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On the nilpotent conjugacy class graph of groups

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Abstract. The nilpotent conjugacy class graph (or NCC-graph) of a group G is a graph whose vertices are the nontrivial conjugacy classes of G such that two distinct vertices x^G and y^G are adjacent if $\langle x', y'\rangle$ is nilpotent for some $x' \in x^G$ and $y' \in y^G$. We discuss on the number of connected components as well as diameter of connected components of these graphs. Also, we consider the induced subgraph $\Gamma_n(G)$ of the NCC-graph with vertices set $\{g^G \mid g \in G \setminus \mathrm{Nil}(G)\}$, where $\mathrm{Nil}(G) = \{g \in G \mid \langle x, g\rangle$ is nilpotent for all $x \in G\}$, and classify all finite non-nilpotent group G with empty and triangle-free NCC-graphs.

Keywords: Triangle-free, conjugacy class, non-nilpotent group, graph

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1 introduction

Herzog, Longobardi and Maj [6] defined the *commuting conjugacy class graph* (or CCC-graph) of a group G as a graph whose vertex set is the set of nontrivial conjugacy classes of G such that two distinct vertices x^G and y^G are adjacent if $\langle x', y'\rangle$ is abelian for some $x' \in x^G$ and $y' \in y^G$. They have determined the connectivity of the CCC-graph of a group G and gave upper bounds for diameter of the corresponding connected components of the graph.

Likewise, we may define the *nilpotent conjugacy class graph* (or NCC-graph) of G, denoted by $\Gamma(G)$, as a simple undirected graph, in which the vertices of $\Gamma(G)$ are the nontrivial conjugacy classes of G and two distinct vertices x^G and y^G are adjacent whenever there exist two elements $x' \in x^G$ and $y' \in y^G$ such that $\langle x', y'\rangle$ is nilpotent. We shall obtain analogues to results in [6] for the case of NCC-graphs by describing the number of their connected components and diameter of each component for a finite or locally finite group. Also, in case of finite solvable groups or supersolvable groups, we shall give structural theorems in the case where the corresponding NCC-graphs are disconnected.

In what follows, we set the following notation. An edge between two distinct vertices x and y is denoted by $x \leftrightarrow y$, while by $x \Leftrightarrow y$ we mean either $x = y$ or $x \leftrightarrow y$. Also, a path between two vertices x and y is denoted by $x \leftrightsquigarrow y$. The distance between two vertices x and y of a graph is denoted by $d(x, y)$. Recall that the diameter of a graph is the maximum distance between its vertices. For a given group G, $\pi(G)$ and $\omega(G)$ stand for the set of prime divisors of $|G|$ and the set of element orders of G, respectively. A graph Γ is called empty graph if and only if Γ has no edges. The commuting graph of a group G, denoted by $\Delta(G)$, is the graph whose vertex set is $G \setminus Z(G)$, $Z(G)$ is the center of G, and two distinct vertices are adjacent if they commute. Also, the prime graph $\Pi(G)$ is the graph whose vertex set is $\pi(G)$ and two distinct vertices $p, q \in \pi(G)$ are adjacent if $pq \in \omega(G)$. As an analogue to the center of the group, we may define the following set

$$\mathrm{Nil}(G) = \{g \in G \mid \langle x, g\rangle \text{ is nilpotent for all } x \in G\}$$

for any group G. It is evident that the vertices in $\mathrm{Nil}(G)$ are adjacent to all other vertices of the NCC-graph $\Gamma(G)$ of G.

In what follows, the ith small group of order n, in SmallGroup library of GAP [3], is denoted by $SG(n, i)$.

2 On the connectivity of $\Gamma(G)$

To prove our main results, we need to evaluate the distance between distinct vertices of the NCC-graph of a group, which we assume to be locally finite.

Lemma 2.1. Let G be a locally finite group and p be a prime number. Then the following statements hold:

(i) If $x, y \in G \setminus \{1\}$ are p-elements, then $d(x, y) \leq 1$.

(ii) If $x, y \in G \setminus \{1\}$ are of non-coprime orders, then $d(x, y) \leq 3$. Moreover $d(x, y) \leq 2$, whenever either x or y is of prime power order.

Proof. (i) Let P be a Sylow p-subgroup of the finite group $\langle x, y\rangle$, which contains x. Then $y^g \in P$ for some $g \in G$ and so $\langle x, y^g\rangle$ is nilpotent. This implies that $d(x, y) \leq 1$.

(ii) Assume $|x| = pm$ and $|y| = pn$ for some prime p and $m, n \in \mathbb{N}$. Then, by part (i), $d(x^m, y^n) \leq 1$. Since $d(x, x^m) \leq 1$ and $d(y, y^n) \leq 1$, we obtain $d(x, y) \leq 3$.

Finally, if $|x|$ is a power of p, then part (i) implies $d(x, y^m) \leq 1$ and so $d(x, y) \leq 2$, as required. QED

Lemma 2.2. Let G be a locally finite group and $x, y \in G$. Suppose p and q are prime divisors of $|x|$ and $|y|$, respectively, and that G has an element of order pq. Then

(i) $d(x, y) \leq 5$, and moreover $d(x, y) \leq 4$ if either x or y is of prime power order;

(ii) If either a Sylow p-subgroup or a Sylow q-subgroup of G is a cyclic or generalized quaternion finite group, then $d(x, y) \leq 4$. Moreover, $d(x, y) \leq 3$ if either x or y is of prime order.

(iii) If both a Sylow p-subgroup and a Sylow q-subgroup of G are either cyclic or generalized quaternion finite groups, then $d(x, y) \leq 3$. Moreover, $d(x, y) \leq 2$ if either x or y is of prime order.

Proof. Let $H = \langle x, y, z\rangle$ and assume $|x| = pm$ and $|y| = pn$ for some $m, n \in \mathbb{N}$. Let $z = z_p z_q \in G$ be such that $|z_p| = p$, $|z_q| = q$ and $z_p z_q = z_q z_p$. By assumption such an element z exists.

(i) From Lemma 2.2(i), we observe that the following path exists between two vertices x^G and y^G:

$$x \Leftrightarrow x^m \Leftrightarrow z_p \leftrightarrow z_q \Leftrightarrow y^m \Leftrightarrow y.$$

Therefore $d(x, y) \leq 5$.

(ii) Assume that a Sylow p-subgroup of G is either a cyclic group or a generalized quaternion finite group. By [10, Theorem 14.3.4], all Sylow p-subgroups of G are conjugate and some conjugate z_p^a of z_p satisfies $\langle z_p^a\rangle = \langle x^m\rangle$. Hence $d(x, z_p) = d(x, z_p^a) \leq 1$ and we get following path between vertices x^G and y^G:

$$x \Leftrightarrow z_p \leftrightarrow z_q \Leftrightarrow y^n \Leftrightarrow y.$$

Thus $d(x, y) \leq 4$.

(iii) Suppose that both a Sylow p-subgroup and a Sylow q-subgroup of G are cyclic or generalized quaternion finite group. As in part (ii) all Sylow p-subgroups (resp. q-subgroup) of G are conjugate in G. Therefore $\langle z_p^a\rangle = \langle x^m\rangle$ and $\langle z_q^b\rangle = \langle y^n\rangle$ for some $a, b \in G$. Thus $d(x, z_p) = d(x, z_p^a) \leq 1$ and $d(z_q, y) = d(z_q^b, y) \leq 1$ yielding $d(x, y) \leq 3$. QED

Lemma 2.3. Let $G = HK$ be a finite group with a normal subgroup H and a subgroup K such that $\Gamma(H)$ and $\Gamma(K)$ are connected. If there exist two elements $h \in H \setminus \{1\}$ and $x \in G \setminus H$ such that h^G and x^G are connected in $\Gamma(G)$, then $\Gamma(G)$ is connected.

Proof. Evidently $\gcd(|x|, |K|) > 1$ so that there exists $k \in K$ such that k^G and x^G are connected via a path in $\Gamma(G)$ by Lemma 2.1(ii). Let a^G and b^G be distinct vertices of $\Gamma(G)$. If $a, b \in H$, then a^G and b^G are connected via a path in $\Gamma(G)$ as $\Gamma(H)$ is connected and $H \subseteq G$. If $a, b \notin H$, then $(|a|, |K|) > 1$ and $(|b|, |K|) > 1$, hence there exist $k', k'' \in K \setminus \{1\}$ such that we have the following path in the graph $\Gamma(G)$:

$$a^G \leftrightsquigarrow k'^G \leftrightsquigarrow k''^G \leftrightsquigarrow b^G.$$

Therefore a^G and b^G are connected via a path in $\Gamma(G)$.

Finally, assume that $a \notin H$ and $b \in H$. Then a^G and k'^G are connected for some $k' \in K$ and we have a path

$$a^G \leftrightsquigarrow k'^G \leftrightsquigarrow k^G \leftrightsquigarrow x^G \leftrightsquigarrow h^G \leftrightsquigarrow b^G,$$

as required. QED

Theorem 2.4. Let G be a finite group and $C_\pi = \{x^G \mid x \text{ is a } \pi\text{-element}\}$ for every set π of primes. The the map $\pi \mapsto C_\pi$ defines a one-to-one correspondence between connected components of the graphs $\Pi(G)$ and $\Gamma(G)$.

Proof. Let $x, y \in G$, and p and q $(p \neq q)$ be prime divisors of $|x|$ and $|y|$, respectively. If $\{p, q\}$ is an edge in $\Pi(G)$, then G contains an element of order pq, from which it follows that x^G and y^G are connected via a path in $\Gamma(G)$ (see Lemma 2.2).

Conversely, assume that $\{x^G, y^G\}$ is an edge in $\Gamma(G)$. Then there exist elements $a \in x^G$ and $b \in y^G$ such that $\langle a, b\rangle$ is nilpotent. Hence G has an element of order pq so that $\{p, q\}$ is an edge in $\Pi(G)$, as required. QED

Theorem 2.5 ([5, Proposition 1])**.** Let G be a finite solvable group and p, q, r be distinct prime divisors of $|G|$. Then $\omega(G) \cap \{pq, pr, qr\} \neq \emptyset$.

Theorem 2.6. Let G be a finite solvable group. Then $\Gamma(G)$ has at most two connected components whose diameters are at most 7.

Proof. By Theorem 2.5, $\Pi(G)$ has at most two connected components and so Theorem 2.4 implies that $\Gamma(G)$ has at most two connected components too.

Suppose $x, y \in G \setminus \{1\}$, and x^G and y^G are connected via a path in $\Gamma(G)$. We show that $d(x, y) \leq 7$. Suppose on the contrary that $d(x, y) = n \geq 8$, and let

$$x^G \leftrightarrow x_1^G \leftrightarrow x_2^G \leftrightarrow \cdots \leftrightarrow x_{n-1}^G \leftrightarrow y^G$$

be a shortest path between x^G and y^G. Without loss of generality, by substituting each x_i with a suitable power, we may assume that $|x_i|$ are primes for all

$1 \leq i \leq n-1$. Since $d(x,y) \geq 8$, it follows from Lemmas 2.1 and 2.2 that there exist two distinct primes p and q, such that $p \big| |x|$ and $q \big| |y|$, and there exists no element of order pq in G.

Let H be a Hall $\{p,q\}$-subgroup of G, and let N be a minimal normal subgroup of H. Then N is an elementary abelian p-group on which the Sylow q-subgroups of H act fixed-point-freely. Then every Sylow q-subgroup of H, and hence of G, is either cyclic or generalized quaternion.

Since $d(x,x_5) = 5$ and $|x_5| = r$, where r is a prime, it follows by Lemmas 2.1 and 2.2 that $r \neq p$ and G has no elements of order pr. Hence G has an element of order qr by Theorem 2.5. Let K be a Hall $\{p,q,r\}$-subgroup of G and M be a minimal normal subgroup of K. If p divides $|M|$, then the Sylow r-subgroups of K act fixed-point-freely on M, which implies that the Sylow r-subgroups of K are either cyclic or generalized quaternion. Hence both a Sylow q-subgroup and a Sylow r-subgroup of G are cyclic or generalized quaternion, and $d(x_5,y) \leq 2$ by Lemma 2.2. Therefore,

$$d(x,y) = d(x,x_5) + d(x_5,y) \leq 5 + 2 = 7,$$

which is a contradiction. If p does not divide $|M|$, then the Sylow p-subgroups of K act fixed-point-freely on M. Thus the Sylow p-subgroups of G are either cyclic or generalized quaternion. Let $|x_4| = s$ (s being a prime). Since $d(x,x_4) = 4$, it follows by Lemmas 2.1 and 2.2 that $s \neq p$ and G has no elements of order ps. Hence, by Theorem 2.5, G has an element of order sq and $d(x_4,y) \leq 3$, by Lemma 2.2. Thus

$$d(x,y) = d(x,x_4) + d(x_4,y) \leq 4 + 3 = 7,$$

which contradicts out assumption. Hence $d(x,y) \leq 7$, as required. QED

Theorem 2.7. Let G be a periodic solvable group. Then $\Gamma(G)$ has at most two connected components whose diameters are at most 7.

Proof. Suppose $\Gamma(G)$ is disconnected and the vertices x^G and y^G of $\Gamma(G)$ belong to different connected components of $\Gamma(G)$. We claim that for any vertex g^G of $\Gamma(G)$, g^G is connected to x^G or y^G via a path in $\Gamma(G)$.

Let $H = \langle x,y,g \rangle$. Since G is locally finite, H is a finite solvable group and x^H, y^H belong to distinct connected components of $\Gamma(H)$. Hence, by Theorem 2.6, g^H is connected to x^H or y^H via a path in $\Gamma(H)$, from which our claim follows. Accordingly, $\Gamma(G)$ has two connected components.

Now assume that G is any periodic solvable group. Let x^G and y^G be two vertices of $\Gamma(G)$ with $d(x,y) = n < \infty$, and let

$$x^G \leftrightarrow x_1^G \leftrightarrow x_2^G \leftrightarrow \cdots \leftrightarrow x_{n-1}^G \leftrightarrow y^G$$

be a shortest path between x^G and y^G. By the definition of $\Gamma(G)$, we may assume that there exist elements $g_1, g_2, \ldots, g_n, h_1, h_2, \ldots, h_n \in G$ such that $\langle x^{g_1}, x_1^{h_1}\rangle$, $\langle x_i^{g_{i+1}}, x_{i+1}^{h_{i+1}}\rangle$ ($i \in \{1, 2, \ldots, n-2\}$), and $\langle x_{n-1}^{g_n}, y^{h_n}\rangle$ are nilpotent subgroups of G.

Let $H = \langle x, y, x_1, x_2, \ldots, x_{n-1}, g_1, g_2, \ldots, g_n, h_1, h_2, \ldots, h_n\rangle$. Then H is a finite solvable group and $d(x, y) = n$ in $\Gamma(H)$ as n was minimal. Therefore, $n \leq 7$ by Theorem 2.6 and the proof is complete. QED

Theorem 2.8 (Theorem 1.1, [8])**.** Suppose that G is a finite group with trivial center. Then every connected component of the commuting graph of G has diameter at most 10. In particular, if the commuting graph of G is connected, then its diameter is at most 10

The next theorem describe the relationship between the commuting graph and the prime graph of G.

Theorem 2.9 (Theorem 3.7, [8])**.** Suppose that G is a finite group with trivial center. Let $\Delta(G)$ be the commuting graph of G and $\Pi(G)$ be the prime graph of G. Let $\Delta(G)/G$ be a set of representatives of G-orbits of connected components of $\Delta(G)$. Then the map $C \longmapsto \pi(C)$ is a bijection between $\Delta(G)/G$ and the connected components of $\Pi(G)$. Furthermore, $\Delta(G)$ is connected if and only if $\Pi(G)$ is connected.

Theorem 2.10. Let G be a finite group. Then $\Gamma(G)$ has at most six connected components whose diameters are at most 10.

Proof. Since G is a finite group, by the main Theorem of [12], $\Pi(G)$ has at most six connected components and so Theorem 2.4 implies that $\Gamma(G)$ has at most six connected components too.

If G has nontrivial center, then $\Gamma(G)$ is connected with diameter at most 2. So, assume that $Z(G) = 1$ and C be a connected component of $\Gamma(G)$. Theorem 2.4 implies that C is a connected component of $\Pi(G)$ and so, by Theorem 2.9, C is a connected component of $\Delta(G)$. Hence, by Theorem 2.8, the diameter of C is at most 10, as claimed. QED

Theorem 2.11. Let G be a locally finite group. Then $\Gamma(G)$ has at most six connected components whose diameters are at most 10.

Proof. It is similar to the proof of Theorem 2.7. QED

3 Characterization of (supersolvable) solvable group with disconnected NCC-graph

In this section, we characterize (supersolvable) solvable groups with disconnected NCC-graph. To this end, we need the following lemma of Abdollahi and Zarrin [1].

Lemma 3.1 (Lemma 3.7, [1])**.** Let G be a group and H be a nilpotent subgroup of G for which $C_G(x) \leq H$ for every $x \in H \setminus \{1\}$. Then $\mathrm{Nil}_G(x) = H$ for every $x \in H \setminus \{1\}$.

Theorem 3.2. Let G be a supersolvable group. Then the graph $\Gamma(G)$ is disconnected if and only if one of the following holds:

(i) If G is an infinite group, then $G = H \rtimes \langle x \rangle$, where $x \in G$, $|x| = 2$ and H is a subgroup of G on which x acts fixed-point-freely;

(ii) If G is a finite group, then $G = H \rtimes K$ is a Frobenius group with kernel H and a cyclic complement K.

Proof. Assume G is a supersolvable group of type (i) or (ii). If G is of type (i), then we claim that $C_G(x^g) = \langle x^g \rangle$ for each $g \in G \setminus \{1\}$. To this end, let $y \in C_G(x^g)$. Then $y^{g^{-1}} = hx$ for some $h \in H$ and $[hx, x] = 1$. Hence, $[h, x] = 1$. But x acts fixed-point-freely on H, which implies that $h = 1$ and consequently $y^{g^{-1}} = x$, as claimed. Thus, by Lemma 3.1, $\mathrm{Nil}_G(x^g) = \langle x^g \rangle$ for every $g \in G \setminus \{1\}$. Therefore $x^G \leftrightarrow y^G$ if and only if $x^G = y^G$ for any $y \in G$. It follows that x^G is an isolated vertex in $\Gamma(G)$ and hence $\Gamma(G)$ is disconnected. If G is of type (ii), then $C_G(k) = K$ for each $k \in K \setminus \{1\}$ and so by Lemma 3.1, $\mathrm{Nil}_G(k) = K$ for every $k \in K \setminus \{1\}$. It implies that there is no path connecting h^G and k^G in $\Gamma(G)$ for each $h \in H \setminus \{1\}$ and $k \in K \setminus \{1\}$. Therefore $\Gamma(G)$ is disconnected in this type too.

To prove the converse, assume G is a supersolvable group for which $\Gamma(G)$ is disconnected. We have two cases:

(i) G is infinite. Utilizing [11, Theorem 7.2.11], G has a normal infinite cyclic subgroup C. Clearly, $|G/C_G(C)| \leq 2$ and $\Gamma(C_G(C))$ is connected. Since $\Gamma(G)$ is disconnected, we must have $|G/C_G(C)| = 2$. Furthermore, if either $g^2 \in C_G(C) \setminus \{1\}$ or g commutes with a nontrivial element of $C_G(C)$ for every element $g \in G \setminus C_G(C)$, then $\Gamma(G)$ is a connected graph, contradicting the assumption. Thus, there exists an element $x \in G \setminus C_G(C)$ such that $x^2 = 1$ and x acts fixed-point-freely on $C_G(C)$, as required.

(ii) G is finite. Then G possesses a normal cyclic subgroup C of prime order for which $G/C_G(C)$ is a cyclic group. Hence there exists $x \in G$ such that $G = C_G(C)\langle x \rangle$. Clearly, $\Gamma(C_G(C))$ and $\Gamma(\langle x \rangle)$ are connected so that Lemma 2.3

yields $C_G(C) \cap \langle x \rangle = 1$ and $\langle x \rangle$ acts fixed-point-freely on $C_G(C)$. The proof is complete. $\boxed{QED}$

Finite solvable groups with a disconnected prime graph were described by Gruenberg and Kegel in [4]. In the following Theorem, we characterize finite solvable groups G with a disconnected NCC-graph.

Lemma 3.3. Let G be a finite solvable group with a disconnected graph $\Gamma(G)$, and let N be a maximal normal subgroup of G such that the subgraph induced by vertices in N is connected in $\Gamma(G)$. Then $G = N \rtimes N_G(K)$, where K is a proper subgroup of G of prime order q, $\gcd(|N|, q) = 1$ and K acts fixed-point-freely on N.

Proof. Let M/N be a minimal normal subgroup of G/N. Then M/N is an elementary abelian q-group for some prime q. Suppose $\gcd(|N|, q) = q$. Then there exists an element $a \in N$ of order q. If $x \in M \setminus N$, then q divide order of x and so x^G and a^G are connected in $\Gamma(G)$ by Lemma 2.1. Thus, by the definition of N, if $x, y \in M \setminus \{1\}$, then x^G is connected to y^G in $\Gamma(G)$, contradicting the maximality of N. Thus $\gcd(|N|, q) = 1$.

Since $(|N|, q) = 1$, $M = N \rtimes K$, where K is an elementary abelian q-group. From the definition of N and Theorem 2.3, it follows that K acts fixed-point-freely on N. Thus $|K| = q$ and Frattini argument implies that, $G = NN_G(K)$. Since M is a Frobenius group with the kernel N and a complement K, $N \cap N_G(K) = N_N(K) = 1$. Thus $G = N \rtimes N_G(K)$, as required. $\boxed{QED}$

Theorem 3.4. Let G be a finite solvable group with a disconnected graph $\Gamma(G)$. Then there exists a nilpotent normal subgroup N of G such that one of the following holds:

(i) $G = N \rtimes H$ is a Frobenius group with the kernel N and a complement H;

(ii) $G = (N \rtimes L) \rtimes \langle x \rangle$, where L is a nontrivial cyclic subgroup of G of odd order which acts fixed-point-freely on N, $x \in N_G(L)$ is such that $\langle x \rangle$ acts fixed-point-freely on L, and there exist $a \in N \setminus \{1\}$ and $i \in \mathbb{N}$ such that $x^i \neq 1$ and $[a, x^i] = 1$.

Conversely, if either (i) or (ii) holds, then $\Gamma(G)$ is disconnected.

Proof. Let N be a maximal normal subgroup of G such that the subgraph induced by vertices in N is connected in $\Gamma(G)$. By Lemma 3.3, $G = N \rtimes N_G(K)$, where $|K| = q$ for some prime q, $\gcd(|N|, q) = 1$ and K acts fixed-point-freely on N.

First suppose that u^G and v^G are connected via a path in $\Gamma(G)$ for all $u, v \in N_G(K) \setminus \{1\}$. Then, by Theorem 2.3 and the definition of $H := N_G(K)$,

$N_G(K)$ acts fixed-point-freely on N. Thus G is a Frobenius group with kernel N and complement H, hence (i) holds in this case.

Next suppose that there exist $u, v \in N_G(K) \setminus \{1\}$ such that u^G and v^G are not connected in $\Gamma(G)$. Write $L = C_G(K)$. Since $\Gamma(L)$ is connected, it follows that L is a proper subgroup of $N_G(K)$ and since $|K| = q$, $N_G(K) = L\langle x\rangle$ for some $x \in N_G(K)$. Hence $L > 1$, $x \in N_G(L)$ and since L is connected, it follows by Theorem 2.3 that $\gcd(|L|, x) = 1$ and $\langle x\rangle$ acts fixed-point-freely on L. Thus $N_G(K) = L \rtimes \langle x\rangle$ and L is nilpotent. As $N \cap N_G(K) = 1$, we obtain $G = (N \rtimes L) \rtimes \langle x\rangle$. Since NL is normal in G, the maximality of N implies, by Theorem 2.3, that L acts fixed-point-freely on N.

Assume L has even order and P is a Sylow 2-subgroup of L. As a characteristic subgroup of the nilpotent group L, we have $P \trianglelefteq N_G(L)$. Suppose that $y \in P$ is an involution. Then P is either a cyclic group or a generalized quaternion group, since NL is a Frobenius group with kernel N and complement L. Therefore $y^x = y$, which contradicts the fact that $\langle x\rangle$ acts fixed-point-freely on L. So, L has odd order. Since L is nilpotent with no elementary subgroups of rank two, L is cyclic.

Finally, we show that there exist $a \in N$ and $i \in \mathbb{N}$ such that $x^i \neq 1$ and $[a, x^i] = 1$. Let $y \in L\setminus\{1\}$. Then Theorem 2.3 implies, by our assumptions, that y^G and x^G belong to different connected components of $\Gamma(G)$. Let $b \in N\setminus\{1\}$. By Theorem 2.6, b^G is connected to y^G or x^G via a path in $\Gamma(G)$. But, by Theorem 2.3 and the maximality of N, there is not path in $\Gamma(G)$ connecting b^G and y^G, so b^G is connected to x^G via a path in $\Gamma(G)$. If $\langle x\rangle$ acts fixed-point-freely on NL, then NL is nilpotent and so is connected, which contradicts the maximality of N. Hence there exist $i \in \mathbb{N}$ such that $x^i \neq 1$ and $a \in N$ such that $ay \neq 1$ and $[x^i, ay] = 1$. It follows that if $b \in N \setminus \{1\}$, then we have $b^G \leftrightarrow x^G \leftrightarrow (ay)^G$ and the maximality of N implies, in view of Theorem 2.3, that $ay \in N \setminus \{1\}$. Thus $y = 1$ and $[x^i, a] = 1$, as claimed. So (ii) holds in this case.

Conversely, assume that either (i) or (ii) holds. If (i) holds, then $\{a^G \mid a \in N \setminus \{1\}\}$ and $\{h^G \mid a \in H \setminus \{1\}\}$ are the two distinct connected components of $\Gamma(G)$.

Assume (ii) holds and let $L = \langle y\rangle$. First we show that $C_G(y) = L$. If $azx^j \in C_G(y)$ with $a \in N$ and $z \in L$, then

$$1 = [y, azx^j] = [y, x^j][y, az]^{x^j} = [y, x^j][y, z]^{x^j}[y, a]^{zx^j} = [y, x^j][y, a]^{zx^j}.$$

But $[y, x^j] \in L$, $[y, a]^{zx^j} \in N$ and $L \cap N = 1$, so $[y, x^j] = [y, a] = 1$. Since $y \neq 1$, it follows from our assumptions that $a = x^j = 1$ and $azx^j = z \in L$, as claimed.

Now, by Lemma 3.1, $\mathrm{Nil}_G(y) = L$ and hence the set $\{y^G \mid y \in L\}$ is a complete connected component of $\Gamma(G)$. Since $x \notin y^G$ for each $y \in L$, it follows from Theorem 2.6 that $\Gamma(G)$ has two connected components in the case too.

The proof is complete. QED

4 Induced subgraph of NCC-graph

In 2009 Herzog et al. considered the induced subgraph $\Gamma_c(G)$ of CCC-graph with vertices set $\{g^G \mid g \in G \setminus Z(G)\}$ and proved that if G is a periodic non-abelian group, then $\Gamma_c(G)$ is an empty graph if and only if G is isomorphic to one of the following groups S_3, D_8 or Q_8. Recently, Mohammadian and et al. [7] classified all finite non-abelian groups G with triangle-free CCC-graphs. In this section, we consider the induced subgraph $\Gamma_n(G)$ of NCC-graph with vertices set $V(\Gamma(G)) \setminus \{g^G \mid g \in \mathrm{Nil}(G)\}$ and classify all finite non-nilpotent groups G with empty and triangle-free NCC-graphs. We begin the section with the following result which shows that we can use $Z^*(G)$ for $\mathrm{Nil}(G)$ when the group G is finite, where $Z^*(G)$ is the hypercenter of finite group G. So, we can assume that vertices set of $\Gamma_n(G)$ is the set $V(\Gamma(G)) \setminus \{(z^*)^G \mid z^* \in Z^*(G)\}$.

Lemma 4.1 ([2, Lemma 3.1])**.** Let G be a finite group. Then the following assertions hold:

(i) $\langle x, Z^*(G)\rangle$ is nilpotent for all $x \in G$;

(ii) $Z^*(G) = \mathrm{Nil}(G)$.

Theorem 4.2 ([6, Theorem 19])**.** Let G be a periodic non-abelian group. Then $\Gamma_c(G)$ is an empty graph if and only if G is isomorphic to one of the groups D_8, Q_8, or S_3.

Theorem 4.3. Let G be a finite non-nilpotent group. Then $\Gamma_n(G)$ is an empty graph if and only if $G \cong S_3$.

Proof. First we show that $Z^*(G) = 1$. To find a contradiction, let $z^* \in Z^*(G)$ be a p-element. Let $x \in G \setminus Z^*(G)$ be a q-element with $q \neq p$. Then $\langle x, xz^*\rangle$ is nilpotent by Lemma 4.1, and since $\Gamma_n(G)$ has no edges, we have $xz^* = x^g$ for some $g \in G$. Thus $|xz^*| = |x|$, which is impossible. Therefore, $Z^*(G) = 1$.

Now $Z(G) = Z^*(G) = 1$ and so $\Gamma_c(G)$ is an induced subgraph of $\Gamma_n(G)$. Thus, Theorem 4.2 implies that $G \cong S_3$. QED

Now, we state two important Lemmas of T. A. Peng [9] which are useful in the proof of main Theorem of this section.

Lemma 4.4 ([9, Lemma 2])**.** Let G be a finite group. A p-subgroup P of G lies in $Z^*(G)$ if and only if $[P, O^p(G)] = 1$, where $O^p(G)$ is a subgroup generated by all p'-elements of G.

Lemma 4.5 ([9, Lemma 3]). Let G be a finite group. If a prime p divides $|Z^*(G)|$, then p divides $|Z(G)|$.

Theorem 4.6 ([7, Theorem 2.3]). Let G be a finite non-abelian group of odd order. Then $\Gamma_c(G)$ is triangle-free if and only if $|G| = 21$ or 27.

Theorem 4.7 ([7, Section 3.2]). Let G be a finite non-abelian group of even order which is not a 2-group. Then $\Gamma_c(G)$ is triangle-free if and only if G satisfies one of the following conditions:

(1) $Z(G) \neq 1$ and G is isomorphic to D_{12} or $T_{12} := \langle a, b \mid a^4 = b^3 = 1, b^a = a^{-1}\rangle$;

(2) G is a centerless group and

- (i) If G is non-solvable, then G is isomorphic to one of the groups $PSL(2, q)$ ($q \in \{4, 7, 9\}$), $PSL(3, 4)$, or $SG(960, 11357)$.
- (ii) If G is solvable, then G is isomorphic to one of the groups S_3, D_{10}, A_4, S_4, $SG(72, 41)$, $SG(192, 1023)$, or $SG(192, 1025)$.

Theorem 4.8. Let G be a finite non-nilpotent group of odd order. Then $\Gamma_n(G)$ is triangle-free if and only if $|G| = 21$.

Proof. First we claim that $Z^*(G) = 1$. To reach to a contradiction, let $z^* \in Z^*(G)$ be a p-element $x \in G \setminus Z^*(G)$ be a q element with $q \neq p$. Then $\{x^G, (x^{-1})^G, (xz^*)^G\}$ induces a triangle, which is a contradiction. Therefore $Z^*(G) = 1$. Now since $Z(G) = Z^*(G) = 1$, $\Gamma_c(G)$ is an induced subgraph of $\Gamma_n(G)$. Thus, by Theorem 4.6, $|G| = 21$, as required. QED

Theorem 4.9. Let G be a finite non-nilpotent group of even order. Then $\Gamma_n(G)$ is triangle-free if and only if G is isomorphic to one of the groups S_3, D_{10}, D_{12}, A_4, T_{12}, or $PSL(2, q)$ ($q \in \{4, 7, 9\}$).

Proof. First, we show that $|x|$ has at most two distinct prime divisors for all $x \in G \setminus Z^*(G)$. Suppose on the contrary that there exists $x \in G \setminus Z^*(G)$ such that $|x| = p^\alpha q^\beta r^\gamma$ for some distinct primes p, q, and r. Since $x \notin Z^*(G)$ we may assume without loss of generality that $x^{p^\alpha}, x^{q^\beta} \notin Z^*(G)$. Then the vertices x^G, $(x^{p^\alpha})^G$, and $(x^{q^\beta})^G$ give rise to a triangle, which is a contradiction.

Now, we claim that $Z^*(G) = Z(G)$. Assume $Z^*(G) \neq 1$. If the elements of G have prime power orders, then G is a p-group, which is a contradiction. Therefore, G has a non-hypercentral element x whose order is not a prime power. Assume that there exists $z^* \in Z^*(G)$ which is not of prime power order. Lemma 4.5, implies that there exists $z \in Z(G)$ which is not of prime power order. Since the order of every element of $G \setminus Z^*(G)$, has at most two distinct prime divisors, we deduce that $\pi(\langle x\rangle) = \pi(\langle z\rangle) = \pi(G) = \{2, p\}$ for some odd prime p. Let Q be a Sylow 2-subgroup of G and suppose there exists $y \in Q \setminus Z^*(G)$. Then there exists an element z' of order p in $Z(G)$ such that the vertices y^G, $(yz')^G$ and $(yz'^{-1})^G$ induce a triangle, which is a contradiction. Thus $Q \leq Z^*(G)$ and by Lemma 4.4, Sylow p-subgroup of G is normal. Therefore G is nilpotent, which is a contradiction.

Hence, every element of $Z^*(G)$ has prime power order and so $Z^*(G)$ is a q-group for prime q. Using Lemma 4.5 again, $Z(G)$ is a q-group. Assume q is odd and let $c \in Z(G)$ be a nontrivial element of prime order q. Since $|G|$ is even, there exists a 2-element $x \in G \setminus Z^*(G)$. Then $\{x^G, (xc)^G, (xc^{-1})^G\}$ induces a triangle unless $(xc)^G = (xc^{-1})^G$. Hence $(xc)^G = (xc^{-1})^G$ and there exists $g \in G$ such that $x^g = xc^{-2}$. But then $|x| = |xc^{-2}| = q|x|$, which is a contradiction. Therefore $Z(G)$ is a 2-group and accordingly $Z^*(G)$ is a 2-group.

Let $a \in Z^*(G) \setminus Z(G)$ and $b \in Z(G)$. Then the vertices y^G, $(yb)^G$ and $(ya)^G$ form a triangle for each p-element $y \in G \setminus Z^*(G)$ unless $(yb)^G = (ya)^G$, in which case $y^g = yab^{-1}$ for some $g \in G$ and so $|y| = |yab^{-1}| = |y||ab^{-1}|$, which is a contradiction. Therefore $Z(G) = Z^*(G)$, as claimed.

Finally, since $Z^*(G)$ coincides with $Z(G)$, $\Gamma_c(G)$ is an induced subgraph of $\Gamma_n(G)$. Thus Theorem 4.7 implies that G is isomorphic to one of the groups in the Theorem 4.7. Using GAP we can show that the $\Gamma_n(G)$ has a triangle if and only if G is isomorphic to one of the groups S_4, $SG(72, 41)$, $SG(192, 1023)$, $SG(192, 1025)$, $SG(960, 11357)$, or $PSL(3, 4)$. The proof is complete. QED

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