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Homogeneous geodesics and g.o. manifolds

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Abstract. Let M be a homogeneous pseudo-Riemannian manifold, affine manifold, or Finsler space. A homogeneous geodesic is an orbit of a 1-parameter group of isometries, respectively, of affine diffeomorphisms. A homogeneous manifold is called a g.o. manifold if all geodesics are homogeneous.

Homogeneous geodesics were studied first in Riemannian manifolds using the algebraic tool called geodesic lemma. This was generalized for Finsler spaces and also for pseudo-Riemannian reductive manifolds, where new phenomena appear. The non-reductive homogeneous pseudo-Riemannian manifolds can be studied in the broader context of homogeneous affine manifolds using the more fundamental affine method based on Killing vector fields. In Finsler geometry, both the algebraic approach to reductive spaces and the affine approach can be used.

The present paper is a survey on the interesting phenomena and examples related with the existence of homogeneous geodesics and with g.o. manifolds in Riemannian, pseudo-Riemannian, affine and Finsler geometry.

Keywords: Homogeneous manifold, homogeneous geodesic, Killing vector field

MSC 2000 classification: primary 53C30, secondary 53C22

1 Introduction and background

Let (M, g) be a pseudo-Riemannian manifold. If there is a connected Lie group $G \subset I_0(M)$ which acts transitively on M as a group of isometries, then M is called a *homogeneous pseudo-Riemannian manifold*. It can be naturally identified with the *pseudo-Riemannian homogeneous space* $(G/H, g)$, where H is the isotropy group of the origin $p \in M$.

If the metric g is positive definite, then $(G/H, g)$ is always a *reductive homogeneous space*: We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H respectively and consider the adjoint representation $\mathrm{Ad}\colon H \times \mathfrak{g} \to \mathfrak{g}$ of H on $\mathfrak{g}$. There exists a *reductive decomposition* of the form $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ where $\mathfrak{m} \subset \mathfrak{g}$ is a vector subspace such that $\mathrm{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$. For a fixed reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ there is the natural identification of $\mathfrak{m} \subset \mathfrak{g} = T_eG$ with the tangent space T_pM via the projection $\pi\colon G \to G/H = M$. Using this natural identification and the scalar product g_p on T_pM, we obtain the invariant scalar product $\langle\, , \rangle$ on $\mathfrak{m}$.

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If the metric g is indefinite, the reductive decomposition may not exist (see for instance [28] or [29] for examples of nonreductive pseudo-Riemannian homogeneous spaces). In such a case, we can study the manifold M using a more fundamental affine method based on Killing vector fields, which was proposed in [13] and [26] and which we shall explain in Section 3 of the present paper.

A geodesic $\gamma(s)$ through the point p is *homogeneous* if it is an orbit of a one-parameter group of isometries. More explicitly, if s is an affine parameter and $\gamma(s)$ is defined in an open interval J, there exists a diffeomorphism $s = \varphi(t)$ between the real line and the open interval J and a nonzero vector $X \in \mathfrak{g}$ such that $\gamma(\varphi(t)) = \exp(tX)(p)$ for all $t \in \mathbb{R}$. The vector X is called a *geodesic vector*. The diffeomorphism $\varphi(t)$ may be nontrivial only for null curves in a properly pseudo-Riemannian manifold.

In the reductive case, geodesic vectors are characterized by the following *geodesic lemma* (see [35] for the Riemannian version, [28] for the first formulation in the pseudo-Riemannian case and [21] for the complete mathematical proof).

Lemma 1. *Let $X \in \mathfrak{g}$. Then the curve $\gamma(t) = \exp(tX)(p)$ is geodesic with respect to some parameter s if and only if*

$$\langle [X, Z]_{\mathfrak{m}}, X_{\mathfrak{m}} \rangle = k \langle X_{\mathfrak{m}}, Z \rangle$$

for all $Z \in \mathfrak{m}$ and for some constant $k \in \mathbb{R}$. If $k = 0$, then t is an affine parameter for this geodesic. If $k \neq 0$, then $s = e^{-kt}$ is an affine parameter for the geodesic. The second case can occur only if the curve $\gamma(t)$ is a null curve in a properly pseudo-Riemannian space.

In the paper [34], O. Kowalski and J. Szenthe proved that any homogeneous Riemannian manifold admits a homogeneous geodesic through the origin. This was the generalization of the result by J. Kajzer [32] for invariant metrics on Lie groups. The examples which prove that the result from [34] is optimal were shown by O. Kowalski and Z. Vlášek in [36]. The generalization to the pseudo-Riemannian (reductive and nonreductive) case was obtained in [15] in the framework of affine geometry which we shall explain in Sections 3 and 4.

In pseudo-Riemannian geometry, null homogeneous geodesics are of particular interest. In [28] and [43], plane-wave limits (Penrose limits) of homogeneous spacetimes along light-like homogeneous geodesics were studied. In [10], G. Calvaruso and R.A. Marinosci described an example of a 3-dimensional Lie group with an invariant Lorentzian metric which does not admit a light-like homogeneous geodesic. Here the standard geodesic lemma was used, because the example is reductive. The affirmative answer on the existence of a light-like homogeneous geodesic in any Lorentzian manifold of even dimension was obtained in [17] by adapting and refining the affine method to the pseudo-Riemannian situation. We shall discuss this result in more detail in Section 4 of the present paper.

A homogeneous pseudo-Riemannian manifold M all of whose geodesics are homogeneous is called a pseudo-Riemannian *g.o. manifold*, or, if the isometry group G and the presentation $M = G/H$ is fixed, a *g.o. space*. Their analogues with noncompact isotropy group are *almost g.o. spaces*. For almost g.o. spaces, geodesics are homogeneous in almost all directions, but there is a singular set of directions, in which geodesics are not homogeneous. For many results and further references on homogeneous geodesics in the reductive case see for example the survey paper [14]. The interesting examples of g.o. spaces and almost g.o. spaces, new developments and further references will be given in Section 2 of the present paper. Further, in Section 3, we shall explain the more general and more fundamental affine method for the study of homogeneous geodesics, we shall present interesting affine g.o. spaces and illustrate some new phenomena related with the affine situation. In Section 4, we shall show how this method is useful in the proof of the existence of a homogeneous geodesic, how it is adapted to the pseudo-Riemannian situation and used for the proof of the existence of a light-like homogeneous geodesic in even dimension. In Section 5, we shall mention recent developments in Finsler geometry related to homogeneous geodesics. Because Finsler homogeneous spaces are reductive, the algebraic method based on the reductive decomposition can be used, however, the affine method can be also adapted to the Finslerian situation.

2 Reductive homogeneous g.o. manifolds

Let $(G/H, g)$ be a pseudo-Riemannian g.o. space with compact isotropy group H and $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ an $\mathrm{Ad}(H)$-invariant decomposition of the Lie algebra $\mathfrak{g}$. One of the techniques used for the characterization of g.o. manifolds and g.o. spaces is based on the idea of "geodesic graph", coming from J. Szenthe [45]. A *geodesic graph* is an $\mathrm{Ad}(H)$-equivariant map $\eta\colon \mathfrak{m} \to \mathfrak{h}$ which is rational on an open dense subset U of $\mathfrak{m}$ and such that $X + \eta(X)$ is a geodesic vector for each $X \in \mathfrak{m}$.

According to Lemma 10 in [45], for every reductive g.o. space $(G/H, g)$ as above, there exists at least one geodesic graph. The construction of canonical and general geodesic graphs is based on geodesic lemma and is described in details in [33] or [20]. For naturally reductive spaces, there exists a linear geodesic graph, and this property can be taken as an equivalent condition with natural reductivity. For general g.o. space, on the open dense subset U of $\mathfrak{m}$ and with respect to a basis $\{E_1, \dots, E_n\}$ of $\mathfrak{m}$ and a basis $\{F_1, \dots, F_h\}$ of $\mathfrak{h}$, the components of a geodesic graph η are rational functions of the coordinates on $\mathfrak{m}$. They are of the form $\eta_k = P_k/P$, where P_k and P are homogeneous polynomials and $\deg(P_k) = \deg(P) + 1$. The *degree* of the geodesic graph η is $\deg(\eta) = \deg(P)$, where P_k and P have no nontrivial common factor. The *degree* of the g.o. space G/H is the minimum of degrees of all geodesic graph on G/H. It may happen that a g.o. manifold M admits more presentations as a homogeneous space G/H and the geodesic graph may be simpler with respect to some bigger group $G' \supset G$ of isometries. Hence we define the *degree* of the g.o. manifold M as the minimum of degrees of all geodesic graphs constructed for all possible g.o. spaces G/H, where $M = G/H$.

The first example of a g.o. space which is not naturally reductive is the 6-dimensional nilpotent group with an invariant metric and it was given by A. Kaplan in [31].

In [35], O. Kowalski and L. Vanhecke proved that every simply connected Riemannian g.o. space $(G/H, g)$ of dimension $n \leq 5$ is a naturally reductive Riemannian manifold. In dimension 5, there are examples such that G/H is a g.o. space of degree 2, but it becomes naturally reductive if we extend the group G (see [35] or [33]). In the same work, the 6-dimensional Riemannian g.o. spaces which are never naturally reductive were classified. One class is two-step nilpotent Lie group with 2-dimensional center, the maximal connected isotropy group is isomorphic to SU(2) or U(2) and metrics depend on three real parameters (these metrics include the Kaplan's example). Second class are manifolds of the form $M = \mathrm{SO}(5)/\mathrm{U}(2)$ or $M = \mathrm{SO}(1,4)/\mathrm{U}(2)$, where SO(5), or SO(1, 4), respectively, is the identity component of the full isometry group and metrics depend on two real parameters.

In dimension 7, a nilpotent example of a g.o. manifold was constructed by C. Gordon in [30]. It is the nilpotent group $N = N \rtimes \mathrm{SU}(2)/\mathrm{SU}(2)$ with a 3-parameter family of left-invariant Riemannian metrics. The 7-dimensional compact example $M = \big(\mathrm{SO}(5) \times \mathrm{SO}(2)\big)/\big(\mathrm{SU}(2) \times \mathrm{SO}(2)_\varphi\big)$ and its dual $M = (\mathrm{SO}(1,4) \times \mathrm{SO}(2))/(\mathrm{SU}(2) \times \mathrm{SO}(2)_\varphi)$ with a 2-parameter family of left-invariant Riemannian metrics was constructed by the present author, O. Kowalski and S. Nikčević in [25] and in special cases considered in [24].

Important class of nilpotent g.o. manifolds are some *H-type groups* and *modified H-type groups*. An H-type group is a 2-step nilpotent Lie group N with a left-invariant metric whose Lie algebra $\mathfrak{n}$ decomposes as $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$ (here $\mathfrak{z}$ is the center and $\mathfrak{v} = \mathfrak{z}^\perp$) and the operators $J_Z\colon \mathfrak{v} \to \mathfrak{v}$ defined by the formula

$$\langle J_Z X, Y\rangle = \langle Z, [X, Y]\rangle, \qquad X, Y \in \mathfrak{v}, Z \in \mathfrak{z}$$

satisfy $J_Z^2 = -\langle Z, Z\rangle \mathrm{Id}$, for each $Z \in \mathfrak{z}$. For modified H-type group, the last identity changes to $J_Z^2 = \lambda(Z)\mathrm{Id}$, for some $\lambda(Z) < 0$.

The 6-dimensional example by A. Kaplan is an H-type group and the 6-dimensional nilpotent examples above correspond to modified H-type groups.

H-type groups which are g.o. manifolds were classified by C. Riehm in [44]. There are 2 series (of dimension $2 + 4n$ with 2-dimensional center and H-type groups of dimension $3 + 4n$ with 3-dimensional center) and 5 additional examples. The dimensions of the 5 additional examples are 13, 14, 15, 23 and 31. Modified H-type groups were investigated by J. Lauret in [38]. He described which of modified H-type groups are g.o. manifolds. For modified H-type groups which are g.o. manifolds, the dimensions are the same, only the H-type metric is generalized to the family of modified H-type metrics in each case.

Geodesic graphs on 6-dimensional and 7-dimensional g.o. spaces mentioned above were described in [12], [33], [33] and the degree is equal to 2. In [20], the present author and O. Kowalski investigated the 13-dimensional generalized Heisenberg group (H-type group) with 5-dimensional center. The 13-dimensional H-type group admits 2 transitive groups of isometries, $G = I_0(M)$ and $G' \subsetneq G$. The isotropy groups H and H' corresponding to G and G', respectively, are $H = \mathrm{SO}(5) \times \mathrm{SO}(2)$ and $H' = \mathrm{SO}(5)$. Hence, the group N admits two presentations $N = G/H$ and $N = G'/H'$ as a homogeneous space and both these spaces are g.o. spaces. In G'/H', geodesic graph is unique and $\deg(G'/H') = 6$. In G/H, there are more geodesic graphs. For the canonical geodesic graph ξ it holds $\deg(\xi) = 6$, but there is a general geodesic graph of degree 3. Hence, $\deg(M) = \deg(G/H) = 3$ (see [20] for details about general geodesic graphs). Unfortunately, the other examples of H-type groups cannot be described by this method. To solve the equations given by geodesic lemma, it is necessary to calculate big determinants. In dimension 13, this was at the limits of the computer possibilities.

In [5] and [6], D. Alekseevsky and A. Arvanitoyeorgos classified Riemannian flag manifolds which are g.o. manifolds. There are two series, namely $\mathrm{SO}(2n+1)/\mathrm{U}(n)$ and $\mathrm{Sp}(n)/\mathrm{U}(1) \cdot \mathrm{Sp}(n-1)$, for $n \geq 2$. For $n = 2$, both these manifolds coincide with the 6-dimensional compact example mentioned above. In [18], the present author described the second manifold in the first series, namely $\mathrm{SO}(7)/\mathrm{U}(3)$, explicitly in terms of Lie algebras and calculated the canonical geodesic graph. The group $\mathrm{SO}(7)$ is the maximal isometry group and the canonical geodesic graph ξ is unique in this manifold and hence it holds $\deg(M) = \deg(\mathrm{SO}(7)/\mathrm{U}(3)) = \deg(\xi) = 4$. For the manifold $\mathrm{Sp}(3)/\mathrm{U}(1) \cdot \mathrm{Sp}(2)$, there is not unique geodesic graph and the computation of the canonical geodesic graph is again too complicated for the computer.

The mentioned classification of flag g.o. manifolds in [5] and [6] was further extended by D. Alekseevsky and Yu.G. Nikonorov in [7] to compact Riemannian manifolds with positive Euler characteristics. Another nice result is description of g.o. metrics on spheres obtained in [40] by Yu.G. Nikonorov. This classification includes the 7-dimensional g.o. manifold from [25] mentioned above. The abstract algebraic resuls about the structure of g.o. spaces were obtained by Yu.G. Nikonorov in the recent paper [41].

We should mention also recent works [3], [4] by A. Arvanitoyeorgos and co-authors on homogeneous geodesics in generalized Wallach spaces and in M-spaces, respectively. These and more related results can be found also in the recent survey paper [2] by A. Arvanitoyeorgos.

In [22], the present author and O. Kowalski generalized the metrics on the above examples and obtained pseudo-Riemannian g.o. spaces (with compact isotropy group). In [23] and [11], the present author with O. Kowalski modified other 6-dimensional and 7-dimensional g.o. spaces and obtained homogeneous spaces with the noncompact isotropy group. There are 6-dimensional manifolds of the form $\mathrm{SO}(2,3)/\mathrm{U}(1,1)$ with the signature $(2,4)$ and $\mathrm{SO}(2,3)/(\mathrm{SU}(1,1)\times\mathbb{R})$ with the signature $(3,3)$. Further, there are 7-dimensional manifolds of the form $(\mathrm{SO}(2,3)\times\mathrm{SO}(2))/(\mathrm{SU}(1,1)\times\mathrm{SO}(2)_\varphi)$ with signatures $(2,5)$ or $(3,4)$ and $(\mathrm{SO}(2,3)\times\mathbb{R})/(\mathrm{SU}(1,1)\times\mathbb{R}_\varphi)$ with the signature $(3,4)$. All these homogeneous spaces are almost g.o. spaces, because the geodesic graph can be defined on an open dense subset $\mathcal{U}\subset\mathfrak{m}$, but not on all of $\mathfrak{m}$. In [11], the behaviour of geodesic graphs on g.o. spaces was compared with geodesic graph on almost g.o. spaces and certain conjectures about this differences were formulated. In particular, in all examples of almost g.o. spaces above, on the null cone, homogeneous geodesics do not need reparametrization. In [8], V. del Barco found an expample of a g.o. nilmanifold, where the homogeneous geodesics on the null cone require the reparametrization.

3 Affine homogeneous manifolds

Let ∇ be an affine connection on a manifold M. Then ∇, or also (M,∇), is said to be *locally homogeneous*, if for each two points $x,y\in M$ there exists a neighborhood $\mathcal{U}$ of x, a neighborhood $\mathcal{V}$ of y and an affine transformation $\varphi:\mathcal{U}\to\mathcal{V}$ such that $\varphi(x)=y$. It means that φ is a (local) diffeomorphism such that

$$\nabla_{\varphi_* X}\varphi_* Y=\varphi_*(\nabla_X Y)$$

holds for every vector fields X,Y defined in $\mathcal{U}$. In a homogeneous affine manifold (M,∇), by a *homogeneous geodesic* we mean a geodesic which is an orbit of a one-parameter group of affine diffeomorphisms. Here the canonical parameter of the group need not be the affine parameter of the geodesic. An *affine g.o. space* is a homogeneous affine manifold (M,∇) such that each geodesic is homogeneous.

A vector field X on an affine manifold (M,∇) is called an *affine Killing vector field* if the Lie derivative $\mathcal{L}_X\nabla$ vanishes, or, equivalently, if X satisfies

the equation
$$[X, \nabla_Y Z] - \nabla_Y [X, Z] - \nabla_{[X,Y]} Z = 0, \tag{3.1}$$
for all vector fields Y, Z. An affine manifold (M, ∇) is locally homogeneous if it admits at least $n = \dim M$ affine Killing vector fields which are linearly independent at each point. Recall that a parametrized curve in a manifold M is said to be *regular* if $\gamma'(t) \neq 0$ for all values of t. It is well-known that, in a homogeneous space $M = G/H$ with an invariant affine connection ∇, each regular orbit of a 1-parameter subgroup $g_t \subset G$ on M is an integral curve of an affine Killing vector field on M. A nonvanishing smooth vector field Z on M is said to be *geodesic along its regular integral curve* γ if the curve $\gamma(t)$ is geodesic up to a possible reparametrization. If all regular integral curves of Z are geodesics up to a reparametrization, then the vector field Z is called *a geodesic vector field*.

Proposition 1 ([26]). *Let Z be a nonvanishing Killing vector field on $M = (G/H, \nabla)$.*

1) *Z is geodesic along its integral curve γ if and only if*
$$\nabla_{Z_{\gamma(t)}} Z = k_\gamma \cdot Z_{\gamma(t)} \tag{3.2}$$
holds along γ, where $k_\gamma \in \mathbb{R}$ is a constant. If $k_\gamma = 0$, then t is the affine parameter of geodesic γ. If $k_\gamma \neq 0$, then the affine parameter of this geodesic is $s = e^{k_\gamma t}$.

2) *Z is a geodesic vector field if and only if*
$$\nabla_Z Z = k \cdot Z \tag{3.3}$$
holds on M. Here k is a smooth function on M, which is constant along integral curves of the vector field Z.

Suppose that there exists a geodesic Killing vector field Z on $M = (G/H, \nabla)$. We will present examples in which for certain geodesic Killing vector fields the function k may have the same value for all integral curves or it may have different values for different integral curves. We also point out that the existence of geodesic affine Killing vector fields is rather limited. For example, a round sphere with the corresponding Levi-Civita connection does *not* admit such a vector field. Still, all geodesics are homogeneous.

We will start in dimension 2, because here we have the following classification of homogeneous affine connections, which was first formulated by B. Opozda in [42] for the torsion-free case and then it was refined and generalized to the case with arbitrary torsion by T. Arias-Marco and O. Kowalski in [1].

Theorem 1. *Let ∇ be a locally homogeneous affine connection with arbitrary torsion on a 2-dimensional manifold M. Then, in a neighborhood $\mathcal{U}$ of each* *point $m \in M$, either ∇ is locally a Levi-Civita connection of the unit sphere or, there is a system (u, v) of local coordinates and constants A, B, C, D, E, F, G, H such that ∇ is expressed in $\mathcal{U}$ by one of the following formulas:*

$$\begin{array}{lll} typeA: & \nabla_{\partial_u}\partial_u = A\,\partial_u + B\,\partial_v, & \nabla_{\partial_u}\partial_v = C\,\partial_u + D\,\partial_v, \\ & \nabla_{\partial_v}\partial_u = E\,\partial_u + F\,\partial_v, & \nabla_{\partial_v}\partial_v = G\,\partial_u + H\,\partial_v, \end{array}$$

$$\begin{array}{lll} typeB: & \nabla_{\partial_u}\partial_u = \frac{A}{u}\,\partial_u + \frac{B}{u}\,\partial_v, & \nabla_{\partial_u}\partial_v = \frac{C}{u}\,\partial_u + \frac{D}{u}\,\partial_v, \\ & \nabla_{\partial_v}\partial_u = \frac{E}{u}\,\partial_u + \frac{F}{u}\,\partial_v, & \nabla_{\partial_v}\partial_v = \frac{G}{u}\,\partial_u + \frac{H}{u}\,\partial_v, \end{array}$$

where not all A, B, C, D, E, F, G, H are zero.

3.1 Connections of type A

First, let us have a connection ∇ of type A, it means with constant Christoffel symbols $A, \dots, H$. According to the equations (3.1), the elementary operators ∂_u, ∂_v are affine Killing vector fields. Hence we can put, in particular, $\mathcal{U} = \mathbb{R}^2[u, v]$, which gives a globally homogeneous space. For the simplicity, we shall consider here just the torsion-free case, which means $E = C$ and $F = D$ in this situation. For the general case, see [26].

We derive the condition under which a vector field $Z = x\,\partial_u + y\,\partial_v$ (where x, y are constants and $(x, y) \neq (0, 0)$) is geodesic. If we calculate the covariant derivative $\nabla_Z Z$ explicitly, the equality $\nabla_Z Z = k \cdot Z$ gives us for $k = 0$ conditions

$$\begin{aligned} A\,x^2 + 2C\,xy + G\,y^2 &= 0, \\ B\,x^2 + 2D\,xy + H\,y^2 &= 0 \end{aligned} \tag{3.4}$$

and for $k \neq 0$, we obtain the condition

$$B\,x^3 + (2D - A)\,x^2y + (H - 2C)\,xy^2 - G\,y^3 = 0. \tag{3.5}$$

A necessary and sufficient condition for the Killing vector field $Z = x\,\partial_u + y\,\partial_v$ to be geodesic is that the pair (x, y) of constants satisfies the condition (3.5). The integral curves of this Killing vector field do not require a reparametrization if (x, y) satisfy also the equations (3.4), see [13] or [26] for details.

The equation (3.5) has always at least one solution. For example, if $B \neq 0$, we can put $y = 1$ and fix x as a solution of the resulting cubic equation. We see that $(\mathbb{R}^2, \nabla)$ always admits at least one geodesic Killing vector field and consequently at least one homogeneous geodesic through each point.

Further, the equation (3.5) is satisfied for all (x, y) if all its coefficients are equal to zero. Hence, for $(\mathbb{R}^2, \nabla)$ to be an affine g.o. space with zero torsion, it is sufficient that

$$B = 0, \quad A = 2D, \quad G = 0, \quad H = 2C. \tag{3.6}$$

If the Christoffel symbols satisfy the relations (3.6) (and thus $(\mathbb{R}^2, \nabla)$ is a g.o. space), then every Killing vector field $Z = x\,\partial_u + y\,\partial_v$ is geodesic. Here the pair (x, y) determines the direction from the origin. Unless $C = D = 0$, geodesics in all directions except that corresponding to $(x, y) = c(C, -D)$, $c \neq 0$, require a reparametrization and the "reparametrization factor" is $k = Dx + Cy$. For an affine g.o. space $(\mathbb{R}^2, \nabla)$ satisfying the formulas (3.6), and such that not all Christoffel symbols are equal to zero, we have $k \neq 0$ whenever (x, y) is not proportional to $(C, -D)$. This shows that, in this case, *almost each* homogeneous geodesic of the space must be reparametrized to recover the affine parameter. This contrasts strongly with the pseudo-Riemannian case where only homogeneous geodesics of the null cone have to be (possibly) reparametrized.

We also remark that, for all affine g.o. spaces of type A, every homogeneous geodesic is an integral curve of a geodesic Killing vector field and the reparametrization is the same for all integral curves of this Killing vector field.

3.2 Connections of type B

Let us continue with the connection ∇ of type B. Then the Christoffel symbols are

$$\begin{aligned} A(u,v) = A/u, \quad B(u,v) = B/u, \quad C(u,v) = C/u, \\ D(u,v) = D/u, \quad G(u,v) = G/u, \quad H(u,v) = H/u, \end{aligned} \tag{3.7}$$

where $A, \ldots, H$ are constants and we always assume $u \neq 0$. One checks directly that the connection admits the affine Killing vector fields ∂_v and $u\partial_u + v\partial_v$ (infinitesimal translations and infinitesimal homotheties). In the following, we shall assume the globally homogeneous case where $\mathcal{U} = \mathcal{H}^+ = \{\mathbb{R}^2(u,v) \,|\, u > 0\}$ and, for the simplicity, we shall consider here just the torsion-free case, which means again $E = C$ and $F = D$ in this situation.

Consider the Killing vector field $Z = x\,\partial_v + y\,(u\partial_u + v\partial_v)$, where x, y are arbitrary parameters. We calculate the covariant derivative $\nabla_Z Z$ and consider the condition (3.2) along the integral curve $\gamma(t)$ of Z. Here we cannot consider the general condition (3.3), because, in these equations, there are too many variables present. Not only x, y which determine the Killing vector field, but also initial conditions of a particular integral curve. See [13] or [26] for details. In the case $k_\gamma = 0$, condition (3.2) gives us the equations

$$\begin{aligned} (A+1)\,c_1^2 + 2C\,c_1c_2 + G\,c_2^2 &= 0, \\ B\,c_1^2 + (2D+1)\,c_1c_2 + H\,c_2^2 &= 0 \end{aligned} \tag{3.8}$$

and in the case $k_\gamma \neq 0$ we obtain the polynomial equation

$$B\,c_1^3 + (2D - A)\,c_1^2c_2 + (H - 2C)\,c_1c_2^2 - G\,c_2^3 = 0. \tag{3.9}$$

These equations are very similar to the equations for connections of type A, but here, the variables are the initial conditions of the particular integral curve of the Killing vector field Z. The variables x, y determining the Killing vector field Z are eliminated from the equations.

For given Killing vector field $Z = x\,\partial_v + y\,(u\partial_u + v\partial_v)$, the integral curve γ determined by (c_1, c_2) is geodesic if (c_1, c_2) satisfy the equation (3.9). The geodesic does not require a reparametrization if (c_1, c_2) satisfy also the equations (3.8).

Again, it can be proved easily that the equation (3.9) has always at least one solution (c_1, c_2). As a consequence, we obtain that for any globally homogeneous affine connection ∇ of type B defined on the domain $\mathcal{H}^+$, any Killing vector field $Z = x\,\partial_v + y\,(u\partial_u + v\partial_v)$ is geodesic along some integral curve γ and the manifold $(\mathcal{H}^+, \nabla)$ admits at least one homogeneous geodesic through each point. The key step in the proof is choosing the pair (x, y) in a way that the integral curve γ, corresponding to (c_1, c_2) which solve the system (3.9), pass through p.

If the coefficients of the connection satisfy the relations

$$B = 0, \quad 2D = A, \quad G = 0, \quad H = 2C, \tag{3.10}$$

then all coefficients of the equation (3.9) are zero and this equation is satisfied for all admissible values (c_1, c_2). Consequently, all Killing vector fields $Z = x\,\partial_v + y\,(u\partial_u + v\partial_v)$ are geodesic and $(\mathcal{H}^+, \nabla)$ is an affine g.o. space. If, moreover, $A = -1$ and $C = 0$, then integral curves do not require a reparametrization. Otherwise, for $C \neq 0$, only integral curves of Killing vector fields with $y \neq 0$ and corresponding to $c_2/c_1 = -(2D+1)/2C$ do not require a reparametrization, and for $C = 0$, only integral curves of the Killing vector field ∂_v do not require a reparametrization. See [13] and [26] for all details.

In the case B, in general, homogeneous geodesics are orbits of Killing vector fields which are geodesic only along certain integral curve γ but they are not geodesic in general. However, for g.o. spaces of type B, homogeneous geodesics are orbits of geodesic Killing fields and the necessary reparametrization is *different* along different integral curves. Anyway, the constrast with the pseudo-Riemannian situation described earlier is still valid here. And, finally, for the 2-sphere with the Levi-Civita connection (which is also a g.o. space), homogeneous geodesics are integral curves of Killing vector fields which are geodesic only along main circles. Hence, this situation is different from both types, A and B. In the paper [13], this reparametrization phenomenon was studied in further details. Special examples were selected and the interesting phenomena were shown explicitly.

An imporant question in this context is about the maximality of the algebra of Killing vector fields. In [26], the affine g.o. manifolds for which the two-dimensional algebra considered above (for both types A and B) is maximal were determined. Surprisingly, for torsion-free g.o. manifolds of type A, this algebra can be always extended. In contrast to this case, there are many torsion-free affine g.o. spaces of type B admitting only two linearly independent Killing vector fields.

4 Existence of homogeneous geodesics

In the context of affine homogeneous manifolds, we have seen in previous section that any affine homogeneous manifold of dimension 2 admits a homogeneous geodesic through arbitrary point. The next step in dimension 3 was done by the present author, O. Kowalski and Z. Vlášek in the paper [27], where certain homogeneous connections on Lie groups were studied. The equations here are already too complicated, but, on the other hand, the equations for different connections are very similar. The obtained existence result was only partial, however, general topology was involved in the proof. The full result, which works in any odd dimension and which can be generalilzed to arbitrary dimension was obtained by the present author in [15] and it is using simple differential topology. The crucial idea is the following:

We consider the Killing vector fields $K_1, \dots, K_n$ which are linearly independent at each point of some neighbourhood $\mathcal{U}$ of p we denote by B the basis $\{K_1(p), \dots, K_n(p)\}$ of T_pM. Then any tangent vector $X \in T_pM$ has coordinates $(x_1, \dots x_n)$ with respect to the basis B and it determines the Killing vector field $X^* = x_1K_1 + \cdots + x_nK_n$ and an integral curve γ of X^* through p. We are going to show that there exists a vector $\bar{X} \in T_pM$ such that the corresponding integral curve is geodesic. We consider the sphere S^{n-1} of vectors $X \in T_pM$ whose coordinates $(x_1, \dots, x_n)$ have norm equal to 1 with respect to the Euclidean scalar product. For each $X \in S^{n-1}$ denote by $v(X)$ the covariant derivative $\nabla_{X^*_{\gamma(t)}} X^*|_{t=0}$ and we denote by $t(X)$ the vector $v(X) - \langle v(X), X\rangle X$. Then $t(X) \perp X$ for each $X \in S^{n-1}$. Clearly, the map $X \mapsto t(X)$ defines a smooth tangent vector field on the sphere S^{n-1}. If n is odd, it follows that there is a vector $\bar{X} \in T_pM$ such that $t(\bar{X}) = 0$ and hence $v(\bar{X}) = k_\gamma \bar{X}$, where $k_\gamma = \langle v(\bar{X}), \bar{X}\rangle$ is a constant. We see immediately that $\nabla_{\bar{X}^*_\gamma} \bar{X}^* = k_\gamma \bar{X}^*_\gamma$ and the corresponding integral curve γ is a local homogeneous geodesic, which can be uniquely prolonged to a global homogeneous geodesic.

This idea can be be further refined for any dimension and we obtain that any homogeneous affine manifold and in particular any homogeneous (reductive or nonreductive) pseudo-Riemannian manifold admits a homogeneous geodesic through arbitrary point. See [15] for all details.

In the paper [17] by the present author, this affine approach was adapted to pseudo-Riemannian and in particular to the Lorentzian situation, using the invariance of the homogeneous metric g and induced pseudo-Riemannian connection ∇. The key property here is that the formula

$$\nabla_{X^*_{\gamma(t)}} X^* \in (X^*_{\gamma(t)})^\perp$$

is valid along the integral curve $\gamma(t)$. The orthogonal complement is considered with respect to the scalar product induced by the metric g. It allows us to use again, for light-like direction X, the projection of the vector $v(X)$ to the tangent space of the unit sphere of light-like directions. It was proved that any homogeneous Lorentzian manifold of even dimension admits a light-like homogeneous geodesic through arbitrary point. This result cannot be generalized to arbitrary dimension. In dimension 3, there are counterexamples known from the paper [10] by G. Calvaruso and R.A. Marinosci, where the classical geodesic lemma was used, because these manifolds are Lie groups with an invariant Lorentzian metric. In [17], one of these counterexamples was described in detail using the affine method.

In [9], the affine method for the study of homogeneous geodesics was applied by G. Calvaruso, A. Fino and A. Zaeim to 4-dimensional nonreductive pseudo-Riemannian homogeneous manifolds, which cannot be described using the standard geodesic lemma. Killing vector fields and homogeneous geodesics of these spaces were determined and, in particular, it was shown that any 4-dimensional nonreductive pseudo-Riemannian homogeneous manifold of signature $(2,2)$ admits a null homogeneous geodesic. The general case with other pseudo-Riemannian signatures is still open.

5 Finsler homogeneous spaces

In a homogeneous Finsler space (M, F), we consider the group of isometries of the Finsler metric F and homogeneous geodesics with respect to the action of this group. The Finslerian version of geodesic lemma was proved by D. Latifi in [37]:

Lemma 2 ([37]). *Let $(G/H, g)$ be a homogeneous Finsler space. The vector $X \in \mathfrak{g}$ is a geodesic vector if and only if it holds*

$$g_{X_\mathfrak{m}}([X, Z]_\mathfrak{m}, X_\mathfrak{m}) = 0$$

for all $Z \in \mathfrak{m}$.

Here g is the fundamental tensor coming from the invariant Minkowski norm on $\mathfrak{m}$ and $g_{X_\mathfrak{m}}$ plays the role of the scalar product in the direction $X_\mathfrak{m}$. In [47], Z. Yan and S. Deng constructed examples of Finsler g.o. metrics and obtained results about special Finsler g.o. metrics. Recently, in the paper [46], Z. Yan claimed the existence of a homogeneous geodesic in any homogeneous Finsler space of odd dimension. The algebraic method developed by O. Kowalski and J. Szenthe in [34] and based on the reductive decomposition was generalized to the Finslerian situation and also differential topology and mappings $\mathbb{S}^n \to \mathbb{S}^n$ were used. However, the proof contains a serious gap and it is not correct.

In the paper [19], the present author adapted the affine method described in previous section to the Finslerian setting and the existence of a homogeneous geodesic in any homogeneous Finsler space of odd dimension was proved correctly. Further, in the paper [16], the present author proved that in a homogeneous Berwald space and in a homogeneous reversible Finsler space a homogeneous geodesic always exists. In the recent paper [48] by Z. Yan and L. Huang, the situation was studied in full generality. Using some ideas from the paper [34] by O. Kowalski and J. Szenthe and a purely Finslerian construction, it was proved that any homogeneous Finsler space admits a homogeneous geodesic.

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