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On the structure of Generalized Symmetric Spaces of $\mathbf{SL_2}(\mathbb{F}_q)$ and $\mathbf{GL_2}(\mathbb{F}_q)$

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Abstract. In this paper we will discuss the generalized symmetric spaces for $\mathrm{SL}_2(\mathbb{F}_q)$ and $\mathrm{GL}_2(\mathbb{F}_q)$. Specifically we will characterize the structure of these spaces and prove that when the characteristic of $\mathbb{F}_q$ is not equal to two the extended generalized symmetric space is equal to the generalized symmetric space for $\mathrm{SL}_2(\mathbb{F}_q)$ and nearly equal for $\mathrm{GL}_2(\mathbb{F}_q)$ for all but one involution.

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1 Introduction

Real symmetric spaces were introduced by Élie Cartan [2] as a special class of homogeneous Riemannian manifolds. These spaces were later generalized by Berger [1] who gave classifications of the irreducible semisimple symmetric

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spaces. Since then a rich and deep theory has developed that plays a key role in many fields of active research such as Lie theory, differential geometry, harmonic analysis and physics. The theory of symmetric spaces has numerous generalizations, e.g., symmetric varieties, symmetric k-varieties, Vinberg's theta-groups, spherical varieties, Gelfand pairs, Bruhat-Tits buildings, Kac-Moody symmetric spaces, and generalized symmetric spaces, which are of importance in many areas of mathematics and physics, including number theory, algebraic geometry, and representation theory. These generalizations provide a vast source of interesting open problems.

Generalized symmetric spaces can be defined as the homogeneous spaces G/H with G an arbitrary group and $H = G^\theta = \{g \in G \mid \theta(g) = g\}$ the fixed-point group of an order n-automorphism θ. Of special interest are automorphisms of order 2 which are also called involutions. If G is an algebraic group defined over a field k and θ an involution defined over k, then these spaces are are also called symmetric k-varieties, which were introduced in [3]. Much is known of the structure of symmetric k-varieties for k algebraically closed and $k = \mathbb{R}$. The goal of this paper is to explore the structure of generalized symmetric spaces for $G = \mathrm{SL}_2(k)$ and $\mathrm{GL}_2(k)$ where $k = \mathbb{F}_q$. Generalized symmetric spaces over finite fields occur in the study of perverse sheaves as in the work of Lusztig in [8] and others.

For involutions there is a natural embedding of the homogeneous spaces G/H in the group G as follows. Let $\tau : G \to G$ be a morphism of G given by $\tau(g) = g\theta(g)^{-1}$ for $g \in G$ where θ is an involution of G. The image is $\tau(G) = Q = \{g\theta(g)^{-1} \mid g \in G\}$. The map τ induces an isomorphism of the coset space G/H onto $\tau(G) = Q$. So, $G/H \cong Q$. If G is an algebraic group defined over an algebraically closed field, then Q is a closed subvariety of G.

Definition 1. The **generalized symmetric space** determined by (G, θ) is $G/H \cong Q = \{g\theta(g)^{-1} \mid g \in G\}$.

Note, if one considers $H\backslash G$ instead of G/H and lets H act on G from the left, then we define $\tau(g) = g^{-1}\theta(g)$.

Definition 2. The **extended symmetric space** determined by (G, θ) is $R = \{g \in G \mid \theta(g) = g^{-1}\}$.

In general, $Q \subseteq R$ and typically $Q \neq R$. For example, the involution $\theta : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$ defined by $\theta(A) = (A^T)^{-1}$, where A^T denotes the transpose of the matrix A, finds Q as the set of matrices of the form $\{AA^T \mid A \in \mathrm{SL}_2(\mathbb{R})\}$ and $R = \{A \in \mathrm{SL}_2(\mathbb{R}) \mid A = A^T\}$. Here, R is the familiar set of symmetric matrices in $\mathrm{SL}_2(\mathbb{R})$, but Q is the set of symmetric positive definite matrices in $\mathrm{SL}_2(\mathbb{R})$. Clearly, $Q \subseteq R$ but $Q \neq R$. We will see in this paper that when we work over a finite field the results are different. The extended symmetric spaces play an important role in generalizing the Cartan decomposition for real reductive groups to reductive algebraic groups defined over an arbitrary field. While for real groups it suffices to use Q for the Cartan decomposition, in the general case one needs the extended symmetric space R.

We begin our analysis using a classification of the involutions of $\mathrm{SL}_2(\mathbb{F}_q)$ and $\mathrm{GL}_2(\mathbb{F}_q)$ as determined by Helminck and Wu in [6]. Each conjugacy class of involutions will define a generalized symmetric space as seen in [4]. Using these conjugacy classes, an examination of Q and R for various involutions allows for further study of the structure of the generalized symmetric spaces. Namely, we can look at the study of orbit decomposition of Q by Borel subgroups, fixed-point groups, toral subgroups and parabolic subgroups. These decompositions lend themselves to the representation theory of the space as seen in [5].

In section 1 we discuss the basic definitions and classification of involutions of the groups $\mathrm{SL}_2(\mathbb{F}_q)$ and $\mathrm{GL}_2(\mathbb{F}_q)$ and give an explicit description of Q and R over a general finite field for a representative involution for these specific groups. Our discussion begins with $\mathrm{SL}_2(\mathbb{F}_q)$. We continue to use those results in the characterization of $\mathrm{GL}_2(\mathbb{F}_q)$. We prove that $Q = R$ for $\mathrm{SL}_2(\mathbb{F}_q)$ when $q \neq 2^t$ in section 1 and discuss Q and R in $\mathrm{GL}_2(\mathbb{F}_q)$ in section 3.

2 The generalized and extended symmetric spaces in $\mathrm{SL}_2(\mathbb{F}_q)$

For any involution of an algebraic group G, the generalized symmetric space Q is contained in the extended generalized symmetric space R because

$$\theta\left(g\theta(g)^{-1}\right) = \theta(g)g^{-1} = \left(g\theta(g)^{-1}\right)^{-1}.$$

However, the reverse containment is rare. For example, if G is a real connected group, Q, the image of τ, is also connected while R often is not. After establishing some preliminary facts, we will show that reverse containment does hold for any involution of the special linear group $\mathrm{SL}_2(\mathbb{F}_q)$ provided that the characteristic of $\mathbb{F}_q$ is not equal to two.

To begin we will consider finite fields $k = \mathbb{F}_q$ of characteristic not equal to two. We will also consider automorphisms θ and σ of a group G to be isomorphic if there exists a $\phi \in \mathrm{Aut}(G)$ such that $\phi\theta\phi^{-1} = \sigma$. The following theorem classifies the involutions for $\mathrm{SL}_2(k)$ over such fields.

Theorem 1 (Helminck, Wu [6, Theorem 1])**.** *Let $G = \mathrm{SL}_2(k)$ with char$(k) \neq 2$ and let $\theta \in \mathrm{Aut}\,(G)$ be an involution. Then θ acts as conjugation by $X_m = \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$ where $m \in k^*$. Furthermore, if $m, n \in k^*$ are in the same square class, then conjugation by X_m is isomorphic to conjugation by X_n.*

Given that $k = \mathbb{F}_q$ is not of characteristic two, we know $\mathbb{F}_q^*/\left(\mathbb{F}_q^*\right)^2$ has just two square classes. Therefore, there are, up to isomorphism, two involutions both acting as conjugation by X_m - one for m a square and one for m a non-square. We will denote the involution that acts as conjugation by X_m as θ_m. The generalized symmetric space of G formed from the action of θ_m will be denoted as $Q_m(G)$ and the extended symmetric space of the same group under the same involution as $R_m(G)$. We omit the subscript m when specifying the involution is unnecessary.

For $G = \mathrm{SL}_2\,(k)$ with char$(k) \neq 2$ we have the following descriptions of the generalized and extended symmetric spaces:

$$\begin{aligned} Q_m\left(\mathrm{SL}_2\,(k)\right) &= \left\{ g\theta(g)^{-1} \middle|\, g \in \mathrm{SL}_2\,(k) \right\} \\ &= \left\{ g \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix} g^{-1} \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}^{-1} \middle|\; g \in \mathrm{SL}_2\,(k) \right\} \text{ and} \\ R_m\left(\mathrm{SL}_2\,(k)\right) &= \left\{ g \in \mathrm{SL}_2\,(k) \middle|\, \theta(g) = g^{-1} \right\} \\ &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2\,(k) \;\middle|\; \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \right\} \\ &= \left\{ \begin{pmatrix} a & b \\ -mb & d \end{pmatrix} \;\middle|\; ad + mb^2 = 1 \right\}. \end{aligned}$$

We can now show these two spaces are equal for $k = \mathbb{F}_q$ with $q \neq 2^t$.

Theorem 2. *For $G = \mathrm{SL}_2(k)$ where $k = \mathbb{F}_q$ with* $\mathrm{char}(k) \neq 2$ *and any involution θ_m of G, $Q(\mathrm{SL}_2(k)) = R(\mathrm{SL}_2(k))$.*

Proof. As was previously shown, $Q(\mathrm{SL}_2(\mathbb{F}_q)) \subseteq R(\mathrm{SL}_2(\mathbb{F}_q))$. Let $r \in R\left(\mathrm{SL}_2\left(\mathbb{F}_q\right)\right)$. Then $r = \begin{pmatrix} a & b \\ -mb & d \end{pmatrix}$ for some a, b, and d in $\mathbb{F}_q$ satisfying

$$1 \quad = \quad ad + mb^2. \tag{2.1}$$

To demonstrate that r is an element of $Q\left(\mathrm{SL}_2\left(\mathbb{F}_q\right)\right)$, we must find some $g \in$

$\mathrm{SL}_2(\mathbb{F}_q)$ such that

$$r = g \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix} g^{-1} \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}^{-1}$$

in accordance to the description of $Q_m(\mathrm{SL}_2(\mathbb{F}_q))$ given above. We proceed with several cases. Though the details of constructing g in each case are not shown, it should be stated that the computations required to provide this presentation involve solving several nonlinear multivariable equations simultaneously.

Case 1. Suppose $a = 0$. Then by equation (2.1) we have $b^2 = m^{-1}$. Therefore, this case only arises for θ_m where m is in the square class. Realizing $b = \pm\sqrt{m^{-1}}$ and letting t be any element in $\mathbb{F}_q^*$ we have:

$$r = \begin{pmatrix} 0 & \pm\sqrt{m^{-1}} \\ \mp\sqrt{m} & d \end{pmatrix}.$$

Then

$$r = g \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix} g^{-1} \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}^{-1} \text{ for } g = \begin{pmatrix} t & \pm t\sqrt{m^{-1}} \\ \pm\frac{\sqrt{m}(dt-t^{-1})}{2} & \frac{dt+t^{-1}}{2} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{F}_q).$$

It is worth noting that in this description we are listing all possible $g \in \mathrm{SL}_2(\mathbb{F}_q)$ such that $r = g(\theta(g))^{-1}$. In practice, if one wanted a specific element g, one could chose a value for t and either sign for $\sqrt{m}$.

Case 2. Suppose $d = 0$. Again, equation (2.1) gives us $b^2 = m^{-1}$. Thus, as in the previous case, we are only considering θ_m where m is in the square class. Realizing $b = \pm\sqrt{m^{-1}}$ and letting t be any element in $\mathbb{F}_q^*$ we have:

$$r = \begin{pmatrix} a & \pm\sqrt{m^{-1}} \\ \mp\sqrt{m} & 0 \end{pmatrix}$$

is in Q for

$$g = \begin{pmatrix} \mp\frac{at+mt^{-1}}{2\sqrt{m}} & \frac{at-mt^{-1}}{2m} \\ t & \mp\frac{t}{\sqrt{m}} \end{pmatrix}.$$

Case 3. Suppose $a \neq 0$, $d \neq 0$ and $-md \in (\mathbb{F}_q^*)^2$. Write $-md = s^2$ for some $s \in \mathbb{F}_q^*$. Thus $d = -m^{-1}s^2$. By equation (2.1) we have $a = ms^{-2}(mb^2 - 1)$. Then

$$r = \begin{pmatrix} ms^{-2}\left(mb^2 - 1\right) & b \\ -mb & -m^{-1}s^2 \end{pmatrix}$$

is in Q for

$$g = \begin{pmatrix} bms^{-1} & s^{-1} \\ -s & 0 \end{pmatrix}.$$

Case 4. Suppose $a \neq 0$, $d \neq 0$ and $-md \notin (\mathbb{F}_q^*)^2$. First we show that $d + m^{-1}x^2 = y^2$ holds for some $x, y \in \mathbb{F}_q$. If d is a square then choose $x = 0$ and $y = \sqrt{d}$. Suppose d is not a square. Then $-m = (-md)(d^{-1})$ is a square because it can be expressed as a product of two non-squares. Every element in a finite field can be expressed as the sum of two squares. Therefore, $d = y^2 + z^2$ for some y and z in $\mathbb{F}_q$. Thus, $d - z^2 = y^2$, or equivalently $d + m^{-1}\left(\sqrt{-mz^2}\right)^2 = y^2$ and we can choose $x = \sqrt{-mz^2}$.

Recall we have chosen $r = \begin{pmatrix} a & b \\ -mb & d \end{pmatrix}$ such that equation (2.1) holds. This equation implies that $a = d^{-1}(1 - mb^2)$ and so for

$$r = \begin{pmatrix} \frac{1-mb^2}{d} & b \\ -mb & d \end{pmatrix}$$

let

$$g = \begin{pmatrix} \frac{xb+y}{d} & \frac{xm^{-1}+by}{d} \\ x & y \end{pmatrix}.$$

We know $g \in \mathrm{SL}_2(\mathbb{F}_q)$ because we chose x and y so that $d + m^{-1}x^2 = y^2$. Thus it follows that $\det(g) = \frac{y^2m - x^2}{dm} = \frac{dm}{dm} = 1$. $\boxed{QED}$

Remark 1. Theorem 2 also holds if one replaces $\mathbb{F}_q$ by any field k of characteristic not 2 for which any element can be written as a sum of two squares.

Remark 2. A similar result does not hold for $k = \mathbb{Q}_p$ the p-adic numbers. We conjecture that $Q_m(\mathrm{SL}_2(\mathbb{Q}_p)) = R_m(\mathrm{SL}_2(\mathbb{Q}_p))$ if and only if $m \in \mathbb{Q}_p^2$.

It is important to note that when $\mathbb{F}_q$ has odd characteristic, then $\mathrm{Inn}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, where $\mathrm{Inn}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is conjugation by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, is conjugate to the involution $\theta(A) = (A^T)^{-1}$ on $\mathrm{SL}_2(\mathbb{F}_q)$; therefore, since $R_m = Q_m$, we have also essentially shown that every symmetric matrix in $\mathrm{SL}_2(\mathbb{F}_q)$ has a Cholesky decomposition gg^T where $g \in \mathrm{SL}_2(\mathbb{F}_q)$. However, a similar result does not hold when $\mathrm{char}(k) = 2$. It was shown in [9] that the only involution of $\mathrm{SL}_2(\mathbb{F}_2)$ is $\mathrm{Inn}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which gives rise to

$$Q(\mathrm{SL}_2(\mathbb{F}_2)) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\} \quad \text{and}$$

$$R(\mathrm{SL}_2(F_2)) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

In this case, the fact that $\mathrm{Inn}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ acts as θ where $\theta(A) = \left(A^T\right)^{-1}$ and $Q \neq R$ mirrors the result of Lempel [7] since no symmetric matrix in $\mathrm{GL}_n(\mathbb{F}_2)$ with zeros on the diagonal can be decomposed into gg^T for some $g \in \mathrm{GL}_n(\mathbb{F}_2)$.

In section 3 we will consider the generalized symmetric space and the extended symmetric space for $\mathrm{GL}_2(\mathbb{F}_q)$, and we will see how the result differs from our result for $\mathrm{SL}_2(\mathbb{F}_q)$.

3 The generalized and extended symmetric spaces for $\mathrm{GL}_2(\mathbb{F}_q)$

The main result of the previous section, that the generalized and extended symmetric spaces of $\mathrm{SL}_2(\mathbb{F}_q)$ are equal when $q \neq 2^t$, is surprising. In this section, we discuss the conjugacy classes of involutions and extend our considerations to the generalized and the extended symmetric spaces for $\mathrm{GL}_2(\mathbb{F}_q)$. We conclude by showing that, while the generalized and extended symmetric spaces are not equal for $\mathrm{GL}_2(\mathbb{F}_q)$ with $q \neq 2^t$, they are quite similar for inner automorphisms. This result stems from the connections between the generalized and extended symmetric spaces for $\mathrm{GL}_2(\mathbb{F}_q)$ and $\mathrm{SL}_2(\mathbb{F}_q)$.

Since an automorphism of $\mathrm{SL}_2(\mathbb{F}_q)$ can be written as the restriction to $\mathrm{SL}_2(\mathbb{F}_q)$ of a inner automorphism of $\mathrm{GL}_2(\mathbb{F}_q)$, using [6], we can determine that all inner automorphisms of $\mathrm{GL}_2(\mathbb{F}_q)$ are conjugate to the two forms discussed in Theorem 1. However, in $\mathrm{GL}_2(\mathbb{F}_q)$ we have an additional automorphism, the involution $\theta(A) = (A^T)^{-1}$ which can easily be seen as outer because it acts non-trivially on the center. These three forms constitute all the conjugacy classes of involutions of $\mathrm{GL}_2(\mathbb{F}_q)$. We will consider these cases in the following propositions.

Proposition 1. *Let $\mathbb{F}_q$ be any finite field of characteristic not equal to 2 and consider the inner involution θ_m of $\mathrm{GL}_2(\mathbb{F}_q)$. Then the generalized symmetric space $Q(\mathrm{SL}_2(\mathbb{F}_q))$ is equal to the generalized symmetric space $Q(\mathrm{GL}_2(\mathbb{F}_q))$.*

Proof. Clearly $Q(\mathrm{SL}_2(\mathbb{F}_q)) \subseteq Q(\mathrm{GL}_2(\mathbb{F}_q))$. Let $x \in Q(\mathrm{GL}_2(\mathbb{F}_q))$. Then $x = g\theta_m(g)^{-1}$ for some $g \in \mathrm{GL}_2(\mathbb{F}_q)$. Thus

$$\begin{aligned}
\det(x) &= \det\left(g\theta_m(g)^{-1}\right)\\
&= \det(g)\det\left(\left(X_m g X_m^{-1}\right)^{-1}\right)\\
&= \det(g)\det\left(X_m g^{-1} X_m^{-1}\right)\\
&= \det(g)\det(g^{-1})
\end{aligned}$$

$$= \quad 1$$

which implies that $x \in \mathrm{SL}_2(\mathbb{F}_q)$. Since $Q(\mathrm{GL}_2(\mathbb{F}_q)) \subseteq R(\mathrm{GL}_2(\mathbb{F}_q))$ we know $x \in R(\mathrm{GL}_2(\mathbb{F}_q))$. However, by $\det(x) = 1$ we can restrict further to conclude $x \in R(\mathrm{SL}_2(\mathbb{F}_q))$. Thus $x \in Q(\mathrm{SL}_2(\mathbb{F}_q))$ by Theorem 2. $\boxed{QED}$

While the analogous result for the extended symmetric spaces of the two groups does not hold, they are quite similar.

Proposition 2. *Let $\mathbb{F}_q$ be any finite field of characteristic not equal to 2 and consider the inner involution θ_m of $\mathrm{GL}_2(\mathbb{F}_q)$. If $m \notin (\mathbb{F}_q^*)^2$, then $R_m(\mathrm{GL}_2(\mathbb{F}_q)) = R_m(\mathrm{SL}_2(\mathbb{F}_q))$. If $m \in (\mathbb{F}_q^*)^2$, then*

$$R_m(\mathrm{GL}_2(\mathbb{F}_q)) = R_m(\mathrm{SL}_2(\mathbb{F}_q)) \cup \left\{ \begin{pmatrix} 0 & \pm\sqrt{m^{-1}} \\ \pm\sqrt{m} & 0 \end{pmatrix} \right\}.$$

Proof. Let $r \in R_m(\mathrm{GL}_2(\mathbb{F}_q))$. Then $\det(X_m r X_m^{-1}) = \det(r^{-1})$, implying $\det(r) = \pm 1$. If $\det(r) = 1$ then clearly $r \in R_m(\mathrm{SL}_2(\mathbb{F}_q))$. Suppose that $r = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det(r) = -1$. Since $r \in R_m(\mathrm{GL}_2(\mathbb{F}_q))$ we have

$$\begin{aligned} X_m r &= r^{-1} X_m \\ \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix} \\ \begin{pmatrix} c & d \\ ma & mb \end{pmatrix} &= \begin{pmatrix} bm & -d \\ -am & c \end{pmatrix}. \end{aligned}$$

Therefore, $a = d = 0$. Because $\det(r) = ad - bc = -1$ we have $bc = 1$. But $c = mb$ and thus $mb^2 = 1$, forcing m to be in the square class. We conclude $\det(r) = -1$ only in the case where $m \in \left(\mathbb{F}_q^*\right)^2$ and in this case $mb^2 = 1$ so that $b = \pm\sqrt{m^{-1}}$ and $c = mb = \pm\sqrt{m}$. Therefore, if $m \notin (\mathbb{F}_q^*)^2$ we have

$$R_m(\mathrm{GL}_2(\mathbb{F}_q)) \subseteq R_m(\mathrm{SL}_2(\mathbb{F}_q)),$$

and if $m \in (\mathbb{F}_q^*)^2$ we have

$$R_m(\mathrm{GL}_2(\mathbb{F}_q)) \subseteq R_m(\mathrm{SL}_2(\mathbb{F}_q)) \cup \left\{ \begin{pmatrix} 0 & \pm\sqrt{m^{-1}} \\ \pm\sqrt{m} & 0 \end{pmatrix} \right\}.$$

The reverse containments can be checked by direct computation. $\boxed{QED}$

Our observations concerning the generalized and extended symmetric spaces of $\mathrm{GL}_2(\mathbb{F}_q)$ lead directly to the following corollary of Theorem 2.

Corollary 1. *For $k = \mathbb{F}_q$, $q \neq 2^t$, and any involution θ_m,*

$$R_m\left(\mathrm{GL}_2(\mathbb{F}_q)\right) = \begin{cases} Q_m\left(\mathrm{GL}_2(\mathbb{F}_q)\right) & \text{if } m \notin \left(\mathbb{F}_q^*\right)^2 \\ Q_m\left(\mathrm{GL}_2(\mathbb{F}_q)\right) \cup \left\{ \begin{pmatrix} 0 & \pm(\sqrt{m^{-1}}) \\ \pm\sqrt{m} & 0 \end{pmatrix} \right\} & \text{if } m \in \left(\mathbb{F}_q^*\right)^2. \end{cases}$$

We see above that R and Q are equal for certain inner involutions and nearly equal for others depending on the conjugacy class of the involution. When we consider the involution $\theta(A) = (A^T)^{-1}$ on $\mathrm{GL}_2(\mathbb{F}_q)$, then

$$\begin{aligned} R =& \{g \in \mathrm{GL}_2(\mathbb{F}_q) | \theta(g) = g^{-1}\} \\ =& \{g \in \mathrm{GL}_2(\mathbb{F}_q) | g^T = g\}. \end{aligned}$$

As mirrored in the introduction, these are symmetric matrices in $\mathrm{GL}_2(\mathbb{F}_q)$; however, $Q = \{gg^T | g \in \mathrm{GL}_2(\mathbb{F}_q)\}$. We know immediately in this case that $|R|$ is much larger than $|Q|$. For a brief proof, consider an element $x \in Q$. Then for some $g \in \mathrm{GL}_2(\mathbb{F}_q)$ since $x = gg^T$ we have

$$\begin{aligned} \det(x) &= \det(gg^T) \\ &= \det(g)\det(g^T) \\ &= \det(g)^2. \end{aligned}$$

Since the determinant of an element in Q must be a square, while the determinant of an element of R can be any non-zero value in the field, Q will always have fewer elements than R and we never achieve equality. Note, for characteristic two, $R \neq Q$ with this automorphism. We demonstrated this case at the end of Section 2.

References

[1] M. Berger, *Les espaces symetriques noncompacts*, Annales Scientifiques De L'E.N.S., **74** (1957) 85-177.

[2] E. Cartan, *Groupes simples clos et ouvert et geometrie Riemannienne*, J. Math. Pure Appl., **8** (1929) 1-33.

[3] A. G. Helminck, *Symmetric k-varieties*, Proc. Sympos. Pure Math, **56** (1993) 233-279.

[4] A. G. Helminck, *On the classification of k-involutions*, Adv. in Math., **153** (2000) 1-117.

[5] A. G Helminck and S.P. Wang, *On Rationality Properties of Involutions of Reductive Groups*, Adv. in Math., **99** (1993) 26-96.

[6] A. G. Helminck and L. Wu, *Classification of Involutions of* $\mathrm{SL}(2,k)$, Communications in Algebra, **30(1)** (2002) 193-203.

[7] A. Lempel, *Matrix Factorizations over* $GF(2)$ *and Trace-Orthogonal Bases of* $GF(2^n)$, SIAM J. Computation, **4(2)** (1975) 175-186.

[8] G. Lusztig, *Quivers, perverse sheaves, and quantized enveloping algebras*, J. Amer. Math.Soc., **4(2)** (1991), 365-421.

[9] N. Schwartz, *k-Involutions of* $\mathrm{SL}(n,k)$ *over fields of characteristic* 2, Communications in Algebra (2017). Advanced online pubblication. doi:10.1080/00927872.20171363219