \emptyset$ and $O_1 \in T$, we may find $M \supset R$ with $M \cap A_1 = \emptyset$. Thus $A_1 \subset H_1$, a contradiction. Now assume $R \cap T \neq \emptyset$. In this case $R \cup T$ is the scheme-theoretic base locus of the linear system $|\mathcal{I}_{R \cup T}(2)|$. Fix any $Y \in |\mathcal{I}_{R \cup T}(2)|$. [4, Lemma 5.1] gives $A_1 \cup (B \setminus \{O_2\}) \subset Y$ and either $O_2 \in Y$ or $O_2 \in B$. Claim 3 gives $O_2 \in Y$. Therefore $W_0 \subset Y$. Since $\mathcal{I}_{R \cup T}(2)$ is spanned, we get $W_0 \subset R \cup T$. Since $\deg(A \cap \langle \{O_1, O_2\} \rangle) = 3$, we get $\langle \{O_1, O_2\} \rangle \subset R \cup T$. Claim 1 implies that T is a smooth conic. Hence $R = \langle \{O_1, O_2\} \rangle = \langle A' \rangle$. Obviously $O_1 \in T$. Since $R \nsubseteq N_1 = \langle T \rangle$, we get $\deg(T \cap A) = 1$. Hence $\deg(A_1 \cap (R \cup T)) \leq 3$, a contradiction.

(c3.2) Assume the existence of a line $L \subset H_1$ such that $\deg(L \cap W_0) \geq d+2$. Remark 2 gives $\deg(L \cap A) \geq 2$. Hence $L = \langle \{O_1, O_2\} \rangle$. Since $\deg(R \cap A) \leq 3$, we have $R \cap (B \setminus B \cap L) \neq \emptyset$. Therefore $R \neq L$. Remark 2 gives $R \cap A \neq \emptyset$. Hence either $O_1 \in R$ or $O_2 \in R$. In both cases we have $R \cap L \neq \emptyset$. Let M be the plane spanned by $L \cup R$. Since $A_1 \nsubseteq M$, [4, Lemma 5.1] gives $h^1(\mathcal{I}_{\mathrm{Res}_M(W_0)}(d-1)) > 0$. Since $\deg(\mathrm{Res}_M(W_0)) \leq 3d+3-(d+2)-(d+1)+1$, we get $\mathrm{Res}_M(W_0) \subset D$ for some line D. Since A_1 is curvilinear and $\mathcal{I}_{D \cup L \cup R}(3)$ is spanned, as in step (c2.1) we get get $A_1 \subset D \cup L \cup R$, a contradiction. QED

Proposition 5. *Assume $d \geq 9$, $m \geq 3$. Let $A_1 \subset \mathbb{P}^m$, $m \geq 3$, be a connected and curvilinear zero-dimensional scheme such that $\deg(A_1) = 4$ and $\dim(\langle A_1 \rangle) = 3$. Set $\{O_1\} := (A_1)_{\mathrm{red}}$. Let A' (resp. A'') be the degree 2 (resp. 3)* *subscheme of A. Fix* $O_2 \in \langle A'' \rangle \setminus \langle A' \rangle$*, set* $A := A_1 \cup \{O_2\}$ *and take any* $P \in \langle A \rangle$ *such that* $P \notin \langle E \rangle$ *for any* $E \subsetneq A$*.*

(i) We have $3d-3 \le r_{m,d}(P) \le 3d-2$*.*

(ii) Fix any $B \subset \mathbb{P}^m$ *such that* $3d-3 \le \sharp(B) \le 3d-2$*,* $P \in \langle \nu_d(B) \rangle$ *and* $P \notin \langle \nu_d(E) \rangle$ *for any* $E \subsetneq B$*. Then there are a smooth conic* $C \subset \langle A'' \rangle$ *and a line* $L \subset \langle A_1 \rangle$ *such that* $L \cap \langle A'' \rangle = \{O_1\}$*,* $A'' \cup \{O_2\} \subset C$*,* $\sharp(B \cap L) = d$ *and* $A \cup B \subset C \cup L$*.*

(iii) We have $r_{m,d}(U) = 3d-3$ *for some* $U \in \mathbb{P}^r$ *whose cactus rank is evinced by A.*

Proof. By concision ([10, Exercise 3.2.2.2]) we may assume $m = 3$. Set $H := \langle A'' \rangle$. Since $O_2 \notin \langle A' \rangle$, we have $\deg(D \cap A'') \le 2$ for all lines $D \subset H$. Therefore $h^0(H, \mathcal{I}_{A'' \cup \{O_2\}}(2)) = 2$ and a general conic C of H containing $A'' \cup \{O_2\}$ is smooth. By Lemma 8 there is a line $L \subset \mathbb{P}^3$ such that $A_1 \subset C \cup L$, $O_1 \in L$ and $L \nsubseteq H$. Therefore $O_2 \notin L$. Since $O_2 \in C$, we get $A \subset C \cup L$. Hence $P \in \langle \nu_d(C \cup L) \rangle$. Therefore there are $P_1 \in \langle \nu_d(C) \rangle$ and $P_2 \in \langle \nu_d(L) \rangle$ such that $P \in \langle \{P_1, P_2\} \rangle$ (we do not claim that $P_1 \ne P_2$, but if $P_1 = P_2$, then $P = P_2$ and hence $r_{m,d}(P) \le d$ by Sylvester's theorem (Remark 1) and we will later get a contradiction with the weaker assumption $r_{m,d}(P) \le 3d-4$). We have $P_1 \in \langle \nu_d(\{O_2\} \cup A'') \rangle$. Since $P \notin \langle \nu_d(A_1) \rangle$, we have $P_1 \notin \langle \nu_d(A'') \rangle$. Assume for the moment $P_1 \in \langle \nu_d(\{O_1\} \cup A') \rangle$. Since P_2 has at most rank d with respect to the rational normal curve $\nu_d(L)$ and every point of $\langle \nu_d(\langle A' \rangle) \rangle$ has at most rank d with respect to the rational normal curve $\nu_d(\langle A' \rangle)$, we would get $r_{m,d}(P) \le 2d+1$; we will later find a contradiction with the weaker assumption $r_{m,d}(P) \le 3d-4$. Now assume $P_1 \notin \langle \nu_d(\{O_1\} \cup A') \rangle$. Since $P \notin \langle \nu_d(A_1) \rangle$, we have $P_1 \notin \langle \nu_d(A'') \rangle$. Therefore P_1 has border rank 4 with respect to the degree $2d$ rational normal curve $\nu_d(C)$. Sylvester's theorem ([8], Remark 1) gives that P_1 has rank $2d-2$ with respect to $\nu_d(C)$. Every point of $\langle \nu_d(L) \rangle$ has rank $\le d$ with respect to the rational normal curve $\nu_d(L)$. Hence $r_{m,d}(P) \le 3d-2$. Take $B \in \mathcal{S}(P)$ and set $W_0 := A \cup B$. We saw that $\deg(W_0) \le 3d+3$. To prove parts (i) and (ii) of Proposition 5 it is sufficient to prove that $\deg(W_0) \ge 3d+2$ and that there are curves C, L as in part (ii). See step (d) for the proof of part (iii).

Claim 1: If $B \subset H \cup L$, then $\sharp(B \cap (L \setminus \{O_1\})) \ge d$.

Proof of Claim 1: Assume $\sharp(B \cap (L \setminus \{O_1\})) \le d-1$. Since $\mathrm{Res}_H(A) = \{O_1\}$, we get $\deg(\mathrm{Res}_H(W_0)) \le d$ and hence $h^1(\mathcal{I}_{\mathrm{Res}_H(W_0)}(d-1)) = 0$. By [4, Lemma 5.1] we get $A_1 \subset H$, a contradiction.

Claim 2: Assume $A_1 \subset C \cup L$ with L a line and C either a reduced conic or the disjoint union of two distinct lines. Then C is a smooth conic, $A'' \subset C$, $H = \langle C \rangle$, $L \nsubseteq H$, and $\{O_1\} = L \cap C = L \cap H$.

Proof of Claim 2: By Claim 1 of the proof of Proposition 4 A_1 is not contained in the union of 3 lines. Hence C must be a smooth conic. Since

A_1 is connected and $\langle A_1\rangle = \mathbb{P}^3$, we have $L \nsubseteq \langle C\rangle$ and the scheme-theoretic intersection $L \cap \langle C\rangle$ is a single point, o. Since A_1 is connected and $\langle A\rangle = \mathbb{P}^3$, we get $o = O_1$. First assume $L = \langle A'\rangle$. Since $\langle C\rangle \nsupseteq L$, we get $\deg(A \cap \langle C\rangle) = 1$ and hence $\mathrm{Res}_{\langle C\rangle}(A) = A''$. Since $\langle A''\rangle$ is a plane, we have $A'' \nsubseteq L$ and hence $A \nsubseteq \langle C\rangle \cup L$. Therefore $A \nsubseteq C \cup L$, a contradiction. Hence $L \neq \langle A''\rangle$. Since $A_1 \subset C \cup L$, we get $A'' \subset C$ and hence $\langle C\rangle = H$.

Claim 3: Assume $W_0 \subset C \cup L$ with C a reduced conic and L a line. Then $\sharp(B \cap C) \geq 2d - 3$, $O_1 \notin B$, $\sharp(B \cap L) = d$ and $\deg(A \cap L) = 1$.

Proof of Claim 3: By Claim 2 C is a smooth conic, $H = \langle C\rangle$ and $A'' \subset C$. Since $A'' \subset C$, we have $\deg(L \cap A_1) = 1$. Since $O_1 \in L$, we have $O_2 \notin L$. Since $A_1 \nsubseteq H$ and $O_2 \notin L$, we have $\deg(L \cap A) = 1$ and hence $\deg(W_0 \cap L) = d + 1$. Assume $\sharp(B \cap C) \leq 2d - 4$. Take a smooth quadric Q containing $C \cup L$. Since $\deg(W_0 \cap C) = 2d$, $\deg(W_0 \cap L) = d+1$ and $\deg(W_0) \leq \deg(W_0 \cap C) + \deg(W_0 \cap L) = 3d + 1$, Lemma 3 gives $h^1(\mathcal{I}_{W_0}(d)) = 0$, a contradiction.

Our goal is to prove the existence of a reduced conic C and a line L such that $W_0 \subset C \cup L$. If we prove the existence of C and L, then we get part (i) by Claim 3.

We repeat the proof of Proposition 4, except that now we also have to handle the smooth conics containing $A_1 \cup \{O_2\}$. Since A is not in linearly general position in $\mathbb{P}^3$, then $e_1 \geq 4$. Steps (a), (b), (c1) works verbatim (they only use the integers e_i and not the position of O_2). The first difference arises in step (c2.3.1). Instead of having $L = \langle A'\rangle$ we have that either $L = \langle A'\rangle$ or $L = \langle \{O_1, O_2\}\rangle$. However in this part of the proof we have $\deg(W_0) = 3d+3$ (i.e. $A \cap B = \emptyset$ and $\sharp(B) = 3d - 2$) and $W_0 \subset T \cup L$ with T a reduced conic. So in this case instead of having a contradiction we just jump to step (d). In step (c2.3.1) we either get a contradiction or get $\deg(W_0) = 3d + 3$ and $W_0 \subset T \cup L$ with T a reduced conic. Claim 3 gives parts (i) and (ii). We rewrite with minimal modifications steps (c3.1) and (c3.2).

(c3.1) Assume the existence of a plane conic $T \subset H_1$ such that $\deg(W_0 \cap T) \geq 2d+2$. We get $e_1 = 2d+2$, $W_0 \cap H_1 \subset T$ and $W_1 \subset R$. Therefore $e_2 = d+1$ and $\deg(W_0) = 3d + 3$. Since $\deg(B \cap T) \leq 2d$ by Remark 2, we have $O_1 \in T$.

First assume $R \cap T = \emptyset$ and in particular $O_1 \notin R$. The linear system $|\mathcal{I}_{T\cup R}(2)|$ is formed by the pencil of all reducible quadrics $H_1 \cup M$ with M a plane containing R. By [4, Lemma 5.1] we get $A_1 \subset H_1 \cup M$. Since $O_1 \notin M$ for a general $M \supset R$, we get $A_1 \subset H_1$, a contradiction. Now assume $R \cap T \neq \emptyset$. In this case $R \cup T$ is the scheme-theoretic base locus of the linear system $|\mathcal{I}_{R\cup T}(2)|$. Fix any $Y \in |\mathcal{I}_{R\cup T}(2)|$. [4, Lemma 5.1] gives $A_1 \cup (B \setminus \{O_2\}) \subset Y$ and either $O_2 \in Y$ or $O_2 \in B$. The case $x = 2$ of Claim 3 of the proof of Proposition 4 gives $W_0 \subset Y$. Since $R \cup T$ is the scheme-theoretic base locus of the linear system $|\mathcal{I}_{R\cup T}(2)|$, we get $W_0 \subset T \cup R$. Apply Remark 2 and Claim 2 to get (i) and (ii).

(c3.2) Assume the existence of a line $L \subset H_1$ such that $\deg(L \cap W_0) \geq d+2$. Remark 2 gives $\deg(L \cap A) \geq 2$. Hence either $L = \langle \{O_1, O_2\} \rangle$ or $L = \langle A' \rangle$. Since $\deg(R \cap A) \leq 3$, we have $R \cap (B \setminus B \cap L) \neq \emptyset$. Therefore $R \neq L$. First assume $R \cap L \neq \emptyset$. Let M be the plane spanned by $L \cup R$. Since $A_1 \nsubseteq M$, [4, Lemma 5.1] gives $h^1(\mathcal{I}_{\mathrm{Res}_M(W_0)}(d-1)) > 0$. Since $\deg(\mathrm{Res}_M(W_0)) \leq 3d+3-(d+2)-(d+1)+1$, [5, Lemma 34] gives $\deg(\mathrm{Res}_M(W_0)) = d+1$ and $\mathrm{Res}_M(W_0) \subset D$ for some line D. Since A_1 is curvilinear, we also get $A_1 \subset D \cup L \cup R$, contradicting Claim 1. Now assume $R \cap L = \emptyset$. Remark 2 gives $R \cap A \neq \emptyset$. Therefore $L = \langle A' \rangle$, $O_2 \in R$ and $O_1 \notin R$. Fix any $o \in B \setminus B \cap (L \cup R)$ and take any $Q \in |\mathcal{I}_{L \cup R \cup \{o\}}(2)|$. Since $\deg(W_0 \cap (L \cup R)) \geq 2d+3$, we have $\deg(\mathrm{Res}_Q(W_0)) \leq d-1$ (this is obviously true even if $B \subset L \cup R$) and hence $h^1(\mathcal{I}_{\mathrm{Res}_Q(W_0)}(d-1)) = 0$. By [4, Lemma 5.1] we get $A_1 \subset Q$. Since $R \cap L = \emptyset$, the base locus of $|\mathcal{I}_{L \cup R \cup \{o\}}(2)|$ is a line D. Hence $A_1 \subset L \cup D \cup R$ (we even have $A_1 \subset L \cup D$, because $O_1 \notin R$), a contradiction.

(d) We saw that for each $B \in \mathcal{S}(P)$ we have $3d-3 \leq \sharp(B) \leq 3d-2$, and that there are C, L such that $W_0 \subset C \cup L$ and $\sharp(B \cap (L \setminus \{O_1\})) = d$ (and hence $\sharp(B \cap C) = \sharp(B) - d$). Take P with $r_{m,d}(P) = 3d-2$ (if any). Therefore $\sharp(B \cap C) = 2d-2$.

Claim 4: $A \cap B = \emptyset$.

Proof of Claim 4: Assume $A \cap B \neq \emptyset$. Remark 2 gives $O_1 \notin B$. Assume $O_2 \in B$ and set $B'' := B \setminus \{O_2\}$. Since $\deg(A \cap B) = 1$, we have $\dim \langle \nu_d(A) \rangle \cap \langle \nu_d(B) \rangle) \geq 1$ and hence $\langle \nu_d(A_1) \rangle \cap \langle \nu_d(B'') \rangle \neq \emptyset$. Take $V \in \langle \nu_d(A_1) \rangle \cap \langle \nu_d(B'') \rangle$ and let $E \subseteq B''$ be the minimal subset of B'' with $V \in \langle \nu_d(E) \rangle$. Since $B \in \mathcal{S}(P)$, then $E \in \mathcal{S}(V)$. Since $P \in \langle \nu_d(E \cup \{O_2\}) \rangle$, we get $\sharp(E) \geq 3d-3$, i.e. $E = B''$. By [5] V has not border rank ≤ 3. Hence A_1 evinces the border rank of V. Therefore $\sharp(B'') = 3d-2$ ([4, Proposition 5.19]), a contradiction.

Since $B \cap A = \emptyset$ (Claim 4), $r_{m,d}(P) = 3d-2$ if and only if $B \cap C$ evinces the rank of a point $P_1 \in \langle \nu_d(C) \rangle$ with border rank 4. Since $h^0(\mathcal{O}_{C \cup L}(d)) = 3d+2$ and $\deg(W_0) = 3d+3$, we see that $\langle \nu_d(B) \rangle \cap \langle \nu_d(L) \rangle$ is a line. Fix $o \in B \cap C$. The set $\langle \nu_d(A) \rangle \cap \langle \nu_d(B \setminus \{o\}) \rangle$ is a point, P'', and $r_{m,d}(P'') \leq 3d-3$. Set $B' := B \setminus \{o\}$.

Claim 5: A evinces the cactus rank of P'' and B' evinces the rank of P''.

Proof of Claim 5: P'' has cactus rank at most 5. Since $d \geq 8$, P'' has cactus rank 5 if and only if $P'' \notin \langle \nu_d(E) \rangle$ for any $E \subsetneq A$. Assume $P'' \in \langle \nu_d(A'' \cup \{O_2\}) \rangle$. We would get $r_{m,d}(P'') \leq (2d-1)+1$ by [5, Theorem 37] and hence $r_{m,d}(P) \leq 2d+1$, contradicting, for instance, Claim 3 and steps (b), (c) of the proof of Proposition 4. Now assume $P'' \in \langle \nu_d(A_1) \rangle \setminus \langle \nu_d(A'') \rangle$. Since $r_{m,d}(P'') = 3d-2$ ([4, Proposition 5.19]), we get $\sharp(B') \geq 3d-2$ and hence $\sharp(B) \geq 3d-1$, a contradiction. Let $B_1 \subseteq B'$ be a minimal subset of B' such that $P' \in \langle \nu_d(B_1) \rangle$. Since $B \in \mathcal{S}(P)$, it is easy to check that $B_1 \in \mathcal{S}(P')$. If $B_1 \subsetneq B'$, then $\sharp(B_1) \ge 3d-3$ (by what we proved in steps (a), (b) and (c)) and hence $\sharp(B) \ge 3d-1$, a contradiction.

Claim 5 shows that $r_{m,d}(P) = 3d-3$ for some P whose cactus rank is evinced by A. QED

Proposition 6. *Fix integers $m \ge 1$, $b \ge 2$, $d \ge 2b+1$ and let $P \in \mathbb{P}^r$ be a point with border rank b whose border rank and cactus rank is evinced by a scheme A with $b-1$ connected components, one of degree 2 and the other ones of degree 1. Write $A = A_1 \sqcup \{O_2, \cdots O_{b-1}\}$ with $\deg(A_1) = 2$ and set $L := \langle A_1 \rangle$. Let c be the number of indices $i \in \{2, \dots, b-1\}$ such that $O_i \in L$. Assume $2b \le 4+3c$. We have $r_{m,d}(P) = d+b-2-2c$ and every $B \in \mathcal{S}(P)$ has a decomposition $B_1 \sqcup B_2$ with $\sharp(B_2) = b-c-2$, $B_2 = \{O_2, \dots, O_{b-1}\} \setminus \{O_2, \dots, O_{b-1}\} \cap L$, $\sharp(B_1) = d-c$, $B_1 \subset L \setminus A_{\mathrm{red}} \cap L$ and $A \cap B_1 = \emptyset$.*

Proof. Set $W_0 := A \cup B$. Since A is not reduced, we have $A \ne B$ and hence $h^1(\mathcal{I}_{A\cup B}(d)) > 0$ ([3, Lemma 1]). The case $m = 1$ (and hence $c = b-2$) of the assertion on $r_{m,d}(P)$ is Sylvester's theorem (Remark 1, [8], [11, Theorem 4.1], [5, Theorem 23]). For $m = 1$ the assertion on $\mathcal{S}(P)$ says only that $A_{\mathrm{red}} \cap B = \emptyset$, which is true by the last part of Remark 1.

Now assume $m > 1$. Take any $B \in \mathcal{S}(P)$. Set $E := A \cap L$ and $F := \{O_2, \dots, O_{b-1}\} \setminus \{O_2, \dots, O_{b-1}\} \cap L$. We have $\sharp(F) = b-2-c$. Since A evinces the cactus rank of P, there are $P_1 \in \langle \nu_d(E) \rangle$ and $P_2 \in \langle \nu_d(F) \rangle$ such that $P \in \langle \{P_1, P_2\} \rangle$, E evinces the cactus rank of P_1 and F evinces the cactus rank of P_2. Sylvester's theorem gives $r_{1,d}(P_1) = d-c$ (Remark 1, [8], [11, Theorem 4.1], [5, Theorem 23]). Since F is reduced, we get $r_{m,d}(P) \le d+b-2-2c$ and hence $\sharp(B) \le d+b-2-2c$. Let $M \subset \mathbb{P}^m$ be a general hyperplane containing L (hence $M = L$ if $m = 1$). Since every non-reduced connected component of A is contained in L, $W_0 \setminus W_0 \cap L$ is a finite set and M is general, we have $M \cap W_0 = W_0 \cap L$ and hence $\sharp(W_0 \setminus W_0 \cap L) \le d+2b-4-3c$. Since $2b \le 4+3c$, we have $\sharp(W_0 \setminus W_0 \cap L) \le d$ and hence $h^1(\mathcal{I}_{\mathrm{Res}_M(W_0)}(d-1)) = 0$. By [4, Lemma 5.1] we get $B \setminus B \cap L = F$. Hence $\deg(W_0) \le d+2+b-2c$. Since B evinces the rank of P, we get $r_{m,d}(P) = b-2-c+r_{m,d}(P_1)$. By concision ([10, Exercise 3.2.2.2]) we have $r_{m,d}(P_1) = r_{1,d}(P_1)$ and every element of $\mathcal{S}(P_1)$ is contained in L. QED

Proposition 7. *Fix integers $d \ge 5$, $x \ge 0$, $y \ge 0$, and assume $x < \lceil (d-2)/2 \rceil$ and $d \ge 2+2y$. Let $A_1 \subset \mathbb{P}^m$, $m \ge 2$, be a curvilinear scheme of degree 3 such that $\dim(\langle A_1 \rangle) = 2$. If $m = 2$, then assume $y = 0$. Let $A' \subset A_1$ be the degree two subscheme of A_1. Set $\{O_1\} := (A_1)_{\mathrm{red}}$ and $L := \langle A' \rangle$. Fix finite sets $E \subset L \setminus \{O_1\}$ and $F \subset \mathbb{P}^m \setminus \langle A_1 \rangle$ with $\sharp(E) = x$ and $\sharp(F) = y$. Set*

$A := A_1 \cup E \cup F$. Fix $P \in \langle \nu_d(A)\rangle$ such that $P \notin \langle \nu_d(U)\rangle$ for any $U \subsetneq A$. Then $r_{m,d}(P) = 2d - 1 - x + y$ and every $B \in \mathcal{S}(P)$ contains F.

Proof. First assume $y = 0$. By concision we may assume $m = 2$. Since $P \notin \langle \nu_d(U)\rangle$ for any $U \subsetneq A$, the set $\langle \nu_d(E) \cup \{P\}\rangle \cap \langle \nu_d(A_1)\rangle$ is a single point, Q_1, and $Q_1 \notin \langle \nu_d(U)\rangle$ for any $U \subsetneq A_1$. Therefore A_1 achieves the cactus rank and the border rank of Q_1. Therefore $r_{m,d}(Q_1) = 2d - 1$ ([5, Theorem 37]). Hence $2d+1-x \le r_{m,d}(P)$. Therefore it is sufficient to prove that $r_{m,d}(P) \le 2d-1-x$.

Let $R \subset \mathbb{P}^2$ be any line such that $O_1 \in R$ and $L \ne R$. Since $\mathrm{Res}_L(A) = \{O_1\} \subset R$, we have $A \subset L \cup R$ and hence $P \in \langle \nu_d(L \cup R)\rangle$. Fix $P_1 \in \langle \nu_d(L)\rangle$ and $P_2 \in \langle \nu_d(R)\rangle$ such that $P \in \langle \{P_1, P_2\}\rangle$. Let A' be the degree two subscheme of A_1.

Claim 1: We have $P_1 \in \langle \nu_d(A' \cup E)\rangle$.

Proof of Claim 1: Assume $P_1 \notin \langle \nu_d(A' \cup E)\rangle$ and call $J \subset L$ any zero-dimensional scheme evincing the border rank, b, of P_1 with respect to the rational normal curve $\nu_d(L)$. Let $K \subset R$ be any zero-dimensional scheme evincing the border rank, b', of P_2 with respect to $\nu_d(R)$. We have $b \le \lfloor (d+2)/2 \rfloor$ and $b' \le \lfloor (d+2)/2 \rfloor$ and hence $\deg(J \cup K) \le b + b' \le 2\lfloor (d+2)/2 \rfloor$ (Remark 1). We have $P \in \langle \nu_d(J \cup K)\rangle$. If $A \nsubseteq J \cup K$, then $h^1(\mathcal{I}_{J\cup K\cup A}(d)) > 0$ ([3, Lemma 1]). We have $\deg(J \cup K \cup A) \le 2\lfloor (d+2)/2 \rfloor + 3 + x \le 2d + 1$. By [5, Lemma 34] we get the existence of a line $T \subset \mathbb{P}^2$ with $\deg(T \cap (J \cup B \cup A)) \ge d + 2$. Since $A \cup J \cup K \subset L \cup R$, then either $T = L$ or $T = R$. Assume $T = R$. Since $T \cap A = \{O_1\}$ as schemes, we get $b' \ge d + 1$, a contradiction. If $T = L$ we get $2 + x + b \ge d + 2$, i.e. $x \ge \lceil (d-2)/2 \rceil$, a contradiction. Now assume $A \subset J \cup K$. Since $R \cap A = \{O_1\}$ as schemes, we get $A' \cup E \subseteq J$ and hence $P_1 \in \langle \nu_d(A' \cup E)\rangle$.

Claim 2: We have $P_1 \notin \langle \nu_d(U)\rangle$ for any $U \subsetneq E$.

Proof of Claim 2: Assume $P_1 \in \langle \nu_d(U)\rangle$ for some $U \subseteq E$. We get $P \in \langle \nu_d(U \cup R)\rangle$. Let $K \subset R$ be a scheme evincing the cactus rank, b', of P_2 with respect to $\nu_d(R)$. Since $A \nsubseteq U \cup K$, [3, Lemma 1] gives $h^1(\mathcal{I}_{U\cup K}(d)) > 0$ and hence $\deg(U \cup K) \ge d+2$. Since $\deg(U \cup K) \le x+1+b' \le x+1+\lfloor (d+2)/2 \rfloor$, we get a contradiction.

By Claims 1 and 2 P_1 has border rank $2+x$ with respect to $\nu_d(L)$. Sylvester's theorem (Remark 1), gives the existence of $B_1 \subset L$ such that $\sharp(B_1) = d-x$ and $P_1 \in \langle \nu_d(B_1)\rangle$. By Sylvester's theorem (Remark 1) applied to $\nu_d(R)$ and P_2 it would be sufficient to prove that P_2 has border rank > 2 with respect to $\nu_d(R)$ for some choice of P_1, P_2. Instead of P_2 we may take any $P_2' \in \langle \{P_1, \nu_d(O_1)\}\rangle \setminus \{\nu_d(O_1)\}$ and then find a new point P_1. We may find P_2' with border rank > 2 unless the border rank of P_2 is evinced by the degree two scheme $\mathbf{v} \subset R$ with O_1 as its support. In this case we would have $P \in \langle \nu_d(\mathbf{v} \cup A' \cup E)\rangle$. Since $\mathbf{v} \cup A'$ is contained in the scheme $2O_1 \subset \mathbb{P}^2$ with $(\mathcal{I}_{O_1})^2$ as its ideal sheaf and since each point of $\langle \nu_d(2O_1)\rangle$ has rank d ([5, Theorem 32]), we would get $r_{m,d}(P) \le d+x < 2d-1-x$, a contradiction.

Now assume $y>0$ and hence $m>2$. By the case $y=0$ we know that $r_{m,d}(P) \le 2d-1-x+y$. Fix any $B \in \mathcal{S}(P)$. We have $\sharp(B) \le 2d-1-x+y$ and hence $W := A \cup B$ has degree $\le 2d+2+2y$. By the case $y=0$ it is sufficient to prove that $F \subset B$. We have $\deg(W) \le 2d+2+2y \le 3d$ and there is no plane containing $A_1 \cup F$. Since $h^1(\mathcal{I}_W(d))>0$ by [3, Lemma 1], Lemma 1 and Remark 3 give that either there is a line D with $\deg(D \cap W_0) \ge d+2$ or there is a reduced conic T with $\deg(T \cap W_0) \ge 2d+2$. Assume the existence of T. Let $H \subset \mathbb{P}^m$ be any hyperplane containing T. Since $\deg(\mathrm{Res}_M(W)) \le d-2$, we have $h^1(\mathcal{I}_{\mathrm{Res}_H(W)}(d-1))=0$. By [4, Lemma 5.1] we get $A_1 \subset M$ and $B\setminus B\cap M = (E\cup F)\setminus(E\cup F)\cap M$. Since M is an arbitrary hyperplane containing T, we get $\langle A_1\rangle = \langle T\rangle$ and $F \subset B$. We also get that $B\setminus F$ evinces the rank of a point with $A_1 \cup E$ evincing its cactus rank and hence $\sharp(B\setminus F) = 2d-1-x$.

Now assume the existence of the line D such that $\deg(D\cap W) \ge d+2$. Since $\sharp(F) = y \le d$, we have $\deg(A_1 \cup E) \ge 2$. Therefore $L \subset \langle A_1\rangle$ and, as we saw using T, we get $F \subset B$. QED

Proposition 8. *Assume $d \ge 9$. Let $A_1, A_2 \subset \mathbb{P}^m$, $m \ge 3$, be disjoint connected curvilinear schemes such that $\deg(A_1)=3$ and $\deg(A_2)=2$. Set $A := A_1 \cup A_2$. Assume $\dim(\langle A\rangle)=3$ and that A is in linearly general position in $\langle A\rangle$, i.e. assume $(A_2)_{\mathrm{red}} \notin \langle A_1\rangle$ and that the line $\langle A_2\rangle$ does not intersect the line spanned by the degree two subscheme of A_1. Fix $P \in \langle \nu_d(A)\rangle$ such that $P \notin \langle \nu_d(U)\rangle$ for any $U \subsetneq A$. Then $r_{m,d}(P) = 3d-3$.*

Proof. By concision we may assume $m=3$. Since A is curvilinear, it has only finitely many subschemes. Hence $A \cup \{Q\}$ is in linearly general position for a general $Q \in \mathbb{P}^3$. The scheme $A\cup\{Q\}$ is contained in a unique rational normal curve C ([9, part (b) of Theorem 1]). Hence $P \in \langle \nu_d(C)\rangle$. Since A is not reduced and $d \ge 4$, Sylvester's theorem says that P has rank $3d+2-\deg(A) = 3d-3$ with respect to the degree $3d$ rational normal curve $\nu_d(C)$ (Remark 1). Therefore $r_{m,d}(P) \le 3d-3$. Assume $r_{m,d}(P) \le 3d-4$ and fix $B \in \mathcal{S}(B)$. Set $W_0 := A\cup B$ and use the proof of [4, Proposition 5.19] (the proof of Proposition 4 was harder (e.g. step (b) in that proof does not occur), because now $\deg(W_0) \le 3d+1$, while $\deg(W_0) \le 3d+2$ in that proof). QED

Proposition 9. *Fix integers $m \ge 2$, $c \in \{0,1,2\}$, $y \ge 0$, and $d \ge \max\{9, 2+2y\}$. If $y>0$, then assume $m \ge 3$. Let $A_1 \subset \mathbb{P}^m$ be a connected curvilinear scheme such that $\deg(A_1)=3$ and $\dim(\langle A_1\rangle)=2$. Set $\{O_1\} := (A_1)_{\mathrm{red}}$. Let A' be the degree 2 zero-dimensional subscheme of A_1. Fix sets $E \subset \langle A_1\rangle \setminus \langle A'\rangle$ and $F \subset \mathbb{P}^m \setminus \langle A_1\rangle$ such that $\sharp(E)=c$ and $\sharp(F)=y$. If $c=2$, then assume*

$O_1 \notin \langle E\rangle$. Set $A := A_1 \cup E \cup F$. Fix $P \in \langle \nu_d(A)\rangle$ such that $P \notin \langle \nu_d(U)\rangle$ for any $O \nsubseteq A$. Then $r_{m,d}(P) = 2d-1-c+y$ and every element of $\mathcal{S}(P)$ contains F

Proof. First assume $y = 0$. By concision we may assume $m = 2$. Our assumptions on E imply the existence of a smooth conic C containing A and hence $P \in \langle \nu_d(C)\rangle$. Since A is not reduced, Sylvester's theorem gives that P has rank $2d-1-c$ with respect to the curve $\nu_d(C)$ (Remark 1). Hence $r_{m,d}(P) \le 2d-1-c$. Assume $r_{m,d}(P) \le 2d-2-c$ and fix $B \in \mathcal{S}(P)$. Set $W := A \cup B$. Since $h^1(\mathcal{I}_W(d)) > 0$ ([3, Lemma 1]) and $\deg(W) \le 2d+1$, there is a line R with $\deg(R \cap W) \ge d+2$. Since $\mathrm{Res}_R(W)$ has degree $\le d-1$, we have $h^1(\mathcal{I}_{\mathrm{Res}_R(W)}(d-1)) = 0$. By [4, Lemma 5.1] we get $A_1 \subset R$, a contradiction.

If $y > 0$, then the proof of Proposition 7 works verbatim. QED

Proposition 10. *Assume $d \ge 9$ and $m \ge 3$. Let $A_1 \subset \mathbb{P}^m$ be a degree 3 connected curvilinear scheme such that $\langle A_1\rangle$ is a plane. Let A' be the degree two subscheme of A_1. Fix $O_2 \in (\langle A_1\rangle \setminus \langle A'\rangle)$ and call A_2 any degree two connected zero-dimensional scheme such that $(A_2)_{\mathrm{red}} = \{O_2\}$ and $A_2 \nsubseteq \langle A_1\rangle$. Set $A := A_1 \cup A_2$. Fix $P \in \langle \nu_d(A)\rangle$ such that $P \notin \langle \nu_d(U)\rangle$ for any $U \subsetneq A$. Then $r_{m,d}(P) = 3d-2$.*

Proof. By concision we may assume $m = 3$. Since $O_2 \in \langle A_1\rangle \setminus \langle A'\rangle$, we have $h^0(\langle A_1\rangle, \mathcal{I}_{A_1 \cup \{O_2\}}(2)) = 2$ and a general conic $C \subset M$ containing $A_1 \cup \{O_2\}$ is smooth. Since $P \in \langle \nu_d(A)\rangle$, but $P \notin \langle \nu_d(U)\rangle$ for any $U \subsetneq A$, the set $\langle \nu_d(A_1 \cup \{O_2\})\rangle \cap \langle \{P\} \cup \nu_d(A_2)\rangle$ is a line containing $\nu_d(O_2)$. Fix any point $P' \ne \nu_d(O_2)$ of this line. We have $P' \notin \langle \nu_d(U)\rangle$ for any $U \subsetneq A_1 \cup \{O_2\}$. Therefore P' has border rank 4 with respect to the degree d rational normal curve $\nu_d(C)$. By Sylvester's theorem (Remark 1) there is $B_1 \subset C$ such that $\sharp(B_1) = 2d-2$ and $P' \in \langle \nu_d(B_1)\rangle$. Since $P' \in \langle \{P\} \cup \nu_d(A_2)\rangle$, there is $P'' \in \langle \nu_d(A_2)\rangle$ such that $P \in \langle \{P', P''\}\rangle$. Since $P'' \in \langle \nu_d(\langle A_2\rangle)\rangle$ and every point of $\langle \nu_d(\langle A_2\rangle)\rangle$ has rank at most d with respect to the degree d rational normal curve $\nu_d(\langle A_2\rangle)$, we get $r_{m,d}(P) \le 3d-2$. Assume $r_{m,d}(P) \le 3d-3$ and take $B \in \mathcal{S}(P)$. Set $W_0 := A \cup B$. We have $h^1(\mathcal{I}_{W_0}(d)) > 0$ ([3, Lemma 1]) and $\deg(W_0) \le 3d+2$.

Claim 1: Assume the existence of a reduced conic C such that $W_0 \subset C \cup \langle A_2\rangle$. Then $\sharp(B \setminus (B \cap \langle A_1\rangle)) \ge d$.

Proof of Claim 1: Assume $\sharp(B \setminus (B \cap \langle A_1\rangle)) \le d-1$. Since $\mathrm{Res}_{\langle A_1\rangle}(W_0) = \{O_2\} \cup (B \setminus (B \cap \langle A_1\rangle))$, we have $\deg(\mathrm{Res}_{\langle A_1\rangle}(W_0)) \le d$ and hence $h^1(\mathcal{I}_{\mathrm{Res}_{\langle A_1\rangle}(W_0)}(d-1)) = 0$. If $h^1(\mathcal{I}_{\langle A_1\rangle \cap W_0}(d)) = 0$, then a residual exact sequence gives $h^1(\mathcal{I}_{W_0}(d)) = 0$, a contradiction. If $h^1(\mathcal{I}_{\langle A_1\rangle \cap W_0}(d)) > 0$, then [4, Lemma 5.1] shows that $A \subset \langle A_1\rangle$, a contradiction.

Claim 2: Assume the existence of a reduced conic C such that $W_0 \subset C \cup \langle A_2\rangle$. Then $\sharp(B \cap C) \ge 2d-2$.

Proof of Claim 2: Since $(A_1)_{\mathrm{red}} \notin \langle A_2\rangle$, then $A_1 \subset C$ and $\langle A_1\rangle = \langle C\rangle$. Claim 1 and Remark 2 gives $O_2 \notin B$ and $\sharp(B \cap \langle A_2\rangle) = d$.

(i) Assume $h^1(\langle A_1\rangle, \mathcal{I}_{W_0\cap\langle A_1\rangle}(d)) > 0$.

(i1) First assume $O_2 \in C$ and that C is a smooth conic. Since $\deg(C \cap A) = 4$ and $\sharp(B \cap C) \le 2d-3$, then $h^1(C, \mathcal{I}_{C\cap W_0}(d)) = 0$ and hence $h^1(\langle A_1\rangle, \mathcal{I}_{W_0\cap\langle A_1\rangle}(d)) = 0$.

(i2) Now assume $O_2 \in C$ and that C is not smooth. Since $O_2 \notin \langle A'\rangle$, we have $\langle C\rangle = \langle\{O_2\} \cup A'\rangle$. Set $M := \langle\{O_2\} \cup A'\rangle$. We have $\mathrm{Res}_M(A) = A'$. Since $\langle A\rangle = \mathbb{P}^3$ and no connected component of A is reduced, [4, Lemma 5.1] gives $h^1(\mathcal{I}_{\mathrm{Res}_M(W_0)}(d-1)) > 0$ and hence $\sharp(B \cap (\langle A'\rangle \setminus \{O_1\})) \ge d-1$. Assume $\sharp(B \cap (\langle\{O_1, O_2\}\rangle \setminus \{O_1, O_2\})) \le d-2$. If $B \cap (\langle\{O_1, O_2\}\rangle \setminus \{O_1, O_2\}) = \emptyset$, then set $\epsilon := \emptyset$. If $B \cap (\langle\{O_1, O_2\}\rangle \setminus \{O_1, O_2\}) \ne \emptyset$, then fix $o \in (B \cap (\langle\{O_1, O_2\}\rangle \setminus \{O_1, O_2\}))$ and set $\epsilon := \{o\}$. There is a smooth quadric $Q \supset \langle A'\rangle\rangle \cup \langle A_2\rangle \cup \epsilon$ and with $A_1 \nsubseteq Q$. Since $\deg(\mathrm{Res}_Q(W_0)) \le d-1$ by our definition of ϵ and the assumption on the integer $\sharp(B \cap (\langle\{O_1, O_2\}\rangle \setminus \{O_1, O_2\}))$, [4, Lemma 5.1] gives a contradiction.

(i3) Now assume $O_2 \notin C$. Assume for the moment $h^1(C, \mathcal{O}_{C\cap W_0}(d)) = 0$. Since $h^1(\langle C\rangle, \mathcal{I}_{O_2}(d-2)) = 0$, the residual exact sequence of the inclusion $\langle A_1\rangle \subset \mathbb{P}^3$ gives $h^1(\langle A_1\rangle, \mathcal{I}_{W_0\cap\langle A_1\rangle}(d)) = 0$, a contradiction. Now assume $h^1(C, \mathcal{O}_{C\cap W_0}(d)) > 0$. If C is smooth, then as before we get $\sharp(B\cap C) \ge 2d-1$, a contradiction. Now assume that C is not smooth. Since $C \supset A_1$, C is the union of $\langle A'\rangle$ and another line $D \subset \langle A_1\rangle$ with $O_1 \in D$. Remark 2 gives $\sharp(B \cap D) \le d$. Hence $h^1(C, \mathcal{O}_{C\cap W_0}(d)) > 0$ only if either $\sharp(B \cap C) = 2d-1$ or $\sharp(B \cap \langle A'\rangle) \ge d$. We may assume that the latter case occurs and that $\sharp(B\cap(D\setminus\{O_1\})) \le d-3$. Fix a general quadric $Q \subset \mathbb{P}^3$ containing the two disjoint lines $\langle A'\rangle$ and $\langle A_2\rangle$. Since $\mathcal{I}_{\langle A'\rangle\cup\langle A_2\rangle}(2)$ is spanned by its global sections, we have $Q \cap (B \cap (D \setminus \{O_1\}) = \emptyset$ and $A_1 \nsubseteq Q$. Hence $\mathrm{Res}_Q(W_0) = \{O_1\} \cup B \cap D$. Since $\deg(\mathrm{Res}_Q(W_0)) \le d-2$, then $h^1(\mathcal{I}_{\mathrm{Res}_Q(W_0)}(d-2)) = 0$. Therefore [4, Lemma 5.1] implies $A_1 \subset Q$, a contradiction.

(ii) Assume $h^1(\langle A_1\rangle, \mathcal{I}_{W_0\cap\langle A_1\rangle}(d)) = 0$. Since $\mathrm{Res}_{\langle A_1\rangle}(W_0) = \{O_2\} \cup (B \cap \langle A_2\rangle)$, we have $h^1(\mathcal{I}_{\mathrm{Res}_{\langle A_1\rangle}(W_0)}(d-1)) = 1$. The residual exact sequence of the inclusion $\langle A_1\rangle \subset \mathbb{P}^3$ gives $h^1(\mathcal{I}_{W_0}(d)) \le 1$ and hence $h^1(\mathcal{I}_{W_0}(d)) = 1$. Grassmann's formula gives $\dim(\langle A\rangle \cap \langle B\rangle) = \deg(A \cap B)$. Set $B_2 := B \cap \langle A_2\rangle$. Since $O_2 \notin B$ and $\sharp(B \cap \langle A_2\rangle) = d$, Grassmann's formula gives that $\langle\nu_d(A)\rangle \cap \langle\nu_d(B)\rangle$ is the linear span of the point $\langle\nu_d(A_2)\rangle \cap \langle\nu_d(B_2)\rangle$ and the set $E := B \cap A \cap C$. Since $P \notin \langle\nu_d(U)\rangle$ for any $U \subsetneq A$, we get $A_1 \subseteq E \subset B$, contradicting the fact that A_1 is not reduced.

(a) By Claims 1 and 2 to get a contradiction to the assumption $\sharp(B) \le 3d-3$ it is sufficient to prove the existence of a reduced conic C such that $W_0 \subset C \cup \langle A_2\rangle$. Since $\langle A_1\rangle$ is a plane, W_0 is not contained in the union of 3 disjoint lines. Since A is not in linearly general position in $\mathbb{P}^3$, A is not contained in a rational normal curve. Since $\langle A\rangle = \mathbb{P}^3$, A is not contained in a plane cubic. Hence any degree 3 reduced curve containing W_0 (if any) has either 1 line or 3 lines as components and in both cases $\langle A_2\rangle$ is one of these lines. Therefore to get a contradiction it is sufficient to prove that W_0 is contained in a reduced degree 3 curve.

(b) Let $H_1 \subset \mathbb{P}^3$ be a plane such that $e_1 := \deg(W_0 \cap H_1)$ is maximal. Set $W_1 := \mathrm{Res}_{H_1}(W_0)$. Fix an integer $i \geq 2$ and assume to have defined the integers e_j, the planes H_j and the scheme W_j, $1 \leq j < i$. Let $H_i \subset \mathbb{P}^3$ be any plane such that $e_i := \deg(H_i \cap W_{i-1})$ is maximal. Set $W_i := \mathrm{Res}_{H_i}(W_{i-1})$. We have $e_i \geq e_{i+1}$ for all i. We look at the residual exact sequences (2). Since $h^1(\mathcal{I}_{W_0}(d)) > 0$, there is an integer $i > 0$ such that $h^1(H_i, \mathcal{I}_{W_{i-1}\cap H_i, H_i}(d+1-i)) > 0$. We call g the first such an integer. Since any zero-dimensional scheme with degree 3 of $\mathbb{P}^3$ is contained in a plane, if $e_i \leq 2$, then $W_i = \emptyset$ and $e_j = 0$ if $j > i$. We have $\sum_i e_i = \deg(W_0) \leq 3d+2$. Since A is not in linearly general position, we have $e_1 \geq 4$. Therefore $g \leq d+1$. Assume $g = d+1$. We get $e_{d+1} \geq 2$. Since $e_1 \geq 4$, we get $e_1 = 4$, and $e_i \geq 3$ for $i \leq d$. Therefore $\deg(W_0) \geq 3d+4$, a contradiction.

(c) Assume $g \leq d$. Since $h^1(\mathcal{O}_{\mathbb{P}^3}(t)) = 0$ for all integers t, we have $e_g > 0$. Recall that $e_1 \geq \cdots \geq e_{g-1} \geq e_g$ and that $e_1 + \cdots + e_g \leq 3d+2$. By [5, Lemma 34] either there is a line $L \subset H_g$ such that $\deg(L \cap W_{g-1}) \geq d+3-g$ or $e_g \geq 2(d+2-g)+2 = 2(d+3-g)$. In the latter case we get $3d+2 \geq 2g(d+3-g)$ and hence $g = 1$. In the former case if $g \geq 2$ we get $e_{g-1} \geq d+4-g$, because A spans $\mathbb{P}^3$. Hence in the former case we get $3d+2 \geq g(d+4-g)-1$ and hence $1 \leq g \leq 3$.

(c1) Assume $g = 3$. We saw that there is a line $L \subset H_3$ such that $\deg(L \cap W_2) \geq d$ and that $e_2 \geq d+1$. Therefore $e_1 = e_2 = d+1$ and $e_3 = d$. Let N_1 be a plane containing L and with $f_1 := \deg(N_1 \cap W_0)$ maximal among the planes containing L. Since $d < f_1 \leq e_1 = d+1$, we have $f_1 = d+1$. Set $Z_0 := W_0$ and $Z_1 := \mathrm{Res}_{N_1}(Z_0)$. Let $N_2 \subset \mathbb{P}^3$ be a plane such that $f_2 := \deg(Z_1 \cap N_0)$ is maximal. Set $Z_2 := \mathrm{Res}_{N_2}(Z_1)$. Fix an integer $i \geq 3$ and assume that we had defined f_j, N_j, Z_j for all $j < i$. Let $N_i \subset \mathbb{P}^3$ be any plane such that $f_i := \deg(N_i \cap Z_{i-1})$ is maximal. Set $Z_i := \mathrm{Res}_{N_2}(Z_{i-1})$. The residual exact sequences like (2) with N_i instead of H_i and Z_i instead of W_i give the existence of an integer $i > 0$ such that $h^1(N_i, \mathcal{I}_{N_i \cap Z_{i-1}}(d+1-i)) > 0$. Let g' be the minimal such an integer. Since A spans $\mathbb{P}^3$ we have $f_1 \geq 1 + \deg(L \cap W_2) \geq d+1$. We have $f_{g'} > 0$ because $h^1(\mathcal{O}_{\mathbb{P}^3}(t)) = 0$ for all integers t. We have $f_i \geq f_{i+1}$ for all $i \geq 2$ and $\sum_{i \geq 2} f_i \leq 3d+2-f_1 \leq 2d+1$. Since $f_i \geq 3$ if $f_{i+1} > 0$, we get $g' \leq d$. Hence either $f_{g'} \geq 2(d+1-g')+2$ or there is a line $R \subset N_{g'}$ with $\deg(R \cap W_{g'-1}) \geq d+3-g'$. In the former case we get $2(g'-1)(d+2-g')+f_1 \leq 3d+2$ with $f_1 = d+1$ and hence $1 \leq g' \leq 2$. In the latter case if $g' \geq 2$ we have

$f_{g'-1} \geq d+4-g'$; hence in the latter case we have $3d+2 \geq g'(d+4-g')-1$ and hence $1 \leq g' \leq 3$.

(c1.1) Assume $g'=3$. Since $f_1 = d+1$, we have $R \cap L = \emptyset$. Fix a general quadric $Q \supset L \cup D$. Since W_0 is curvilinear and $\mathcal{I}_{L\cup R}(2)$ is spanned by its global sections, we have $W_0 \cap Q = W_0 \cap (L \cup R)$. If $h^1(Q, \mathcal{I}_{L\cup R}(d)) > 0$, we immediately get that either $\deg(R \cap W_0) \geq d+2$ (false because $e_3 = d$) or $\deg(L \cap W_0) \geq d+2$ (false because $f_1 = d+1$). The residual sequence of $Q \subset \mathbb{P}^3$ gives $h^1(\mathcal{I}_{\mathrm{Res}_Q(W_0)}(d-2)) > 0$. Since $\deg(\mathrm{Res}_Q(W_0)) \leq 3d+2-d-d$, there is a line D with $\deg(\mathrm{Res}_Q(W_0)) \geq d$. Since $W_0 = A \cup B$ with B reduced, we have $D \neq R$ and $D \neq L$. Since $e_1 < 2d-1$, we have $D \cap R = D \cap L = \emptyset$. Let T be the only quadric containing $L \cup R \cup D$. T is smooth. Call $|\mathcal{O}_Q(1,0)|$ the ruling of T containing L, R and D. Since $\deg(W_0) - \deg(W_0 \cap (L \cup R \cup D)) \leq 2$, we have $h^1(Q, \mathcal{I}_{\mathrm{Res}_{L\cup R\cup D}(W_0)}(d-2,d)) = 0$. Thus [4, Lemma 5.1] gives $W_0 \cup L \cup R \cup D$, contradicting the connectedness of A_1 and that $\langle A_1 \rangle$ is a plane.

(c1.2) Assume $g'=2$. Since $f_2 \leq \deg(Z_1) \leq 2d+1$, either there is a line $R \subset N_2$ with $\deg(R \cap Z_1) \geq d+1$ or there is a conic T with $\deg(T \cap Z_1) \geq f_2 = \deg(Z_1) = 2d$ and $T \supset Z_1$. The latter case cannot occur, because $e_1 < 2d$. Hence R exists. Since $\deg(R \cap W_0) \geq \deg(R \cap Z_1) \geq d+1$ and W_0 is not contained in a line, we get $e_1 \geq d+2$, a contradiction.

(c1.3) Assume $g'=1$.

(c1.3.1) Assume $\deg(L \cap W_0) \geq d+2$. Since A spans $\mathbb{P}^3$, we have $e_1 \geq f_1 \geq 1+\deg(L \cap W_0) \geq d+3$, a contradiction.

(c1.3.2) Assume $\deg(L \cap W_0) \leq d+1$. Since $f_1 \leq e_1 \leq d+2 \leq 2d+1$, Lemma 1 (or [5, Lemma 34]) gives the existence of a line $D \subset N_1$ with $\deg(D \cap W_0) \geq d+2$. Since $L \neq D$, $L \cup D \subset N_1$ and $\deg(L \cap W_0) \geq d$, we get $e_1 \geq f_1 \geq 2d+1$, a contradiction.

(c2) Assume $g=2$. We saw in step (c) that there is a line $L \subset H_2$ such that $\deg(L \cap W_1) \geq d+1$. Hence $e_1 \leq 2d+1$. Let N_1 be a plane containing L and with $f_1 := \deg(N_1 \cap W_0)$ maximal among the planes containing L. Define N_i,f_i,Z_i, g' as in step (c1). In particular $f_i \geq f_{i+1}$ for all $i \geq 2$. We have $f_1 \geq d+2$ (because $W_0 \nsubseteq L$) and $f_2 + \cdots + f_{g'} \leq 3d+2-f_1 \leq 2d$. Since $f_i \geq 3$ if $f_{i+1} > 0$, we get $g' \leq d$. Hence either $f_{g'} \geq 2(d+1-g')+2 = 2(d+2-g')$ or there is a line $R \subset N_{g'}$ with $\deg(R \cap Z_{g'-1}) \geq d+3-g'$. In the former case we get that either $g'=1$ or $2(g'-1)(d+2-g') \leq 2d$; thus $1 \leq g' \leq 2$. In the latter case if $g' \geq 23$ we have $f_{g'-1} \geq d+4-g'$, because $Z_{g'-2}$ is not contained in the line Rn and $f_{g'-1}$ satisfies a maximality condition. Hence in the latter case if $g' \geq 3$ we have $2d \geq (g'-2)(d+4-g')+d+3-g'$. Thus in the latter case we have $1 \leq g' \leq 3$.

(c2.1) Assume $g'=2$, $f_2 \geq 2d$ and the non-existence of a line R such that $\deg(R \cap Z_1) \geq d+1$. Since $e_1 \geq f_2$, we get $e_2 \leq d+1$. Since $\deg(L \cap W_1) \geq d+1$, we get $\deg(W_1) = d + 1$ and $W_1 \subset L$. We also get $f_1 = d + 2$ and hence (since $W_0 \nsubseteq L$) we have $\deg(L \cap W_0) = d + 1$. Since $f_1 = d + 2$, $f_2 \ge 2d$ and $\deg(W_0) \le 3d+2$, we get $f_2 = 2d$ and $Z_1 \subset N_2$. Since $h^1(N_2, \mathcal{I}_{Z_1 \cap N_2}(d-1)) > 0$ and there is no line R with $\deg(R \cap W_1) \ge d + 1$, Lemma 2 gives the existence of a plane conic $E \subset N_2$ containing Z_2. Recall that it is sufficient to prove that $W_0 \subset E \cup L$ with E a reduced conic. Since the sum of the degrees of the unreduced connected components of W_0 is at most 5 and $d \ge 9$, E is not a double line. Since $\deg(E \cap W_1) \ge 6$ and B is reduced, L is not a component of E. By step (a) it is sufficient to prove that $W_0 \subset L \cup E$. Let $M \subset \mathbb{P}^3$ be a general plane containing L. Since W_0 is curvilinear and M is general, we have $W_0 \cap M = W_0 \cap M$. Since $N_1 \supset L$ and $\deg(\mathrm{Res}_{N_1}(W_0) \cap N_2) = 2d$, we have $\deg(W_0 \cap (M \cup N_2)) \ge \deg(L \cap W_0) + f_2 \ge 3d+1$. Hence $\deg(\mathrm{Res}_{N_2 \cup M}(W_0)) \le 1$ and so $h^1(\mathcal{I}_{\mathrm{Res}_{N_2 \cup M}(W_0)}(d-2)) = 0$. Since B is a finite set, [4, Lemma 5.1] (applied to the degree 2 surface $N_2 \cup M$, not a hyperplane) gives $W_0 \subset N_2 \cup M$. Hence $\mathrm{Res}_{N_2}(W_0) \subset M$. Since $W_0 \cap M = W_0 \cap L$, we get $\mathrm{Res}_{N_2}(W_0) \subset L$. Since $N_2 \cap W_0 = E \cap W_0$, if W_0 were a finite set, we would have $W_0 \subset E \cup L$. We at least have $(W_0)_{\mathrm{red}} \subset E \cup L$ and to get $W_0 \subset E \cup L$ it is sufficient to prove that $A_1 \subset E \cup L$ and $A_2 \subset E \cup L$. For an arbitrary zero-dimensional scheme $W_0 \subset N_2 \cup L$ with $W_0 \cap N_2 \subset E$, we have $W_0 \subset E \cup L$ if $E \cap L \cap W_0 = \emptyset$. Thus we may assume $E \cap L \cap W_0 \neq \emptyset$. In particular $E \cap L \neq \emptyset$. Since $L \nsubseteq N_2$, the scheme $E \cap L$ is a point, o, with its reduced structure. We have $h^0(\mathcal{I}_{E \cup L}(2)) = 3$ Since E is not a double line, the general quadric Q containing $E \cup L$ is smooth, unless E is reducible and $L \cap E$ is the singular point of E, i.e. unless $E \cup L$ is a non-coplanar union of 3 lines through o. In the latter case $\mathcal{I}_{E \cup L}(2)$ is spanned by its global sections, because for any $S \subset \mathbb{P}^2$ with $\sharp(S) = 3$ and S not contained in a line the sheaf $\mathcal{I}_{S,\mathbb{P}^2}(2)$ is globally generated and the cone $E \cup L$ is an arithmetically Cohen-Macaulay curve. If o is not the singular point of E, call Q a smooth quadric containing $E \cup L$ and $|\mathcal{O}_Q(2,1)|$ the linear system of Q such that $E \cup L \in |\mathcal{O}_Q(2,1)|$. Since the line bundle $\mathcal{O}_Q(0,1)$ is spanned by its global sections, even in this case we see that $\mathcal{I}_{E \cup L}(2)$ is globally generated. Take a general quadric $T \supset E \cup L$. Since $\deg(\mathrm{Res}_T(W_0)) \le 5 \le d-2$, we have $h^1(\mathcal{I}_{\mathrm{Res}_T(W_0)}(d-2)) = 0$. Since B is a finite set and $A_{\mathrm{red}} \subset E \cup L$, [4, Lemma 5.1] gives $W_0 \subset T$. Since this is true for a general quadric $T \supset E \cup L$, we have $W_0 \subset E \cup L$.

(c2.2) Assume $g' = 3$. We saw the existence of a line $R \subset \mathbb{P}^3$ such that $\deg(R \cap Z_2) \ge d$. Since $f_2 \ge f_3 \ge d$, we get $f_1 \le d + 3$. Since $\deg(R \cap A) \le 3$, we get $R \cap (B \setminus B \cap L) \neq \emptyset$ and hence $R \neq L$. First assume $R \cap L \neq \emptyset$ and hence $R \cup L$ is contained in a plane. We get $f_1 \ge (d + 1) + d - 1$, a contradiction. Now assume $R \cap L = \emptyset$. Let $Q \subset \mathbb{P}^3$ we a general quadric surface containing $R \cup L$. Since W_0 is curvilinear, $\mathcal{I}_{R \cup L}(2)$ is spanned by its global sections and Q

is general, we have $Q \cap W_0 = (R \cup L) \cap W_0$. Since $\deg(\mathrm{Res}_Q(W_0)) \le d+1$. We continue as in step (c1.1).

(c2.3) Assume $g' = 2$. By step (c2.1) there is a line $R \subset \mathbb{P}^3$ such that $\deg(R \cap Z_2) \ge d+1$. Since $\deg(R \cap A) \le 3$, we get $R \cap (B \setminus B \cap L) \ne \emptyset$ and hence $R \ne L$.

(c2.3.1) Assume $R \cap L \ne \emptyset$, then $f_1 \ge (d+1)+(d+1)-1 = 2d+1$. Since $A \nsubseteq N_1$, [4, Lemma 5.1] gives $h^1(\mathcal{I}_{Z_1}(d-1)) > 0$. Since $\deg(Z_1) \le 3d+3-f_1 \le 2d-1$, [5, Lemma 34] gives the existence of a line $D \subset \mathbb{P}^3$ such that $\deg(D \cap Z_1) \ge d+1$. Using $|\mathcal{I}_{R\cup L\cup D}(3)|$ and [4, Lemma 5.1] we get $A \subset R \cup L \cup D$, contradicting step (a).

(c2.3) Assume $g' = 1$.

(c2.3.1) Assume $\deg(L \cap W_0) \ge d+2$. Since $A \nsubseteq N_1$, [4, Lemma 5.1] gives $h^1(\mathcal{I}_{Z_1}(d-1)) > 0$. Since A spans $\mathbb{P}^3$, we have $f_1 \ge d+3$ and hence $\deg(Z_1) \le 2(d-1)+1$. Therefore there is a line $R \subset \mathbb{P}^3$ such that $\deg(R \cap Z_1) \ge d+1$. If $B \subset L \cup R$, then set $\epsilon := \emptyset$. If $B \nsubseteq L \cup R$, then fix $o \in (B \setminus B \cap (R \cup L))$ and set $\epsilon := \{o\}$. Let Q be any quadric containing $R \cup L$. Since $\deg(W_0 \cap (R \cup L)) \ge (d+1)+(d+2)-1$ and $\deg(A) \le d-1$, we have $\deg(\mathrm{Res}_Q(W_0)) \le d-1$ and hence $h^1(\mathcal{I}_{\mathrm{Res}_Q(W_0)}(d-2)) = 0$. Therefore [4, Lemma 5.1] gives $W_0 \subset Q$. If $L \cap R = \emptyset$, then $\mathcal{I}_{L\cup R\cup\epsilon}(2)$ is spanned and hence varying Q, we get $A \subset L \cup R$, a contradiction. If $L \cap R \ne \emptyset$, it is sufficient to take as o a point of B not in the plane $\langle L \cup R \rangle$ (it exists by concision [10, Exercise 3.2.2.2]).

(c2.3.2) Assume $\deg(L \cap W_0) \le d+1$. Since $f_1 \le e_1 \le 2d+1$, there is a line $D \subset N_1$ with $\deg(D \cap W_0) \ge d+2$. We have $D \ne L$ and hence $e_1 \ge f_1 \ge (d+1)+(d+2)-1 = 2d+2$, a contradiction.

(c3) Assume $g = 1$. Since A spans $\mathbb{P}^3$, [4, Lemma 5.1] gives $h^1(\mathcal{I}_{W_1}(d-1)) > 0$. Therefore $\deg(W_1) \ge d+1$ and hence $e_1 \le 2d+1$. Since $h^1(H_1, \mathcal{I}_{W_0 \cap H_1}(d)) > 0$, we have $e_1 \ge d+2$ and hence $\deg(W_1) \le 2d$. By Lemma 1 either there is a line $L \subset \mathbb{P}^3$ such that $\deg(L \cap W_1) \ge d+1$ or there is a plane conic T with $\deg(T \cap W_1) \ge 2d$. The latter case does not arise, because it would imply $e_1 \ge 2d$ and hence $\deg(W_1) \le d+2 < \deg(T \cap W_1)$. Therefore there is a line $L \subset \mathbb{P}^3$ such that $\deg(L \cap W_1) \ge d+1$. We continue as in step (c2). QED

Proposition 11. *Assume $d \ge 9$ and $m \ge 3$. Let $A_1 \subset \mathbb{P}^m$ be a degree 3 connected curvilinear scheme such that $\langle A_1 \rangle$ is a plane. Let A' be the degree two subscheme of A_1. Set $\{O_1\} := (A_1)_{\mathrm{red}}$. Fix $O_2 \in (\langle A' \rangle \setminus \{O_1\})$ and call A_2 any degree two connected zero-dimensional scheme such that $(A_2)_{\mathrm{red}} = \{O_2\}$ and $A_2 \nsubseteq \langle A_1 \rangle$. Set $A := A_1 \cup A_2$. Fix $P \in \langle \nu_d(A) \rangle$ such that $P \notin \langle \nu_d(U) \rangle$ for any $U \subsetneq A$. Then $r_{m,d}(P) = 3d-2$.*

Proof. By concision we may assume $m = 3$. There are $P' \in \langle \nu_d(A_1 \cup \{O_2\} \rangle$ and $P'' \in \langle \nu_d(A_2) \rangle$. By Sylvester's theorem (Remark 1), P'' has rank at most d with respect to the degree d rational normal curve $\nu_d(\langle A_2\rangle)$. Hence to prove that $r_{m,d}(P) \le 3d-2$ it is sufficient to prove that $r_{m,d}(P') \le 2d-2$. This is true by the case $x=1$ and $y=0$ of Proposition 7. Therefore $r_{m,d}(P) \le 3d-2$. Assume $r_{m,d}(P) \le 3d-3$, take $B \in \mathcal{S}(P)$ and set $W_0 := A \cup B$. We have $h^1(\mathcal{I}_{W_0}(d)) > 0$ ([3, Lemma 1]) and $\deg(W_0) \le 3d+2$.

Since $\langle A\rangle = \mathbb{P}^3$, A is not contained in a plane curve of degree 3. Since A is not in linearly general position in $\mathbb{P}^3$, it is not contained in a rational normal curve. Since $\deg(A \cap \langle A'\rangle) = 3$ and $O_1 \notin \langle A_2\rangle$, the only reduced degree 3 curves containing A are the union of 3 lines: they are the union of $\langle A'\rangle$, $\langle A_2\rangle$ and a line of $\langle A_1\rangle$ containing O_1 and different from $\langle A'\rangle$. Assume $W_0 \subset \langle A_1\rangle \cup \langle A_2\rangle$. As in Claim 1 of the proof of Proposition 10 we see that $O_2 \notin B$ and that $\sharp(B \cap \langle A_2\rangle) = d$. Now assume that W_0 contained in the union of 3 different lines, $\langle A'\rangle$, $\langle A_2\rangle$ and a line R of $\langle A_1\rangle$ containing O_1 and different from $\langle A'\rangle$.

(i) Assume $h^1(\langle A_1\rangle, \mathcal{I}_{W_0\cap\langle A_1\rangle}(d)) > 0$. If $\deg(W_0 \cap \langle A_1\rangle) \ge 2d+2$, then we get $\sharp(B \cap \langle A_1\rangle) \ge 2d-2$, because $\deg(A \cap \langle A_1\rangle) = 4$. Now assume $\deg(W_0 \cap \langle A_1\rangle) \le 2d+1$. By [5, Lemma 34] there is a line $L \subset \langle A_1\rangle$ such that $\deg(L \cap W_0) \ge d+2$. Since $\langle A'\rangle$ is the only line D of $\langle A_1\rangle$ with $\deg(D \cap A) \ge 2$ and $\deg(\langle A'\rangle \cap A) = 3$, Remark 2 gives $L = \langle A'\rangle$ and $\sharp(B \cap (L \setminus \{O_1\}) \ge d-1$. Set $M := \langle \{O_1\} \cup A_2\rangle$. M is a plane and $\mathrm{Res}_M(A) = \{O_1\}$. Since $A_1 \nsubseteq M$, [4, Lemma 5.1] gives $h^1(\mathcal{I}_{\mathrm{Res}_M(W_0)}(d-1)) > 0$ and hence $\sharp(B \cap (R \setminus \{O_1\}) \ge d$. Therefore $\sharp(B) \ge 3d-1$, a contradiction.

(ii) Assume $h^1(\langle A_1\rangle, \mathcal{I}_{W_0\cap\langle A_1\rangle}(d)) = 0$. Since $\mathrm{Res}_{\langle A_1\rangle}(W_0) = \{O_2\} \cup (B \cap \langle A_2\rangle)$, we have $h^1(\mathcal{I}_{\mathrm{Res}_{\langle A_1\rangle}(W_0)}(d-1)) = 1$. A residual exact sequence gives $h^1(\mathcal{I}_{W_0}(d)) \le 1$ and hence $h^1(\mathcal{I}_{W_0}(d)) = 1$. Grassmann's formula gives $\dim(\langle A\rangle \cap \langle B\rangle) = \deg(A \cap B)$. Set $B_2 := B \cap \langle A_2\rangle$. Since $O_2 \notin B$ and $\sharp(B \cap \langle A_2\rangle) = d$, Grassmann's formula gives that $\langle \nu_d(A)\rangle \cap \langle \nu_d(B)\rangle$ is the linear span of the point $\langle \nu_d(A_2)\rangle \cap \langle \nu_d(B_2)\rangle$ and the set $E := B \cap (A_1 \cup \{O_2\})$. Since $P \notin \langle \nu_d(U)\rangle$ for any $U \subsetneq A$, we get $A_1 \subseteq E \subset B$, contradicting the fact that A_1 is not reduced.

Then we continue as in the proof of Proposition 10. QED

Proposition 12. *Assume $d \ge 9$ and $m \ge 3$. Let $A_1 \subset \mathbb{P}^m$ be a degree 3 connected curvilinear scheme such that $\langle A_1\rangle$ is a plane. Set $\{O_1\} := (A_1)_{\mathrm{red}}$. Fix a degree 2 connected zero-dimensional scheme A_2 such that $O_1 \in \langle A_2\rangle$ and $O_1 \ne O_2$, where $\{O_2\} := (A_2)_{\mathrm{red}}$. Set $A := A_1 \cup A_2$. Fix $P \in \langle \nu_d(A)\rangle$ such that $P \notin \langle \nu_d(U)\rangle$ for any $U \subsetneq A$. Then $r_{m,d}(P) = 3d-2$.*

Proof. By concision we may assume $m = 3$.

Claim 1: We have $r_{m,d}(P) \le 3d-2$.

Proof of Claim 1: Since $\nu_d(A)$ is linearly independent and $A_1 \cap A_2 = \emptyset$, there are unique points $P_i \in \langle \nu_d(A_i)\rangle$ such that $P \in \langle \{P_1, P_2\}\rangle$. Since $P \notin \langle \nu_d(E)\rangle$ for any $E \subsetneq A$, then $P_i \notin \langle \nu_d(E)\rangle$ for any $E \subsetneq A_i$. Fix any

$P_i' \in \langle\{P_2, \nu_d(O_1)\}\rangle \setminus \{P_2, \nu_d(O_1)\}$ (it exists, because $O_1 \neq O_2$ and $\nu_d(A_2 \cup \{O_2\})$ is linearly independent). The parenthetical remark implies $P_2 \notin \langle\nu_d(A_2)\rangle$. Therefore P_2 has border rank and cactus rank 3 with respect to the degree d rational normal curve $\nu_d(\langle A_2\rangle)$. Sylvester's theorem gives a set $B_2 \subset \langle A_2\rangle$ such that $P_2' \in \langle\nu_d(B_2)\rangle$ (Remark 1). Fix a smooth conic $C \subset \langle A_1\rangle$ containing A_1. We have $P \in \langle\{P_2', \nu_d(O_1), P_1\}\rangle$. Since $\{\nu_d(O_1), P_1\} \subset \langle\nu_d(A_1)\rangle$. Therefore there is $P_1' \in \langle\nu_d(A_1)\rangle$ such that $P \in \{P_2', P_1'\}$. Let A' be the degree two subscheme of A_1 and assume $P_1' \in \langle\nu_d(A')\rangle$. Since $\{O_1\} \subset A'$, we get $P \in \langle\{P_1\} \cup \nu_d(A')\rangle \subset \langle\nu_d(A' \cup A')\rangle$, a contradiction. Therefore $P_1' \nsubseteq \langle\nu_d(E)\rangle$ for any $E \subsetneq A_1$. Sylvester's theorem gives the existence of $B_1 \subset C$ such that $\sharp(B_1) = 2d-1$ and $P_2' \in \langle\nu_d(B_1)\rangle$ (Remark 1). Since $P \in \langle\nu_d(B_1 \cup B_2)\rangle$ and $\sharp(B_1 \cup B_2) \leq 3d-2$, we get $r_{m,d}(P) \leq 3d-2$.

Assume $r_{m,d}(P) \leq 3d-3$, take $B \in \mathcal{S}(P)$ and set $W_0 := A \cup B$. We have $\deg(W_0) \leq 3d+2$.

Claim 3: Assume $B \subset \langle A_1\rangle \cup \langle A_2\rangle$. Then $\sharp(B \cap (\langle A_2\rangle \setminus \{O_2\})) \geq d-1$.

Proof of Claim 3: Since $A \subset \langle A_1\rangle \cup \langle A_2\rangle$, we have $W_0 \subset \langle A_1\rangle \cup \langle A_2\rangle$. Assume $\sharp(B \cap (\langle A_2\rangle \setminus \{O_2\})) \leq d-2$. We get $h^1(\mathcal{I}_{\mathrm{Res}_{\langle A_1\rangle}(W_0)}(d-1)) = 0$, contradicting [4, Lemma 5.1], because A_2 is connected, not reduced and $A_2 \nsubseteq \langle A_1\rangle$.

(a) Assume $B \subset \langle A_1\rangle \cup \langle A_2\rangle$. In this step we prove that either $\sharp(B \cap \langle A_1\rangle) \geq 2d-1$ or $\sharp(B \cap (\langle A_2\rangle \setminus \{O_2\})) = d$ and $\sharp(B \cap (\langle A_1\rangle \setminus \{O_1\})) \geq 2d-2$. Assume $\sharp(B \cap \langle A_1\rangle) \leq 2d-2$. Set $J := B \cap (\langle A_2\rangle \setminus \{O_2\})$.

(a1) Assume $\sharp(B \cap (\langle A_1\rangle \setminus \{O_1\})) \leq 2d-3$. Let $M \subset \mathbb{P}^3$ be the plane spanned by A_2 and the degree two subscheme of A_1. Since $A \nsubseteq N$ and no connected component of A is reduced, [4, Lemma 5.1] gives $h^1(\mathcal{I}_{\mathrm{Res}_N(W_0)}(d-1)) > 0$. Since $\deg(\mathrm{Res}_M(W_0)) \leq 2d-2$, [5, Lemma 34] gives the existence of a line D such that $\deg(D \cap \mathrm{Res}_N(W_0)) \geq d+1$. Since $A_2 \cap \mathrm{Res}_N(W_0) = \emptyset$, Remark 2 gives $O_1 \in D$, $O_1 \notin B$ and $\sharp(B \cap D) = d$. Since $D \cap (B \setminus B \cap \langle A_2\rangle) \neq \emptyset$, we have $D \neq \langle A_2\rangle$. Therefore $N := \langle A_2 \cup D\rangle$ is a plane. Since $\deg(\mathrm{Res}_N(W_0)) \leq 2 + (2d-3) - d$, we have $h^1(\mathcal{I}_{\mathrm{Res}_N(W_0)}(d-1)) = 0$, contradicting [4, Lemma 5.1], because $N \neq \langle A_1\rangle$.

(a2) Assume $h^1(\langle A_1\rangle, \mathcal{I}_{\langle A_1\rangle \cap W_0}(d)) > 0$. Since $\deg(A_1) = 3$ and $O_2 \notin \langle A_1\rangle$, there is a line $D \subset \langle A_1\rangle$ such that $\deg(W_0 \cap D) \geq d+2$. Remark 2 gives that D is spanned by the degree 2 subscheme of A_1, that $O_1 \notin B$ and that $\sharp(B \cap D) = d$. Set $M := \langle D \cup A_2\rangle$. M is a plane and $\deg(M \cap W_0) \geq 4 + d + d - 1$ by Claim 2. Since $\deg(W_0) \leq 3d+2$, we get $h^1(\mathcal{I}_{\mathrm{Res}_M(W_0)}(d-1)) = 0$ and hence (by [4, Lemma 5.1]) $A_1 \subset M$, a contradiction.

(a3) Assume $h^1(\langle A_1\rangle, \mathcal{I}_{\langle A_1\rangle \cap W_0}(d)) = 0$. By the residual exact sequence of the inclusion $\langle A_1\rangle \subset \mathbb{P}^3$ we get $h^1(\mathcal{I}_{W_0}(d)) \leq h^1(\langle A_2\rangle, \mathcal{O}_{\langle A_2\rangle}(d-1)(-J-A_2))$. Remark 2 gives $\sharp(J) \leq d$. Since $h^1(\mathcal{I}_{W_0}(d)) > 0$, we get $\sharp(J) \geq d-1$. By step (a1) we have $\sharp(B \cap (\langle A_1\rangle \setminus \{O_1\})) \ge 2d-2$. Hence if $\sharp(J) \ge d$, then step (a) is proved. Therefore we may assume $\sharp(J) = d-1$ and hence $h^1(\mathcal{I}_{W_0}(d)) = h^1(\langle A_2\rangle, \mathcal{O}_{\langle A_2\rangle}(d-1)(-J-A_2)) = 1$. Since $A \cap B = (\{O_1\} \cup A_2) \cap (B \cap \langle A_2\rangle)$, Grassmann's formula gives $\langle \nu_d(A)\rangle \cap \langle \nu_d(B)\rangle = \langle \nu_d(\{O_1\} \cup A_2)\rangle \cap \langle \nu_d(\{O_1\} \cup (B \cap \langle A_2\rangle))\rangle \subset \langle \nu_d(\{O_1\} \cup A_2)\rangle$. Therefore $P \in \langle \nu_d(\{O_1\} \cup A_2)\rangle$, a contradiction.

By Claims 1 and 2 and step (a) to prove Proposition 12 it is sufficient to either get a contradiction or to show that $W_0 \subset \langle A_1\rangle \cup \langle A_2\rangle$. We follow the proof of Proposition 10 using the same labels for the proofs. We define H_i, e_i, W_i, g as in the proof of Proposition 10 and get (as in that proof) that $1 \le g \le 3$.

(c1) Assume $g = 3$ and take the line $L \subset H_3$ such that $\deg(L \cap W_2) \ge d$. We have $W_2 \subset L$, $\deg(W_2 \cap L) = d$ and $e_1 = e_2 = d+1$ and $d+1 \le e_1 \le d+2$. Define f_i, N_i, Z_i, g' as in the quoted proof and get $1 \le g' \le 3$.

(c1.1) Assume $g' = 3$. We saw (in step (c1) of the quoted proof) the existence of a line $R \subset N_3$ such that $\deg(Z_2 \cap R) \ge d$. Since $\deg(D \cap A) \le 3$ for each line D and $B \cap Z_1 \subset B \setminus B \cap L$, we get $R \cap (B \setminus B \cap L) \ne \emptyset$. Therefore $R \ne L$. We have $R \cap L = \emptyset$ because $e_1 < 2d-1$. Fix a general $Q' \in |\mathcal{I}_{L\cup R}(2)|$. Since $R \cap L = \emptyset$, Q' is smooth. Since A is curvilinear, we also have $B \cap Q' = B \cap (L \cup R)$ and $Q' \cap A = A \cap (L \cup R)$ (as schemes). [4, Lemma 5.1] gives $h^1(\mathcal{I}_{\mathrm{Res}_{Q'}(W_0)}(d-2)) > 0$. Since $\deg(\mathrm{Res}_{Q'}(W_0)) \le d+2 \le 2(d-2)+1$, there is a line $D \subset \mathbb{P}^3$ such that $\deg(D \cap \mathrm{Res}_{Q'}(W_0)) \ge d$. Since $D \cap (B \setminus B \cap (L \cup R)) \ne \emptyset$, we get $D \ne R$ and $D \ne L$. Fix a general $U \in |\mathcal{I}_{D\cup R\cup L}(3)|$. Since $\deg(D \cap (L \cup R)) \le 2$, then $\deg(\mathrm{Res}_U(W_0)) \le 3d+2-d-d-d+2$ and hence $h^1(\mathcal{I}_{\mathrm{Res}_U(W_0)}(d-3)) = 0$. Hence $W_0 \subset U$ ([4, Lemma 5.1]). Since $\mathcal{I}_{D\cup R\cup L)}(3)$ is spanned, for general U we have $U \cap W_0 = W_0 \cap (L \cup D \cup R)$. Hence $W_0 \subset L \cup D \cup R$. Since $A \subset L \cup D \cup R$ and $L \cap R = \emptyset$, we get $D = \langle A_2\rangle$ and that one of the lines L, R, say R' contains O_1, while the other one, say L', contains O_2, but not O_1. We get $A_1 \subset D \cup R$, contradicting the fact that $\langle A_2\rangle \nsubseteq \langle A_1\rangle$.

(c1.2) Assume $g' = 2$. We saw (in step (c1) of the quoted proof) the existence of a line $R \subset \mathbb{P}^3$ such that $\deg(R \cap Z_2) \ge d$. Since $\deg(R \cap A) \le 3$, we get $R \cap (B \setminus B \cap L) \ne \emptyset$ and hence $R \ne L$. Since $e_1 = d+1$, then $R \cap L = \emptyset$. We continue as in steps (c1.1).

(c1.3) Assume $g' = 1$. Since $e_1 = d+1$, we have $f_1 \le d+1$. Hence $h^1(N_1, \mathcal{I}_{W_0\cap N_1}(d)) = 0$, a contradiction.

(c2) Assume $g = 2$. We saw that there is a line $L \subset H_2$ such that $\deg(L \cap W_1) \ge d+1$. Hence $e_1 \le 2d+1$. Let N_1 be a plane containing L and with $f_1 := \deg(N_1 \cap W_0)$ maximal among the planes containing L. Define N_i, f_i, Z_i, g' as in step (c1). Since $f_i \ge 3$ if $f_{i+1} > 0$, we get $g' \le d$. If $g' \ge 2$ we have $f_1 \ge \deg(L \cap W_0)+1 \ge d+2$, because $W_0 \nsubseteq L$. Hence either $f_{g'} \ge 2(d+1-g')+2$ or there is a line $R \subset N_{g'}$ with $\deg(R \cap Z_{g'-1}) \ge d+3-g'$. In the former case if $g' \ge 2$ we get $2(g'-1)(d+2-g') + f_1 \le 3d+3$ and hence $1 \le g' \le 2$. In the latter case if $g' \geq 2$ we have $f_{g'-1} \geq d+4-g'$; hence in the latter case we have $3d+3 \geq g'(d+4-g')-1$ and hence $1 \leq g' \leq 3$.

(c2.1) Assume $g' = 3$. We saw the existence of a line $R \subset \mathbb{P}^3$ such that $\deg(R \cap Z_2) \geq d$. Since $f_2 \geq f_3 \geq d$, we get $f_1 \leq d+3$. Since $\deg(R \cap A) \leq 3$, we get $R \cap (B \setminus B \cap L) \neq \emptyset$ and hence $R \neq L$. First assume $R \cap L \neq \emptyset$. We get $f_1 \geq (d+1)+d-1$, a contradiction. Now assume $R \cap L = \emptyset$. We continue as in step (c1.1).

(c2.2) Assume $g' = 2$ and the existence of a line $R \subset \mathbb{P}^3$ such that $\deg(R \cap Z_2) \geq d+1$. Since $\deg(R \cap A) \leq 3$, we get $R \cap (B \setminus B \cap L) \neq \emptyset$ and hence $R \neq L$.

(c2.2.1) Assume $R \cap L \neq \emptyset$, then $f_1 \geq (d+1)+(d+1)-1 = 2d+1$. Since $A \nsubseteq N_1$, [4, Lemma 5.1] gives $h^1(\mathcal{I}_{Z_1}(d-1)) > 0$. Since $\deg(Z_1) \leq 3d+3-f_1 \leq 2d-1$, [5, Lemma 34] gives the existence of a line $D \subset \mathbb{P}^3$ such that $\deg(D \cap Z_1) \geq d+1$. Using $|\mathcal{I}_{R \cup L \cup D}(3)|$ and [4, Lemma 5.1] we get $A \subset R \cup L \cup D$. Therefore one of the lines L, R, D (call it R') is the line $\langle A_2 \rangle$. The other two lines, say L' and D', must contain O_1 and being contained in $\langle A_1 \rangle$. Claim 2 and step (a) conclude the proof.

(c2.2.2) Assume $g' = 2$, $f_2 \geq 2d$ and the non-existence of a line $R \subset N_2$ with $\deg(Z_1 \cap R) \geq d+1$. We saw that $\deg(L \cap W_0) = d+1$, $f_1 = d+2$, $f_2 = 2d$ and $Z_1 \subset N_2$. Since $h^1(N_2, \mathcal{I}_{Z_1}(d-1)) > 0$, Lemma 2 gives the existence of a conic $E \subset N_2$ such that $Z_1 \subset E$. Since the sum of the degrees of the non-reduced connected components of W_0 is $5 < d$, E is a reduced conic. Let M be the plane spanned by L and one of the points, α, of $E \cap B$. Since $f_1 = d+2$ and $\deg(N_1 \cap W_0) \geq \deg(M \cap W_0)$ by the definition of N_1, we have $\deg(M \cap W_0) \geq\geq d+2$, because $\alpha \notin L$, and $\deg(M \cap W_0) \leq e_1 < 2d$. Since $\deg(L \cap W_0) = d+1$, [5, Lemma 34] gives $h^1(M, \mathcal{I}_{W_0 \cap M}(d)) = 0$. We have $h^1(\mathcal{I}_{\mathrm{Res}_M}(W_0)(d-1)) = 0$, because $\alpha \notin \mathrm{Res}_M(W_0)$ and so $\deg(\mathrm{Res}_M(W_0) \cap E) < 2d$. The residual exact sequence of $M \subset \mathbb{P}^3$ gives a contradiction.

(c2.3) Assume $g' = 1$.

(c2.3.1) Assume $\deg(L \cap W_0) \geq d+2$. Since $A \nsubseteq N_1$, [4, Lemma 5.1] gives $h^1(\mathcal{I}_{Z_1}(d-1)) > 0$. Since A spans $\mathbb{P}^3$, we have $f_1 \geq d+3$ and hence $\deg(Z_1) \leq 2(d-1)+1$. Therefore there is a line $R \subset \mathbb{P}^3$ such that $\deg(R \cap Z_1) \geq d+1$. If $B \subset L \cup R$, then set $\epsilon := \emptyset$. If $B \nsubseteq L \cup R$, then fix $o \in (B \setminus B \cap (R \cup L))$ and set $\epsilon := \{o\}$. Let Q be any quadric containing $R \cup L \cup \epsilon$. Since $\deg(W_0 \cap (R \cup L)) \geq (d+1)+(d+2)-1$ and $\deg(A) \leq d-1$, we have $\deg(\mathrm{Res}_Q(W_0)) \leq d-1$ and hence $h^1(\mathcal{I}_{\mathrm{Res}_Q(W_0)}(d-2)) = 0$. Therefore [4, Lemma 5.1] gives $W_0 \subset Q$. If $L \cap R = \emptyset$, then $\mathcal{I}_{L \cup R \cup \epsilon}(2)$ is spanned and hence varying Q, we get $A \subset L \cup R$, a contradiction. If $L \cap R \neq \emptyset$, it is sufficient to take as o a point of B not in the plane $\langle L \cup R \rangle$ (it exists by concision [10, Exercise 3.2.2.2]).

(c2.3.2) Assume $\deg(L \cap W_0) \leq d+1$. Since $f_1 \leq e_1 \leq 2d+1$, there is a line $D \subset N_1$ with $\deg(D \cap W_0) \geq d+2$. We have $D \neq L$ and hence

$e_1 \geq f_1 \geq (d+1)+(d+2)-1 = 2d+2$, a contradiction.

(c3) Assume $g = 1$. Since A spans $\mathbb{P}^3$, [4, Lemma 5.1] gives $h^1(\mathcal{I}_{W_1}(d-1)) > 0$. Therefore $\deg(W_1) \geq d+1$ and hence $e_1 \leq 2d+1$. Since $h^1(H_1, \mathcal{I}_{W_0 \cap H_1}(d)) > 0$, we have $e_1 \geq d+2$ and hence $\deg(W_1) \leq 2d$. By Lemma 1 either there is a line $L \subset \mathbb{P}^3$ such that $\deg(L \cap W_1) \geq d+1$ or there is a plane conic T with $\deg(T \cap W_1) \geq 2d$. The latter case does not arise, because it would imply $e_1 \geq 2d$ and hence $\deg(W_1) \leq d+2 < \deg(T \cap W_1)$. Therefore there is a line $L \subset \mathbb{P}^3$ such that $\deg(L \cap W_1) \geq d+1$. We continue as in step (c2). QED

4 Other results

The following example with $n = 3$ describes the schemes A appearing in the statement of Proposition 13.

Example 1. Let $A \subset \mathbb{P}^n$ be a connected zero-dimensional scheme such that $\deg(A) = n+2$ and $\langle A \rangle = \mathbb{P}^n$. Set $\{O\} := A_{\mathrm{red}}$. Assume that A is not in linearly general position. By [9, Theorem 1.3] we have $2O \subset A$ (where $2O$ is the closed subscheme of $\mathbb{P}^n$ with $(\mathcal{I}_P)^2$ as its ideal sheaf) and $\mathcal{O}_{A,O}$ is Gorenstein. Fix any hyperplane $H \subset \mathbb{P}^n$. If $O \notin H$, then $H \cap A = \emptyset$. Now assume $O \in H$. Since A is in linearly general position, we have $\deg(H \cap A) \leq n$. Since $A \supset 2O$ and $O \in H$, we have $\deg(H \cap A) \geq n$. Therefore $A \cap H = 2O \cap H$. Hence $h^1(H, \mathcal{I}_{A \cap H}(2)) = 0$ and $\deg(\mathrm{Res}_H(A)) = 2$. Hence $h^1(\mathcal{I}_{\mathrm{Res}_H(A)}(1)) = 0$. The residual exact sequence of the inclusion $H \subset \mathbb{P}^n$ gives $h^1(\mathcal{I}_A(2)) = 0$.

Proposition 13. *Assume $m \geq 3$, $d \geq 9$, and take a 3-dimensional linear subspace $\mathbb{H} \subseteq \mathbb{P}^m$. Let $A \subset \mathbb{H}$ be a connected degree 5 scheme not curvilinear and in linearly general position in $\mathbb{H}$. Then $r_{m,d}(P) \leq 4d-2$ for every $P \in \mathbb{P}^r$ whose cactus rank is evinced by A.*

Proof. By concision we may assume $m = 3$. Set $\{O\} := A_{\mathrm{red}}$. We have $h^0(\mathcal{I}_A(2)) = 5$ and $h^1(\mathcal{I}_A(2)) = 0$ (Example 1). By Castelnuovo-Mumford's lemma $\mathcal{I}_A(3)$ is spanned. Let $H \subset \mathbb{P}^3$ be a hyperplane. If $O \notin H$, then no reducible quadric with H as a component contains A. Now assume $O \in H$. Since $\deg(H \cap A) = 3$, the scheme $\mathrm{Res}_H(A)$ has degree two and hence $h^0(\mathcal{I}_{\mathrm{Res}_H(A)}(1)) = 2$. Therefore a dimensional count gives that a general $Q \in |\mathcal{I}_A(2)|$ is irreducible. Each $Q \in |\mathcal{I}_A(2)|$ is singular at O. Take another general $Q' \in |\mathcal{I}_A(2)|$ and set $T := Q \cap Q'$.

Claim 1: T is the union of 4 distinct lines.

Proof of Claim 1: Since Q and Q' are irreducible and with O as their singular point, T_{red} is a union of at most 4 lines through O and $T = T_{\mathrm{red}}$ if and only if T has no multiple line. We have $h^0(Q, \mathcal{I}_A(2)) = 4$ and for each line $D \subset Q$ (it is a line through O) we have $h^0(Q, \mathcal{I}_{A \cup D}(2)) = 3$. The effective Weil divisor

$2D$ of Q is a Cartier divisor and $2D \in |\mathcal{O}_Q(1)|$. Therefore $h^0(Q, \mathcal{I}_{A\cup 2D}(2)) = h^0(Q, \mathcal{I}_A(1)) = 0$. So Claim 1 is true just taking any $Q' \neq Q$.

By Claim 1 T is a reduced and connected curve. Write $T = L_1 \cup L_2 \cup L_3 \cup L_4$ with each L_i a line. Let $\{2O, L_i\}$ be the degree two effective divisor of L_i with O as its support. We have $A \subset T$ and hence $P \in \langle \nu_d(T) \rangle$. Therefore there are $P_i \in \langle \nu_d(L_i) \rangle$ such that $P \in \langle \{P_1, P_2, P_3, P_4\} \rangle$. Let r_i be the rank of P_i with respect to the rational normal curve $\nu_d(L_i)$. Sylvester's theorem gives that $r_i \le d$ and that equality holds if and only if P_i has cactus rank 2 and its cactus rank is not evinced by a reduced scheme (Remark 1). Let E_i be the only scheme evincing the cactus rank of P_i with respect to the rational normal curve $\nu_d(L_i)$. The points $P_1, \dots, P_4$ are not uniquely determined by P. Take $U_1 \in \langle \nu_d(L_2 \cup L_3 \cup L_4) \rangle$ such that $P \in \{P_1, U_1\}$. Since $p_a(T) = 1$, we have $\{2O, L_1\} \subset L_2 \cup L_3 \cup L_4$ and hence changing U_1 we may move P_1 to a general point $P_i' \in \langle \nu_d(\{2O_i, L_i\} \cup E_1) \rangle$. Therefore (changing simultaneously $P_1, \dots, P_4$ to some $P_1', \dots, P_4'$) we may take $\deg(E_i) \ge 3$, unless $E_i = \{2O_i, L_i\}$. Therefore to prove Proposition 13 it is sufficient to prove that (changing simultaneously $P_1, \dots, P_4$) for at most two indices i we have $E_i = \{2O_i, L_i\}$. We cannot have $E_i = \{2O_i, L_i\}$ for all i, because it would imply $P \in \langle \nu(2O) \rangle$, where $2O$ is the closed subscheme of $\mathbb{P}^3$ with $(\mathcal{I}_O)^2$ as its ideal sheaf; this would imply that P has border rank two by the proof of [5, Theorem 37] or by [6, Lemma 2.3] and the fact the $2O$ is not Gorenstein. Assume that $E_i = \{2O_i, L_i\}$ for 3 indices i, say $i = 1, 2, 3$. If $r_4 \le d - 2$, then Proposition 13 holds. Therefore we may assume $\deg(E_4) = 3$. We have $2O = E_1 \cup E_2 \cup E_3$. Since $\deg(L_i \cap E_i) > 2$ and A is in linearly general position, we have $A \neq 2O \cup E_4$. Since $2O \subset A$, we have $\deg(A \cup (2O \cup E_4)) \le 8$. Since $d \ge 7$, and $P \in \langle \nu_d(A) \rangle \cap \langle \nu_d(2O \cup E_4) \rangle$, we get a contradiction. QED

Lemma 9. *Assume $d \ge 7$ and that the degree 5 scheme evincing the cactus rank of P is not connected. Then $r_{m,d}(P) \le 3d - 1$.*

Proof. Write $A_1, \dots, A_s$, $s \ge 2$, be the connected components of A. Set $a_i := \deg(A_i)$ and assume $a_i \ge a_j$ for all $j \ge 2$. We have $5 = a_1 + \dots + a_s$. There is $P_i \in \langle \nu_d(A_i) \rangle$ such that $P \in \langle \{P_1 \dots, P_s\} \rangle$ and hence $r_{m,d}(P) \le \sum_{i=1}^{s} r_{m,d}(P_i)$. If $a_i = 1$, then $r_{m,d}(P_i) = 1$. If $2 \le a_i \le 4$, then $r_{m,d}(P_i) \le (a_i - 1)d - a_i + 2$ ([5] and [4]). Therefore in all cases we get $r_{m,d}(P) \le 3d - 1$ (we get a stronger inequality, unless $s = 2$). QED

Proposition 14. *Assume the existence of a 3-dimensional linear space $\mathbb{H} \subseteq \mathbb{P}^m$ such that $A \subset \mathbb{H}$ and A is not in linearly general position in $\mathbb{H}$. Then $r_{m,d}(P) \le 4d - 2$ for every $P \in \mathbb{P}^r$ whose cactus rank is evinced by A.*

Proof. By [2, Theorem 1] we may assume $\dim(\langle A\rangle) = 3$. Hence by concision we may assume $m = 3$. By Lemma 9 we may assume that A is connected. Set $\{O\} := A_{\mathrm{red}}$. Since A is not in linearly general position in $\mathbb{P}^3$, there is a plane $H \subset \mathbb{P}^3$ such that $\deg(A \cap H) = 4$. Since $\mathrm{Res}_H(A) = \{O\}$, we have $A \subset H \cup M$ for every plane M containing O. Since $h^0(\mathcal{I}_O(1)) = 3$ and $h^0(\mathcal{I}_A(2)) \geq 5$, a general $Q \in |\mathcal{I}_A(2)|$ has not H as a component. Fix a general plane M containing O and take $P_1 \in \langle \nu_d(Q \cap H)\rangle$ and $P_2 \in \langle \nu_d(Q' \cap M)\rangle$. Since M is general, $Q' \cap M$ is a reduced conic. Hence [11, Proposition 5.1] gives $r_{m,d}(P) \leq 2$. Since P_1 has cactus rank ≤ 4, the case $n = 2$ of [4] and concision gives $r_{m,d}(P_1) \leq 2d - 2$. Hence $r_{m,d}(P) \leq 4d - 2$. QED

5 Proof of Theorem 1

Since $d \geq 4$ and P has border rank 5, there is a zero-dimensional scheme $A \subset \mathbb{P}^m$ such that $\deg(A) = 5$, A is smoothable, $P \in \langle \nu_d(A)\rangle$ and $P \notin \langle \nu_d(E)\rangle$ for any $E \subsetneq A$ ([7, Lemma 2.6], [6, Proposition 2.5]). If A is reduced, then $r_{m,d}(P) = 5$ (we are assuming that P has border rank 5 and hence $r_{m,d}(P) \geq 5$). Since $\deg(A) = 5$, we have $\dim(\langle A\rangle) \leq \min\{m, 4\}$. By concision ([10, Exercise 3.2.2.2]) we may assume $m = \dim(\langle A\rangle)$. The case $\dim(\langle A\rangle) = 4$ is the main result of [2] (but for this paper we only need the easier upper bound for the rank). If A is not connected, then $r_{m,d}(P) \leq 3d-1$ by Lemma 9. If $\dim(\langle A\rangle) = 2$, then $r_{m,d}(P) \leq 3d$ (Lemma 2). Therefore we may assume $\dim(\langle A\rangle) = 3$ and that A is connected. If A is in linearly general position in $\langle A\rangle$ and not curvilinear, then $r_{m,d}(P) \leq 4d - 2$ by Proposition 13. If A is connected, curvilinear and in linearly general position in $\mathbb{P}^3$, then $r_{m,d}(P) = 3d - 3$ (Proposition 3). If A is not in linearly general position in $\mathbb{P}^3$, then $r_{m,d}(P) \leq 4d - 2$ (Proposition 14). QED

Acknowledgements. Many thanks to a referee for several very useful suggestions.

References

[1] E. Ballico: *Subsets of the variety $X \subset \mathbb{P}^n$ evincing the X-rank of a point of $\mathbb{P}^n$*, Houston J. Math. **42** (2016), no. 3, 803–824.

[2] E. Ballico: *The stratification by rank for homogeneous polynomials with border rank 5 which essentially depend on at least 5 variables*, Acta Math. Vietnam. **42** (2017), 509–531.

[3] E. Ballico, A. Bernardi: *Decomposition of homogeneous polynomials with low rank*, Math. Z. **271** (2012), 1141–1149.

[4] E. Ballico, A. Bernardi: *Stratification of the fourth secant variety of Veronese variety via the symmetric rank*, Adv. Pure Appl. Math. **4** (2013), no. 2, 215–250.

[5] A. Bernardi, A. Gimigliano, M. Idà: *Computing symmetric rank for symmetric tensors*, J. Symbolic Comput. **46** (2011), no. 1, 34–53.

[6] W. Buczyńska, J. Buczyński: Secant varieties to high degree Veronese reembeddings, catalecticant matrices and smoothable Gorenstein schemes, J. Algebraic Geom. **23** (2014) 63–90.

[7] J. Buczyński, A. Ginensky, J. M. Landsberg: *Determinantal equations for secant varieties and the Eisenbud-Koh-Stillman conjecture*, J. London Math. Soc. (2) **88** (2013) 1–24.

[8] G. Comas, M. Seiguer: *On the rank of a binary form*, Found. Comp. Math. **11** (2011), no. 1, 65–78.

[9] D. Eisenbud, J. Harris: *Finite projective schemes in linearly general position*, J. Algebraic Geom. **1** (1992), no. 1, 15–30.

[10] J. M. Landsberg: Tensors: Geometry and Applications, Graduate Studies in Mathematics, Vol. 128, Amer. Math. Soc. Providence, 2012.

[11] J. M. Landsberg, Z. Teitler: *On the ranks and border ranks of symmetric tensors*, Found. Comput. Math. **10** (2010), no. 3, 339–366.