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On Autonilpotent and Autosoluble Groups

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Abstract. In this paper, we investigate the properties of lower autonilpotent groups and we determine the structure of autonilpotent groups. We also determine necessary condition for a group to be autosoluble group.

Keywords: Lower Autonilpotent Groups, Autonilpotent Groups, Autosoluble Groups.

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1 Introduction

Let G' and $Z(G)$ denote the derived group and center of a group G, respectively. Let $Aut(G)$ and $Inn(G)$ denote the group of all automorphisms of G and the group of all inner automorphisms of G, respectively. Let $G^* = K(G) =< [g, \alpha] = g^{-1}\alpha(g) | g \in G, \alpha \in Aut(G) >$, be the autocommutator subgroup of G and $L(G) = \{g \in G | \alpha(g) = g, \alpha \in Aut(G)\}$, be the autocenter of G. The concepts of autocommutator and autocenter have been studied in [2]. The autocommutator of higher weight is defined inductively in [3] as follows:
$[g, \alpha_1, \alpha_2, ..., \alpha_i] = [[g, \alpha_1, \alpha_2, ..., \alpha_{i-1}], \alpha_i]$, for all $\alpha_1, \alpha_2, ..., \alpha_i \in Aut(G)$, $i \geq 2$.

The autocommutator subgroup of weight i is defined as:
$K_i(G) = [G, \underbrace{Aut(G), Aut(G), ..., Aut(G)}_{i\,times}] =< [g, \alpha_1, \alpha_2, ..., \alpha_i] | g \in G, \alpha_1, \alpha_2, ..., \alpha_i \in Aut(G) >$.
Clearly $K_i(G)$ is a characteristic subgroup of G, for all $i \geq 1$. Thus, we have a descending chain of autocommutator subgroups $G = K_0(G) \supseteq K_1(G) \supseteq K_2(G) \supseteq ... \supseteq K_i(G) \supseteq ...$, called the lower autocentral series of G. Similarly to [3] the upper autocentral series also has been defined as follows: $1 = L_0(G) \subseteq L_1(G) = L(G) \subseteq ... \subseteq L_n(G) \subseteq ...$, where $\frac{L_n(G)}{L_{n-1}(G)} = L(\frac{G}{L_{n-1}(G)})$. In particular if we take

the group of inner automorphisms, we obtain the usual lower and upper central series of G. A group G is said to be autonilpotent group of class n if $L_n(G) = G$ and $L_{n-1}(G) < G$, for some natural number n. It has been proved in [3] that a finite abelian group is autonilpotent if and only if it is a cyclic 2-group. It is easy to check that if G is an autonilpotent group of class n, then $K_n(G) = 1$. In [4], the authors gave the concept of A-nilpotency. We use the term lower autonilpotent group instead of A-nilpotent group in the present paper as the concept is derived from the lower autocentral series. A group G is said to be lower autonilpotent of class n if $K_n(G) = 1$ and $K_{n-1}(G) \neq 1$. Thus an autonilpotent group is always lower autonilpotent but the converse need not be true. For example the dihedral group of order 8 is lower autonilpotent of class 3, but not autonilpotent. Autosoluble groups have already been discussed in [5]. The authors define a descending series $G = K^{(0)}(G) \geq K^{(1)}(G) \geq \cdots \geq K^{(n)}(G) = 1$ of subgroups of G inductively as follows: $K^{(0)}(G) = G, K^{(1)}(G) = K(G)$ and $K^{(n)}(G) =< [g, \alpha] | g \in K^{(n-1)}(G), \alpha \in Aut(K^{(n-1)}(G)) >$, for all $n \geq 2$. A group G is said to be autosoluble of length n if $K^{(n)}(G) = 1$ and $K^{(n-1)}(G) \neq 1$. It is clear that $G^{(n)} \leq \gamma_{n+1}(G) \leq K_n(G) \leq K^{(n)}(G)$ and $L_n(G) \leq Z_n(G)$, for all natural numbers n, where $G^{(n)}, \gamma_n(G)$ and $Z_n(G)$ denote the terms of the derived series, lower central series and the upper central series of G respectively. Thus we find that a lower autonilpotent group of class n is nilpotent of class at most n and an autosoluble group of length n is soluble of derived length at most n and also nilpotent of class at most n. The second section of this paper deals with the properties of lower autonilpotent groups and the third section is devoted to the study of autonilpotent and autosoluble groups. The main results that are proved in this paper are the following:

Theorem 3.2. Let G be an lower autonilpotent abelian group. Then G is a finite 2-group.

Theorem 3.1. A group G is autonilpotent if and only if it is a cyclic 2-group.

Theorem 3.4. A finite autosoluble group is a 2-group.

Theorem 3.6. If G is a finite autosoluble of length 2, then $G \cong C_4$.

2 Properties of lower autonilpotent group

In this section we aim to present some results on lower autonilpotent groups. These results provide some necessary conditions for a group to be lower autonilpotent.

"Unless specifically mentioned, throughout the paper all groups are considered assumed nontrivial "

Proposition 1. Let G be a lower autonilpotent group, then $L(G) \neq 1$.

Proof. Let G be a lower autonilpotent of class n. Then $K_n(G) = 1$ and $K_{n-1}(G) \neq 1$, for some $n \geq 1$. Since $K_n(G) = [K_{n-1}(G), Aut(G)]$, it follows that $K_{n-1}(G) \leq L(G)$. QED

Proposition 2. For any nontrivial group G, $K(G) = 1$ implies $L(G) = G \cong C_2$.

The proof is straight forward.

Remark 3. In a lower autonilpotent group, the lower autocentral factors are central.

Lemma 4. Let G be a lower autonilpotent group. Then $L(G)$ intersects non trivially with every nontrivial characteristic subgroup of G.

Proof. Let $1 \neq N$ be a characteristic subgroup of G. Suppose G is a lower autonilpotent group of class n. Therefore $K_n(G) = 1$, and $K_{n-1}(G) \neq 1$. Since $K_n(G) = 1$, therefore $K_{n-1}(G) \leq L(G)$. Suppose $N \cap L(G) = 1$. Thus $N \cap K_{n-1}(G) = 1$. Since $N \cap K_0(G) = N \cap G \neq 1$, it follows that there exists an integer $i \geq 0$ such that $N \cap K_i(G) \neq 1$ but $N \cap K_{i+1}(G) = 1$. Now $[K_i(G) \cap N, Aut(G)] \subseteq K_{i+1}(G) \cap N = 1$, implies $N \cap K_i(G) \leq L(G)$. Therefore $N \cap L(G) \neq 1$. QED

Corollary 5. A minimal characteristic subgroup of a lower autonilpotent group is contained in $L(G)$.

Lemma 6. In a finite lower autonilpotent group G, the autocenter $L(G)$ of G intersects every maximal subgroup of G.

Proof. Let G be a lower autonilpotent group and M be a maximal subgroup of G. We know that every autonilpotent group is nilpotent, therefore G is nilpotent and thus M is normal and G/M is of prime order and hence abelian. It follows that $G' \leq M$. Since G' is a characteristic subgroup of G, by the above lemma, $L(G) \cap G' \neq 1$. Thus $L(G) \cap M \neq 1$. QED

Lemma 7. If G is a finite lower autonilpotent group, then every prime divisor of the order of G divides the order of $L(G)$.

Proof. Since G is lower autonilpotent, therefore it is nilpotent. Let the prime p divides the order of G. Let P be a Sylow p-subgroup of G. Then P is characteristic in G. It follows that $L(G) \cap P \neq 1$. Therefore p divides the order of $L(G)$. QED

Proposition 8. If G is a finite lower autonilpotent p-group, then $L(G) \leq \Phi(G)$, where $\Phi(G)$ is the Frattini subgroup of G.

Proof. Suppose there exists $x \in G$ such that $x \in L(G)$ but $x \notin \Phi(G)$. Then there exists a maximal subgroup M of G such that $x \notin M$. Therefore $G = M < x >$. Since G is nilpotent, therefore G/M is of prime order. Also $L(G) \cap M \neq 1$, there exists an element $z \in M \cap L(G)$ of order p. Since every element of G has a unique representation of the form mx^k, $m \in M, 0 \leq k \leq p-1$. It follows that the map

$$\phi : G \to G \text{ defined by}$$
$$\phi(mx^k) = mx^k z^k, \text{ where } k \in \{0, 1, 2, ..., p-1\}.$$

is an automorphism of G. We have $\phi(x) = xz$. Since $x \in L(G)$, therefore $\phi(x) = x$. It follows that $z = 1$, a contradiction. QED

Proposition 9. If G is a finite lower autonilpotent group, then $L(G) \leq \Phi(G)$.

Proof. Since G is nilpotent and therefore is the direct product of its Sylow p-subgroups and each Sylow p-subgroup is characteristic in G. Therefore the autocenter and the Frattini subgroup of G is the direct product of the autocenters and the Frattini subgroups of the corresponding Sylow p-subgroups of G, respectively. By the above proposition, the result follows. QED

Remark 10. The lower autonilpotent groups in which $L(G)$ and $Z(G)$ are identical are purely non-abelian, as $Z(G) = L(G) \leq \Phi(G)$.

Lemma 11. Let G be a lower autonilpotent group of class 2. Then

(1) $G^* \leq L(G)$.

(2) If $exp(G/L(G)) = m$, then $exp(G^*) = m$

Proof. Let $g \in G$, $\alpha \in Aut(G)$ then $[g, \alpha] \in G^*$. Since G is lower autonilpotent of class 2, $K_2(G) = 1$ implies $[g, \alpha, \beta] = [[g, \alpha], \beta] = 1$, for all $\beta \in Aut(G)$.
Thus, $g^{-1}\alpha(g) \in L(G)$, for all $g \in G$ and for all $\alpha \in Aut(G)$. Therefore (1) holds. Since G^* is generated by elements of the form $g^{-1}\alpha(g)$, the result follows.

To prove (2), suppose $exp(G/L(G)) = n$ and $exp(G^*) = m$. Since G^* is an abelian group generated by the elements of the form $g^{-1}\alpha(g), g \in G, \alpha \in Aut(G)$, therefore $exp(G^*) = m$ implies that $(g^{-1}\alpha(g))^m = 1$, for all $g \in G$ and

$\alpha \in Aut(G)$. Since $(g^{-1}\alpha(g))^m = g^{-m}\alpha(g)^m$

For all $g \in G$ and $\alpha \in Aut(G)$ we have $g^{-m}\alpha(g)^m = 1$, and so $\alpha(g^m) = g^m$. This implies $g^m \in L(G)$, for all $g \in G$ and so $exp(G/L(G)) \leq exp(G^*)$.
Now $exp(G/L(G)) = n$, therefore $g^n \in L(G)$, for all $g \in G$. It follows that $\alpha(g^n) = x^n$ and so $({}^{g}-1\alpha(g))^n = 1$, for all $g \in G, \alpha \in Aut(G)$. Therefore, $exp(G^*) \leq exp(G/L(G))$.

Thus, $exp(G^*) = exp(G/L(G))$. QED

Proposition 12. Let G be a group. Then,

(1) If $L(G)$ is torsion free, then the factor groups $\frac{L_i(G)}{L_{i-1}(G)}$ and subgroups $L_i(G)$ are torsion free for each $i \geq 1$,

(2) If $G/K_1(G)$ is torsion, then so is each lower autocentral factor $K_i(G)/K_{i+1}(G)$,

(3) If G is a lower autonilpotent group, then G is torsion free if and only if L(G) and $K(G) \cap Z(G)$ are torsion free,

(4) If G is a lower autonilpotent group, then G is torsion if and only if $G/K_1(G)$ is torsion.

Proof. (1) Let $x \in L_i(G)$. Then $x^{-1}\alpha(x) \in L_{i-1}(G)$, for all $\alpha \in Aut(G)$. Suppose $i = 2$. Then $x^{-1}\alpha(x) \in L(G)$, for all $\alpha \in Aut(G)$. Since $|x| = |\alpha(x)|$ and $L(G)$ is torsion free as well as abelian , therefore $|x|$ should be infinite as $|x^{-1}\alpha(x)| \leq |x|$ Therefore $L_2(G)$ is torsion free. Now it can be proved by induction that each $L_i(G)$ is torsion free. Let $xL_{i-1}(G) \in \frac{L_i(G)}{L_{i-1}(G)}$ .Then $x^{-1}\alpha(x) \in L_{i-1}(G)$ and $|x|$ is infinite. We have $x^{-1}\alpha(x) \in L_{i-1}(G)$ is torsion free for each $i \geq 1$.

(2) We prove the result for $i = 1$ and the result for higher values of i will follow on the same lines. Let $xK_2(G) \in K_1(G)/K_2(G)$, where $x \in K_1(G)$. Since the lower autocentral factor $K_1(G)/K_2(G)$ is central, therefore it is sufficient to prove that generators are of finite order. Suppose $xK_2(G) = g^{-1}\alpha(g)K_2(G)$, for some $g \in G, \alpha \in Aut(G)$. Since $G/K_1(G)$ is torsion, there exists a positive integer m, such that $g^m \in K_1(G)$. We have $x^mK_2(G) = (g^{-1}\alpha(g))^mK_2(G) = g^{-m}\alpha(g^m)K_2(G) = K_2(G)$. Thus $K_1(G)/K_2(G)$ is a torsion group.

(3) It is clear if G is torsion free. Then $L(G)$ and $K(G) \cap Z(G)$ are torsion free. Conversely, suppose $L(G)$ and $K(G) \cap Z(G)$ are torsion free. Since G is nilpotent, $G/L(G)$ is nilpotent. We prove that $Z(G/L(G))$ is torsion free. Suppose $xL(G) \in Z(G/L(G))$ is of finite order, say m, then $x^m \in L(G)$. Now for any $y \in G$, we have $xy = yxt$, for some $t \in L(G)$. Thus, $x^my = yx^mt^m$. Since

$x^m \in L(G) \leq Z(G)$, we get $t^m = 1$, but then $t = 1$. This implies $xy = yx$. Therefore $x \in Z(G)$. We claim $x \in L(G)$. Suppose there exists $\alpha \in Aut(G)$ such that $\alpha(x) \neq x$. We have $1 \neq x^{-1}\alpha(x) \in K(G) \cap Z(G)$. We get $\alpha(x) = xz$, for some $1 \neq z \in Z(G) \cap K(G)$. This implies $x^m = \alpha(x^m) = x^m z^m$. It follows that $z^m = 1$, which implies $K(G) \cap Z(G)$ is not torsion free, a contradiction. Thus $x \in L(G)$. Hence $Z(G/L(G))$ is torsion free, but then $G/L(G)$ is torsion free, also $L(G)$ is torsion free. Hence G is torsion free.

(4) Let G be a lower autonilpotent group of class n. Suppose G is torsion, this implies $G/K_1(G)$ is torsion. Conversely, suppose $G/K_1(G)$ is torsion. We prove that $G/K_i(G)$ is torsion, for each $i \geq 2$. We prove the result for $i = 2$, and the claim will follow by a simple induction on i. Now by(2) $K_1(G)/K_2(G)$ is torsion, also $G/K_1(G)$ is torsion, and this implies $G/K_2(G)$ is torsion. In particular $G = G/K_n(G)$ is torsion. QED

Corollary 13. If G is a group such that $exp(L(G)) = m$, then $\frac{L_i(G)}{L_{i-1}(G)}$ has exponent dividing m.

Theorem 14. Let M be a maximal characteristic subgroup of G such that $G^* \leq M$. Then M is maximal in G. In particular, if G is lower autonilpotent, then every maximal characteristic subgroup is maximal.

Proof. Take $x \in G \backslash M$, and let $A =< \alpha(x) | \alpha \in Aut(G) >$. Then A is a characteristic subgroup of G and A is not contained in M. Therefore $G = MA$. Now $x^{-1}\alpha(x) \in G^* \leq M$, for all $\alpha \in Aut(G)$. Therefore $A \leq < x > M$. It follows that G/M is cyclic and G/M is also characteristically simple, but then $|G/M| = p$, for some prime p. Hence M is a maximal subgroup of G. QED

Theorem 15. If H is a characteristic subgroup of a group G such that H and G/H are lower autonilpotent, then G is lower autonilpotent.

Proof. Let H and G/H be lower autonilpotent groups of classes m and n, respectively. Let $x \in G$, and $\alpha_1, \alpha_2, ..., \alpha_n \in Aut(G)$. Then $\bar{\alpha}_1, \bar{\alpha}_2, ..., \bar{\alpha}_n \in Aut(G/H)$, where $\bar{\alpha}(xH) = \alpha(x)H$, for $\alpha \in Aut(H)$. Then $[xH, \bar{\alpha}_1, \bar{\alpha}_2, ..., \bar{\alpha}_n] = H$. This implies $[x, \alpha_1, \alpha_2, ..., \alpha_n] \in H$. Thus $K_n(G) \leq H$. Since $K_m(H) = 1$ and H is characteristic in G, we have $K_{n+m}(G) = 1$. QED

Definition 16. [Autonormalizer] Let G be a group and H be a subgroup of G. The subset $\overline{N_G(H)} = \{x \in G | [x, \alpha] \in H$, for all $\alpha \in Aut(G)\}$ of G is called the autonormalizer of H in G.

We prove that $\overline{N_G(H)}$ is a subgroup of G contained in $N_G(H)$(the normalizer of H in G).

Since $x \in \overline{N_G(H)}$ implies $[x, \alpha] \in H$, for all $\alpha \in Aut(G)$, and this implies that $[x, f_h] \in H$, for all $f_h \in Inn(G), h \in H$. It follows that $x^{-1}h^{-1}xh = x^{-1}(f_h(x)) \in H$, thus $x^{-1}h^{-1}x \in H$. Thus $x^{-1}Hx \leq H$. We have $x^{-1}Hx \leq H$, for all $x \in \overline{N_G(H)}$. Let $x, y \in \overline{N_G(H)}$. Then

$$\begin{aligned} [xy, \alpha] &= (xy)^{-1}\alpha(xy) \\ &= y^{-1}x^{-1}\alpha(x)\alpha(y) \\ &= y^{-1}x^{-1}\alpha(x)yy^{-1}\alpha(y) \\ &= (y^{-1}x^{-1}\alpha(x)y)(y^{-1}\alpha(y)) \end{aligned}$$

Since $y^{-1}x^{-1}\alpha(x)y \in H$, therefore $[xy, \alpha] \in H$. We have $xy \in \overline{N_G(H)}$. Also $[x, \alpha] = x^{-1}\alpha(x) \in H$, implies that $x^{-1}\alpha(x)x \in Hx$. From this it follows that $(f_x\alpha)(x) \in H$, where f_x is inner automorphism induced by x. Hence $\beta(x) \in Hx$, for all $\beta \in Aut(G)$. We get $\beta(x)x^{-1} \in H$, this gives $x\beta(x^{-1}) = [x^{-1}, \beta] \in H$, for all $\beta \in Aut(G)$. Thus we get $x^{-1} \in \overline{N_G(H)}$, showing that $\overline{N_G(H)}$ is a subgroup of G. Also for $x \in \overline{N_G(H)}$, $x^{-1}Hx \leq H$ and $\overline{N_G(H)}$ is a subgroup of G, therefore $x^{-1}Hx = H$. Hence $x \in N_G(H)$. Thus $\overline{N_G(H)}$ becomes a subgroup of G contained in $N_G(H)$
Clearly if H is a characteristic subgroup of G, then $\overline{N_G(H)}$ is a characteristic subgroup of G containing H.

Proposition 17. A finite group G is lower autonilpotent if and only if every proper characteristic subgroup of G is a proper subgroup of its autonormalizer.

Proof. Let G be a lower autonilpotent group of class n. Then $K_0(G) = G$ and $K_n(G) = 1$. Suppose H is a proper characteristic subgroup of G. There exists a positive integer i, such that $K_i(G) \leq H$ but $K_{i-1}(G)$ is not contained in H. Thus, there exists $x \in K_{i-1}(G)$ such that $x \notin H$. Then $[x, \alpha] \in [K_{i-1}(G), \alpha] \leq K_i(G) \leq H$, for all $\alpha \in Aut(G)$. Therefore $x \in \overline{N_G(H)}$.

Conversely, suppose every proper characteristic subgroup of G is a proper subgroup of its autonormalizer. Define $H_0 = 1$ and $H_1 = \overline{N_G(H_0)}$ and $H_i = \overline{N_G(H_{i-1})}$ for $i \geq 1$. Thus we have a proper ascending chain $\{H_i\}_{i \geq 0}$ of characteristic subgroups of G. Since G is finite, there exists a positive integer n such that $H_n = G$. Since $G = H_n = \overline{N_G(H_{n-1})}$, we have $K_1(G) \leq H_{n-1}$. Now a simple induction shows that $K_i(G) \leq H_{n-i}$, for all i,$1 \leq i \leq n$. In particular, $K_n(G) \leq H_0 = 1$. Therefore G is lower autonilpotent.

QED

3 Autonilpotent groups and Autosoluble groups

This section is mainly devoted to find the structure of an autonilpotent group, an abelian lower autonilpotent group and a finite autosoluble group. M. R.R. Moghaddam [3] proved that a finite abelian group is autonilpotent if and only if it is a cyclic 2-group. We prove the following theorem:

Theorem 18. A group G is autonilpotent if and only if it is a cyclic 2-group.

To prove this result, we need the following ,

Theorem 19. Let G be an abelian lower autonilpotent group. Then G is a finite 2-group.

Proof. Let G be an abelian lower autonilpotent group of class n. First we prove that G is a 2-group with bounded exponent. Let $x \in G$. Since the map α which sends an element to its inverse is an automorphism of G, we have $x^{2^n} \in K_n(G) = 1$. Thus G is a 2-group with bounded exponent.
Further we prove that G is a finite group. Suppose $exp(G) = 2^m$. Then $G = \Sigma_{\lambda \in \Lambda} A_\lambda$ is the direct sum of cyclic subgroups A_λ each of order less than equal to 2^m. We may write $G = B_1 \oplus B_2 \oplus ... \oplus B_m$, where each B_i is the direct sum of cyclic groups each of order 2^i. We prove that each B_i is a cyclic group of order 2^i. Suppose $B_i =< x_1 > \oplus < x_2 > \oplus B'$, where $|x_1| = |x_2| = 2^i$ by collecting $A'_\lambda s$ of same same order . Then $G =< x_1 > \times < x_2 > \oplus B$, for some subgroup B of G. Define a map,

$$f : G \to G \ , \quad \text{by } f(x_1^r x_2^s) = x_1^r x_2^{r+s}, \ 0 \leq r, s \leq 2^i \text{ and identity on } B.$$

Then f is an automorphism of G. We have $x_2 = x_1^{-1} f(x_1) \in K_1(G)$, this gives $x_2 \in K_1(G)$. Similarly, by interchanging the roles of x_1 and x_2, we have $x_1 \in K_1(G)$. We get $x_1, x_2 \in K_2(G)$. Proceeding in this way, we have both $x_1, x_2 \in K_n(G)$, a contradiction. Therefore each B_i is cyclic of order 2^i, but then G is a finite group and hence G is a finite 2-group. QED

Lemma 20. If G is autonilpotent of class n, then $G/L(G)$ is autonilpotent of class at the most $n - 1$.

Proof. It is easy to verify that $L_i(G/L_1(G)) = L_{i+1}(G)/L(G)$ by induction on i. Thus the result follows. QED

Proof. (*Theorem 3.1*) Let G be an autonilpotent group of class n. We prove the result by induction on n. For $n = 1$, $G = L_1(G) = L(G)$, therefore $G \cong C_2$. Therefore the result is true when $n = 1$. Suppose $n \geq 2$. Now $G/L(G)$ is autonilpotent group of class at most $n - 1$, by induction hypothesis, $G/L(G)$

is a cyclic 2-group. Since $L(G) \leq Z(G)$, G is an abelian group. Thus G is an abelian group in which $K_n(G) = 1$. Using the theorem 19, G is a finite abelian 2-group. This implies by [3], that G is a cyclic 2-group.

QED

3.1 Autosoluble Groups

Further we characterize finite autosoluble groups. F.Parvaneh and M. R. R. Moghaddam [5] proved that a finite abelian group is autosoluble if and only if it is a cyclic 2-group. We find the structure of finite autosoluble groups.

Theorem 21. A finite autosoluble group is a 2-group.

Proof. Let G be a finite autosoluble group of length n. Then $K^{(n)}(G) = 1$. Suppose $n = 1$. Then $K^{(1)}(G) = K_1(G) = 1$, this implies $L(G) = G$, and therefore $G \cong C_2$. For $n = 2$, $K^{(2)}(G) = 1$ implies $[K^{(1)}(G), Aut(K^{(1)}(G))] = 1$, thus $K^{(1)}(G) \cong C_2$. Since $\gamma_2(G) \leq K^{(1)}(G)$, either $\gamma_2(G) = 1$ or $\gamma_2(G) \cong C_2$. Thus $\gamma_2(G)$ is a 2- group. In the first case G is abelian, therefore [5] G is a cyclic 2-group. In the later case $|G|$ is even. Since G is autosoluble, therefore G is nilpotent and hence G is the direct product of its Sylow p-subgroups and each Sylow p-subgroup is characteristic. Let $G = A \times G_1 \times G_2 \times ... \times G_k$, where A is the Sylow 2-subgroup of G and G_i's are the Sylow subgroups of G of odd prime power order. Since $\gamma_2(G) \cong C_2$, we have $\gamma_2(G_i)$ is trivial for each i, $1 \leq i \leq k$. Therefore the G_i's are abelian. Let $\alpha \in Aut(G_i)$ be the automorphism which sends elements to their inverses, then this automorphism can be extended to an automorphism of G. Now if $x \in G_i$, we get $x^{-2} = [x, \alpha] \in K(G) = K^{(1)}(G)$. Thus $x \in K(G) \cong C_2$, for the order of x is odd, a contradiction. Therefore each G_i. is trivial. Thus G is a 2-group. Suppose $n \geq 2$. Since G is autosoluble of length n, $K^{(1)}(G) = K(G)$ is an autosoluble group of length $n-1$. By induction $K(G)$ is a 2-group. As $\gamma_2(G) \leq K(G)$, it follows that $\gamma_2(G)$ is a 2-group, now by proceeding on the same lines as above, we get that G is a 2-group. QED

We know that an extra special p-group is the central product of non-abelian subgroups of order p^3 [6]. For $p = 2$, it is either the central product of $D_8's$ (Dihedral group of order 8) or a central product of $D_8's$ and a single Q_8(Quaternion group of order 8).

Lemma 22. Let G be an extra special 2-group. Then any automorphism of its central factor can be extended to an automorphism of G.

Proof. We prove the result for G which is the central product of two subgroups. Let G be the central product of G_1 and G_2. Therefore $G_1 \cap G_2 \leq Z(G) =$

$G' \cong C_2$. Let $\sigma_1 : G_1 \to G_1$ be an automorphism of G_1. Define $\sigma : G \to G$ by $\sigma(g_1 g_2) = \sigma_1(g_1)g_2$. Since $Z(G)$ is a characteristic subgroup and $|Z(G)| = 2$, it follows that σ fixes the elements of $Z(G)$ and therefore σ is an automorphism of G, QED

Theorem 23. If G is a finite autosoluble of length 2, then $G \cong C_4$.

Proof. Let G be a finite autosoluble group of length 2. If G is abelian, then by [5] G is a cyclic 2-group. Since $K^{(2)}(G) = 1$, we have $G \cong C_4$.
We claim that there is no non-abelian autosoluble group of length 2. Suppose this is not true. We take G to be non-abelian autosoluble group of length 2 having smallest order. By the preceding theorem, G is a 2-group. Since $K^{(2)}(G) = 1$, we have $\gamma_2(G) = K(G) \cong C_2$. Also G is nilpotent group of class 2. Therefore $\gamma_2(G) \leq Z(G)$. Now if $\gamma_2(G) = Z(G)$, then G is an extra special 2-group. Therefore G is central product of D_8s or Q_8s, and any automorphism of central factor can be extended to automorphism of G. Therefore $K^{(2)}(G) \neq 1$, as $K^{(2)}(D_8) \neq 1$, $K^{(2)}(Q_8) \neq 1$, a contradiction. Therefore $Z(G) \neq \gamma_2(G)$. Hence $\gamma_2(G) < Z(G)$.
Let $z \in Z(G)$ be of order 2. Let M be maximal subgroup of G.Then $G = M < x >$, for any $x \notin M$. Since G is a nilpotent 2- group, we have $|G/M| = 2$, and so $x^2 \in M$. The map f which sends mx^i to $mx^i z^i$, for $i = 0, 1$ is an automorphism of G. So, $z = [x, f] \in K(G) = \gamma_2(G)$. Thus each element in $Z(G)$ of order 2 belong to $\gamma_2(G)$.
Further we prove $Z(G)$ is cyclic. Suppose $Z(G) =< x_1 > \times < x_2 > \times ... \times < x_n > (n \geq 2)$. Let $|x_i| = 2^{k_i}, 1 \leq i \leq n$.. Without loss of generality, suppose $k_1, k_2 > 1$. Then $|x_1^{2^{k_1-1}}| = 2 = |x_2^{2^{k_2-1}}|$, therefore lies in $\gamma_2(G)$, this implies both are same, as $\gamma_2(G)$ has only one nontrivial element. Thus $x_1^{2^{k_1-1}} = x_2^{2^{k_2-1}} \in< x_1 > \cap < x_2 >= 1$, a contradiction. Therefore $Z(G)$ must be cyclic and $|Z(G)| \geq 4$.
Next we claim $exp(G/\gamma_2(G)) \geq 4$. Suppose $G/\gamma_2(G)$ is an elementary abelian 2-group. Then each element in G has order at most 4. In particular $|Z(G)| = 4$ as $Z(G)$ is cyclic. Let $Z(G) =< x >$. Now $G/\gamma_2(G) = Z(G)/\gamma_2(G) \times A/\gamma_2(G)$. Therefore $G = A < x >$, where $x^2 \in \gamma_2(G)$. Let A be abelian, then G is abelian, a contradiction. Therefore A is non-abelian, hence $1 \neq \gamma_2(A) = \gamma_2(G)$. Any automorphism of A can be extended to an automorphism of G. For if $\sigma_1 : A \to A$ be an automorphism of A. Define $\sigma : G \to G$ by $\sigma(ax^i) = \sigma_1(a)x^i$, for $i = 0, 1$. Then it is easy to check σ is automorphism of G. Thus $K^{(2)}(G) = 1$, this implies $K^{(2)}(A) = 1$. Since $|A| < |G|$, and G is non-abelian autosoluble of length 2 of least order, this is a contradiction, hence $exp(G/\gamma_2(G)) \geq 4$. Suppose $G/\gamma_2(G) =< y_1\gamma_2(G) > \times < y_2\gamma_2(G) > \times ... < y_r\gamma_2(G) >$, where $|y_1\gamma_2(G)| = 2^k, k \geq 2$. Take $M =< y_2, y_3, ...y_r, \gamma_2(G) >$. Then $G = M < y_1 >$,

$y_1^{2^k} \in \gamma_2(G) \leq M$. Choose an element z in $Z(G)$ of order 4. The map $\beta : G \to G$, defined by $\beta(my_1^i) = my_1^i z^i$, is an automorphism of G. Now $\beta(y_1) = y_1 z$, so $z \in K(G) = \gamma_2(G)$, a contradiction. Therefore the only possibility for G is to be abelian. $\boxed{QED}$

Remark: (1) A lower autonilpotent group need not be autosoluble group. $C_4 \times C_2$ is an example of such a kind of the group. Here $K_3(G) = 1$, but the group is not autosoluble.
(2) An autosoluble group need not be a cyclic 2-group. D_8 is an example of such a kind of the group.

We conclude our paper with the following questions which are helpful for the readers to carry out further research in this area.
1. Does there exist any odd order lower autonilpotent group?
2. What are the sufficient conditions for a group to be autosoluble (resp. lower autonilpotent)?
3. Are there only finitely many groups up to isomorphisms which are autosoluble (resp.lower autonilpotent) of class n, for each natural number n.

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