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New annuli for the zeros of polynomials

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Abstract. The purpose of this paper is to present a new annulus containing all the zeros of a polynomial using a new identity involving generalized Fibonacci numbers. This new annulus is related to two parameters. By examples, it is shown that this new annulus is more certain than the known annuli.

Keywords: Complex polynomial, location of the zeros, generalized Fibonacci numbers

MSC 2000 classification: primary 30C10, secondary 30C15, 11B39

Introduction

Let k and t be nonzero real numbers. Generalized Fibonacci numbers $F_{k,t,n}$ are defined by

$$F_{k,t,n} = kF_{k,t,n-1} + tF_{k,t,n-2} \ (n \geq 2), \tag{1}$$

with the initial values $F_{k,t,0} = 0$, $F_{k,t,1} = 1$. For $k = 1$ and $t = 1$ we obtain the well-known Fibonacci numbers F_n; for $k = 2$ and $t = 1$ we obtain the Pell numbers P_n. Generalized Lucas numbers $L_{k,t,n}$ are defined by

$$L_{k,t,n} = kL_{k,t,n-1} + tL_{k,t,n-2} \ (n \geq 2), \tag{2}$$

with the initial values $L_{k,t,0} = 2$, $L_{k,t,1} = k$. For $k = 1$ and $t = 1$ we obtain the Lucas numbers L_n (for more details see [7], [8], [11] and the references therein).

For the generalized Fibonacci number $F_{k,t,n}$ we have

$$F_{k,t,n} = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \ k = \alpha + \beta \text{ and } t = -\alpha\beta, \tag{3}$$

where $\alpha = \dfrac{k + \sqrt{k^2 + 4t}}{2}$ and $\beta = \dfrac{k - \sqrt{k^2 + 4t}}{2}$ [7].

Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$ be a non-constant complex polynomial of degree n. Recently, it has been obtained some new annuli containing all the zeros of a given polynomial $P(z)$ using some identities related to above number sequences. It is known that polynomials are important in many scientific areas and their properties have been investigated by many authors since the time of Gauss and Cauchy. For example, Gauss showed that a polynomial has no zeros outside certain circles [9]. Cauchy improved the result of Gauss [2]. Diaz-Barrero obtained two new bounds for the moduli of the zeros involving binomial coefficients and Fibonacci numbers [3]. Also he gave a ring shaped region containing all the zeros of a polynomial [4]. Diaz-Barrero and Egozcue introduced a new bound for the zeros of polynomials in term of binomial coefficients and Pell numbers [5]. Bidkham, Zireh and Mezerji proved a result concerning the location of the zeros of polynomials in an annulus involving binomial coefficients and (k,t)-Fibonacci numbers [1]. Rather and Mattoo found a new annulus containing all the zeros of the polynomials involving binomial coefficients and new number sequences [10]. Dalal and Govil presented a result providing an annulus containing all the zeros of a polynomial [6].

In this paper, motivated by the above studies, we present a new annulus for the zeros of polynomials. In Section 1, we recall some known annuli containing all the zeros of a polynomial. In Section 2, we prove a new identity using the generalized Fibonacci number sequence. Using this identity we give a new annulus containing all the zeros of polynomials. The comparison of this new annulus with known annuli is made by examples. Consequently, we see that this new annulus is significantly more certain than the known annuli. Finally, we give some comparisons of this new annulus with respect to the parameters k and t using some examples.

1 Preliminaries

In this section we recall some known theorems and corollaries in the literature. For example, Gauss introduced the following result on the investigation of the region containing the zeros of a polynomial.

Theorem 1.1. [9] Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$ be a non-constant complex polynomial of degree n. Then all the zeros of $P(z)$ lie in the disc

$$D_1 = \left\{ z \in \mathbb{C} : |z| = R \right\},$$

where $R = \max\limits_{1 \le k \le n} \left\{ n\sqrt{2}\, |a_k| \right\}^{\frac{1}{k}}$.

Cauchy [2] improved the above result of Gauss by proving the following theorems:

Theorem 1.2. [2] Let

$$P(z) = z^n + \sum_{k=0}^{n-1} a_k z^k,\ a_k\ \in \mathbb{C}$$

be a complex polynomial of degree n. All the zeros of the polynomial $P(z)$ lie in the disc

$$D_2 = \left\{z : |z| < \gamma\right\} \subset \left\{z : |z| < 1 + \beta\right\},$$

where $\beta = \max\limits_{0 \le k \le n-1} |a_k|$ and γ is the unique positive root of the real coefficient equation

$$z^n - |a_{n-1}|\, z^{n-1} - ... - |a_1|\, z - |a_0| = 0.$$

Theorem 1.3. [9] Let $P(z) = \sum\limits_{k=0}^{n} a_k z^k$ $(a_n \neq 0)$ be a non-constant complex polynomial of degree n. Then its all zeros lie in the circle $D_3 = \left\{z \in \mathbb{C} : |z| \le \theta\right\}$, where θ is the positive root of the equation

$$|a_n|\, z^n - |a_{n-1}|\, z^{n-1} - ... - |a_1|\, z - |a_0| = 0.$$

In [3] it was proved the following theorem using some identities involving Fibonacci numbers and binomial coefficients. Here $C(n,k) = \frac{n!}{k!(n-k)!}$ is the binomial coefficient.

Theorem 1.4. [3] Let $P(z) = \sum\limits_{k=0}^{n} a_k z^k$ be a non-constant complex polynomial of degree n. Then its all zeros lie in the discs $D_4 = \left\{z \in \mathbb{C} : |z| \le r_1\right\}$ or $D_5 = \left\{z \in \mathbb{C} : |z| \le r_2\right\}$, where

$$r_1 = \max_{1 \le k \le n} \left\{ \left(\frac{2^{n-1}\ C\left(n+1, 2\right)}{k^2 C\left(n, k\right)} \left|a_{n-k}\right| \right)^{\frac{1}{k}} \right\}$$

and

$$r_2 = \max_{1 \le k \le n} \left\{ \left(\frac{F_{3n}}{2^k F_k\ C\left(n, k\right)} \left|a_{n-k}\right| \right)^{\frac{1}{k}} \right\}.$$

In [4] it was obtained a new annulus using some identities involving Fibonacci numbers and binomial coefficients.

Theorem 1.5. [4] Let $P(z) = \sum\limits_{k=0}^{n} a_k z^k$ $(a_k \neq 0, 0 \leq k \leq n)$ be a non-constant complex polynomial of degree n. Then its all zeros lie in the annulus

$$D_6 = \left\{ z \in \mathbb{C} : r_1 \leq |z| \leq r_2 \right\}, \tag{4}$$

where

$$r_1 = \frac{3}{2} \min_{1 \leq k \leq n} \left\{ \frac{2^n F_k \ C(n,k)}{F_{4n}} \left| \frac{a_0}{a_k} \right| \right\}^{\frac{1}{k}} \tag{5}$$

and

$$r_2 = \frac{2}{3} \max_{1 \leq k \leq n} \left\{ \frac{F_{4n}}{2^n F_k \ C(n,k)} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{\frac{1}{k}}. \tag{6}$$

In [5] Diaz-Barrero defined a new number sequence and obtained a new annulus.

Theorem 1.6. [5] Let $P(z) = \sum\limits_{k=0}^{n} a_k z^k$ $(a_k \neq 0, 0 \leq k \leq n)$ be a non-constant complex polynomial. Then, for $j \geq 2$, all its zeros lie in the annulus

$$D_7 = \left\{ z \in \mathbb{C} : r_1 \leq |z| \leq r_2 \right\}, \tag{7}$$

where

$$r_1 = \min_{1 \leq k \leq n} \left\{ \frac{C(n,k) \, A_k B_j^k \left(b B_{j-1} \right)^{n-k}}{A_{jn}} \left| \frac{a_0}{a_k} \right| \right\}^{\frac{1}{k}} \tag{8}$$

and

$$r_2 = \max_{1 \leq k \leq n} \left\{ \frac{A_{jn}}{C(n,k) \, A_k B_j^k \left(b B_{j-1} \right)^{n-k}} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{\frac{1}{k}}. \tag{9}$$

Here $A_n = cr^n + ds^n$ and $B_n = \sum\limits_{k=0}^{n-1} r^k s^{n-1-k}$, where c, d are real constants and r, s are the roots of the equation $x^2 - ax - b = 0$ in which a, b are strictly positive real numbers. Then $\sum\limits_{k=0}^{n} C(n,k) \, A_k B_j^k \left(b B_{j-1} \right)^{n-k} = A_{jn}$ for $j \geq 2$.

Corollary 1.1. [5] Let $P(z) = \sum\limits_{k=0}^{n} a_k z^k$ $(a_k \neq 0, 0 \leq k \leq n)$ be a non-constant complex polynomial. Then all its zeros lie in the annulus

$$D_8 = \left\{ z \in \mathbb{C} : r_1 \leq |z| \leq r_2 \right\},$$

where

$$r_1 = \min_{1\le k\le n}\left\{\frac{2^k P_k C(n,k)}{P_{2n}}\left|\frac{a_0}{a_k}\right|\right\}^{\frac{1}{k}}$$

and

$$r_2 = \max_{1\le k\le n}\left\{\frac{P_{2n}}{2^k P_k C(n,k)}\left|\frac{a_{n-k}}{a_n}\right|\right\}^{\frac{1}{k}}.$$

More recently, it was given a new annulus by Bidkham, A. Zireh and H. A. Soleiman Mezerji using generalized Fibonacci numbers [1].

Theorem 1.7. [1] Let $P(z) = \sum\limits_{k=0}^{n} a_k z^k$ $(a_k \neq 0, 0 \le k \le n)$ be a non-constant complex polynomial of degree n. Then for $j \ge 1$, all the zeros of $P(z)$ lie in the annulus

$$D_9 = \left\{z \in \mathbb{C} : r_1 \le |z| \le r_2\right\}, \tag{10}$$

where

$$r_1 = \min_{1\le k\le n}\left\{\frac{C(n,k)F_{t,s,k}\left(F_{t,s,2^j}\right)^k\left(sF_{t,s,2^j-1}\right)^{n-k}}{F_{t,s,2^j n}}\left|\frac{a_0}{a_k}\right|\right\}^{\frac{1}{k}} \tag{11}$$

and

$$r_2 = \max_{1\le k\le n}\left\{\frac{F_{t,s,2^j n}}{C(n,k)F_{t,s,k}\left(F_{t,s,2^j}\right)^k\left(sF_{t,s,2^j-1}\right)^{n-k}}\left|\frac{a_{n-k}}{a_n}\right|\right\}^{\frac{1}{k}}. \tag{12}$$

Corollary 1.2. [1] Let $P(z) = \sum\limits_{k=0}^{n} a_k z^k$ $(a_k \neq 0, 0 \le k \le n)$ be a non-constant complex polynomial. Then for $j \ge 1$, all its zeros lie in the annulus

$$D_{10} = \left\{z \in \mathbb{C} : r_1 \le |z| \le r_2\right\},$$

where

$$r_1 = \min_{1\le k\le n}\left\{\frac{C(n,k)P_k\left(P_{2^j}\right)^k\left(P_{2^j-1}\right)^{n-k}}{P_{2^j n}}\left|\frac{a_0}{a_k}\right|\right\}^{\frac{1}{k}}$$

and

$$r_2 = \max_{1\le k\le n}\left\{\frac{P_{2^j n}}{C(n,k)P_k\left(P_{2^j}\right)^k\left(P_{2^j-1}\right)^{n-k}}\left|\frac{a_{n-k}}{a_n}\right|\right\}^{\frac{1}{k}}.$$

In [10] it was given a new annulus using a new number sequence.

Theorem 1.8. [10] Let $P(z) = \sum\limits_{k=0}^{n} a_k z^k$ $(a_k \neq 0, 0 \leq k \leq n)$ be a non-constant complex polynomial degree n. Then all its zeros lie in the annulus

$$D_{11} = \left\{ z \in \mathbb{C} : r_1 \leq |z| \leq r_2 \right\}, \tag{13}$$

where

$$r_1 = \frac{uv + 2w}{uvw + w^2} \min_{1 \leq k \leq n} \left\{ \frac{\left(uvw + w^2\right)^n u^{\xi(k)} \left(uv\right)^{\left\lfloor \frac{k}{2} \right\rfloor} F_k^{(u,v,w)} C(n,k)}{F_{4n}^{(u,v,w)}} \left| \frac{a_0}{a_k} \right| \right\}^{\frac{1}{k}} \tag{14}$$

and

$$r_2 = \frac{abc + c^2}{ab + 2c} \max_{1 \leq k \leq n} \left\{ \frac{F_{4n}^{(a,b,c)}}{\left(abc + c^2\right)^n a^{\xi(k)} \left(ab\right)^{\left\lfloor \frac{k}{2} \right\rfloor} F_k^{(a,b,c)} C(n,k)} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{\frac{1}{k}}, \tag{15}$$

where a, b, c, u, v, w are any positive real numbers, $\xi(k) := k - 2\left\lfloor \frac{k}{2} \right\rfloor$ and $F_m^{(a,b,c)}$ is defined as in [10].

Using a new number sequence, Dalal and Govil proved the following theorem.

Theorem 1.9. [6] Let $A_k > 0$ for $1 \leq k \leq n$ with $\sum\limits_{k=1}^{n} A_k = 1$. If $P(z) = \sum\limits_{k=0}^{n} a_k z^k$ $(a_k \neq 0, 0 \leq k \leq n)$ is a non-constant complex polynomial of degree n, then all the zeros of $P(z)$ lie in the annulus

$$D_{12} = \left\{ z \in \mathbb{C} : r_1 \leq |z| \leq r_2 \right\},$$

where

$$r_1 = \min_{1 \leq k \leq n} \left\{ A_k \left| \frac{a_0}{a_k} \right| \right\}^{\frac{1}{k}}$$

and

$$r_2 = \max_{1 \leq k \leq n} \left\{ \frac{1}{A_k} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{\frac{1}{k}}.$$

Corollary 1.3. [6] Let $P(z) = \sum\limits_{k=0}^{n} a_k z^k$ $(a_k \neq 0, 0 \leq k \leq n)$ be a non-constant complex polynomial of degree n. Then all the zeros of $P(z)$ lie in the annulus

$$D_{13} = \left\{ z \in \mathbb{C} : r_1 \leq |z| \leq r_2 \right\}, \tag{16}$$

where

$$r_1 = \min_{1\le k\le n}\left\{\frac{L_k}{L_{n+2}-3}\left|\frac{a_0}{a_k}\right|\right\}^{\frac{1}{k}} \tag{17}$$

and

$$r_2 = \max_{1\le k\le n}\left\{\frac{L_{n+2}-3}{L_k}\left|\frac{a_{n-k}}{a_n}\right|\right\}^{\frac{1}{k}}. \tag{18}$$

Corollary 1.4. [6] Let $P(z)=\sum\limits_{k=0}^{n} a_k z^k$ $(a_k\neq 0, 0\le k\le n)$ be a non-constant complex polynomial of degree n. Then all the zeros of $P(z)$ lie in the annulus

$$D_{14}=\left\{z\in\mathbb{C}: r_1\le |z|\le r_2\right\}, \tag{19}$$

where

$$r_1 = \min_{1\le k\le n}\left\{\frac{C_{k-1}C_{n-k}}{C_n}\left|\frac{a_0}{a_k}\right|\right\}^{\frac{1}{k}} \tag{20}$$

and

$$r_2 = \max_{1\le k\le n}\left\{\frac{C_n}{C_{k-1}C_{n-k}}\left|\frac{a_{n-k}}{a_n}\right|\right\}^{\frac{1}{k}}. \tag{21}$$

Here, C_n is the nth Catalan number defined by $C_n=\frac{C(2n,n)}{n+1}$.

2 Main Results

In this section we obtain a new annulus for the zeros of polynomials using generalized Fibonacci numbers. To do this we prove the following proposition. Let us consider the nth generalized Fibonacci number $F_{k,t,n}$ defined in (1).

Proposition 2.1. We have

$$\sum_{i=1}^{n} k t^{n-i} F_{k,t,i}^2 = F_{k,t,n}F_{k,t,n+1}. \tag{22}$$

Proof. At first, we consider the case that n is an odd positive integer. Then using the equation (3) we get

$$\begin{aligned}
&\sum_{i=1}^{n} k\ t^{n-i}\ F_{k,t,i}^{2} = \sum_{i=1}^{n} (\alpha+\beta)(-\alpha\beta)^{n-i}\left(\frac{\alpha^{i}-\beta^{i}}{\alpha-\beta}\right)^{2} \\
&= \sum_{i=1}^{n} \frac{(-1)^{n-i}\left(\alpha^{n+i+1}\beta^{n-i}-2\alpha^{n+1}\beta^{n}+\alpha^{n-i+1}\beta^{n+i}+\alpha^{n+i}\beta^{n-i+1}-2\alpha^{n}\beta^{n+1}+\alpha^{n-i}\beta^{n+i+1}\right)}{(\alpha-\beta)^{2}} \\
&= \frac{1}{(\alpha-\beta)^{2}}\left[\begin{array}{c}
+(\alpha^{n+2}\beta^{n-1}-2\alpha^{n+1}\beta^{n}+\alpha^{n}\beta^{n+1}+\alpha^{n+1}\beta^{n}-2\alpha^{n}\beta^{n+1}+\alpha^{n-1}\beta^{n+2}) \\
-\left(\alpha^{n+3}\beta^{n-2}-2\alpha^{n+1}\beta^{n}+\alpha^{n-1}\beta^{n+2}+\alpha^{n+2}\beta^{n-1}-2\alpha^{n}\beta^{n+1}+\alpha^{n-2}\beta^{n+3}\right) \\
+\left(\alpha^{n+4}\beta^{n-3}-2\alpha^{n+1}\beta^{n}+\alpha^{n-2}\beta^{n+3}+\alpha^{n+3}\beta^{n-2}-2\alpha^{n}\beta^{n+1}+\alpha^{n-3}\beta^{n+4}\right) \\
-\left(\alpha^{n+5}\beta^{n-4}-2\alpha^{n+1}\beta^{n}+\alpha^{n-3}\beta^{n+4}+\alpha^{n+4}\beta^{n-3}-2\alpha^{n}\beta^{n+1}+\alpha^{n-4}\beta^{n+5}\right) \\
. \\
. \\
. \\
+\left(\alpha^{2n-3}\beta^{4}-2\alpha^{n+1}\beta^{n}+\alpha^{5}\beta^{2n-4}+\alpha^{2n-4}\beta^{5}-2\alpha^{n}\beta^{n+1}+\alpha^{4}\beta^{2n-3}\right) \\
-\left(\alpha^{2n-2}\beta^{3}-2\alpha^{n+1}\beta^{n}+\alpha^{4}\beta^{2n-3}+\alpha^{2n-3}\beta^{4}-2\alpha^{n}\beta^{n+1}+\alpha^{3}\beta^{2n-2}\right) \\
+\left(\alpha^{2n-1}\beta^{2}-2\alpha^{n+1}\beta^{n}+\alpha^{3}\beta^{2n-2}+\alpha^{2n-2}\beta^{3}-2\alpha^{n}\beta^{n+1}+\alpha^{2}\beta^{2n-1}\right) \\
-\left(\alpha^{2n}\beta-2\alpha^{n+1}\beta^{n}+\alpha^{2}\beta^{2n-1}+\alpha^{2n-1}\beta^{2}-2\alpha^{n}\beta^{n+1}+\alpha\beta^{2n}\right) \\
+\left(\alpha^{2n+1}-2\alpha^{n+1}\beta^{n}+\alpha\beta^{2n}+\alpha^{2n}\beta-2\alpha^{n}\beta^{n+1}+\beta^{2n+1}\right)
\end{array}\right] \\
&= \frac{1}{(\alpha-\beta)^{2}}\left[\alpha^{2n+1}-\alpha^{n+1}\beta^{n}-\alpha^{n}\beta^{n+1}+\beta^{2n+1}\right] \\
&= F_{k,t,n}\ F_{k,t,n+1}.
\end{aligned}$$

The case that n is an even positive integer can be proved by a similar way. Consequently, we have proved the identity (22). $\boxed{QED}$

In order to determine a new annulus containing the zeros of a given polynomial, we use the identity (22). We give the following theorem.

Theorem 2.1. Let $P(z)=\sum\limits_{k=0}^{n} a_k z^k$ $(a_k \neq 0, 0 \leq k \leq n)$ be a non-constant complex polynomial of degree n. Then all the zeros of $P(z)$ lie in the annulus

$$D_{15}=\left\{z\in\mathbb{C} : r_1 \leq |z| \leq r_2\right\}, \tag{23}$$

where

$$r_1=\min_{1\leq i\leq n}\left\{\frac{kt^{n-i}F_{k,t,i}^{2}}{F_{k,t,n}F_{k,t,n+1}}\left|\frac{a_0}{a_i}\right|\right\}^{\frac{1}{i}} \tag{24}$$

and

$$r_2=\max_{1\leq i\leq n}\left\{\frac{F_{k,t,n}F_{k,t,n+1}}{kt^{n-i}F_{k,t,i}^{2}}\left|\frac{a_{n-i}}{a_n}\right|\right\}^{\frac{1}{i}}. \tag{25}$$

Proof. To prove the inequality $r_1 \leq |z|$ we use the equation (24). We have

$$r_1=\min_{1\leq i\leq n}\left\{\frac{kt^{n-i}F_{k,t,i}^{2}}{F_{k,t,n}F_{k,t,n+1}}\left|\frac{a_0}{a_i}\right|\right\}^{\frac{1}{i}},$$

$$r_1 \leq \left\{ \frac{kt^{n-i}F_{k,t,i}^2}{F_{k,t,n}F_{k,t,n+1}} \left| \frac{a_0}{a_i} \right| \right\}^{\frac{1}{i}}$$

and so

$$r_1^i \leq \left\{ \frac{kt^{n-i}F_{k,t,i}^2}{F_{k,t,n}F_{k,t,n+1}} \left| \frac{a_0}{a_i} \right| \right\}, \ i = 1,\ 2,\ ...,\ n. \tag{26}$$

Let us assume that $|z| < r_1$. In this case, if we use the inequalities $|x| - |y| \leq \big||x| - |y|\big| \leq |x + y|$, $|x + y| \leq |x| + |y|$ and $|xy| = |x|\,|y|$, respectively, we have

$$\begin{aligned} \big|P(z)\big| &= \left|\sum_{i=0}^{n} a_i z^i\right| \\ &\geq \left|a_0 + a_1 z + a_2 z^2 + ... + a_n z^n\right| \\ &\geq |a_0| - \left|a_1 z + a_2 z^2 + ... + a_n z^n\right| \\ &\geq |a_0| - \left(|a_1 z| + \left|a_2 z^2\right| + ... + |a_n z^n|\right) \\ &\geq |a_0| - \sum_{i=1}^{n} |a_i|\,|z|^i \\ &> |a_0| - \sum_{i=1}^{n} |a_i|\, r_1^i \\ &= |a_0| \left(1 - \sum_{i=1}^{n} \left|\frac{a_i}{a_0}\right| r_1^i\right). \end{aligned} \tag{27}$$

Substituting (26) into (27), we have

$$\begin{aligned} \big|P(z)\big| &> |a_0| \left(1 - \sum_{i=1}^{n} \left|\frac{a_i}{a_0}\right| r_1^i\right) \\ &\geq |a_0| \left(1 - \sum_{i=1}^{n} \left|\frac{a_i}{a_0}\right| \frac{kt^{n-i}F_{k,t,i}^2}{F_{k,t,n}F_{k,t,n+1}} \left|\frac{a_0}{a_i}\right|\right) \\ &\geq |a_0| \left(1 - \sum_{i=1}^{n} \frac{kt^{n-i}F_{k,t,i}^2}{F_{k,t,n}F_{k,t,n+1}}\right) = 0. \end{aligned}$$

Hence we have seen that there are no roots of $P(z)$ in $\big\{z \in \mathbb{C} : r_1 < |z|\big\}$.

To prove the second part of the theorem, we can write

$$\begin{aligned} \left|P(z)\right| &= \left|\sum_{i=0}^{n} a_i z^i\right| \\ &\geq |a_n z^n| - \sum_{i=1}^{n} |a_{n-i}|\,|z|^{n-i} \\ &= |a_n z^n| \left(1 - \sum_{i=1}^{n} \left|\frac{a_{n-i}}{a_n}\right| \frac{1}{|z|^i}\right). \end{aligned}$$

From the equation (25) it follows that

$$\left|\frac{a_{n-i}}{a_n}\right| \leq \frac{kt^{n-i}F_{k,t,i}^2}{F_{k,t,n}F_{k,t,n+1}} r_2^i,\ \ i = 1, 2, ..., n$$

and so by simple computations we find

$$\begin{aligned} \left|P(r_2)\right| &> |a_n r_2^n| \left(1 - \sum_{i=1}^{n} \left|\frac{a_{n-i}}{a_n}\right| \frac{1}{r_2^i}\right) \\ &\geq |a_n r_2^n| \left(1 - \sum_{i=1}^{n} \left(\frac{kt^{n-i}F_{k,t,i}^2}{F_{k,t,n}F_{k,t,n+1}} r_2^i\right) \frac{1}{r_2^i}\right) \\ &= |a_n|\, r_2^n \left[1 - \sum_{i=1}^{n} \left(\frac{kt^{n-i}F_{k,t,i}^2}{F_{k,t,n}F_{k,t,n+1}}\right)\right] = 0. \end{aligned}$$

This proves the second part of the theorem. $\boxed{QED}$

Remark 2.1. In Theorem 2.1, if we take $A_k = \frac{kt^{n-i}F_{k,t,i}^2}{F_{k,t,n}F_{k,t,n+1}}$ we obtain $\sum\limits_{k=1}^{n} A_k = 1$ and so Theorem 2.1 is a special case of Theorem 1.9. But, in the following examples, we see that our annulus is significantly more certain than the known annuli.

Now we give some examples to make a comparison among this new annulus and the known annuli given in Section 1.

Example 2.1. Let us consider the polynomial

$$P_1(z) = 5z^5 - 0.002z^4 - 0.03z^3 + 0.05z^2 - 0.01z + 100.$$

Using Theorem 1.5, Theorem 1.6 ($a = b = c = d = 1$ and $j = 2$), Theorem 1.7 ($j = 2, k = 1$ and $t = 2$), Theorem 1.8 ($a = b = \frac{1}{5}$,$c = \frac{2}{5}$,$u = v = \frac{1}{2}$ and $w = \frac{3}{8}$), Theorem 2.1 ($k = 2$ and $t = 1$), Corollary 1.3 and Corollary 1.4, we obtain the corresponding regions for the polynomial $P_1(z)$ (see Table 1).

Example 2.2. Let us consider the following polynomial

$$P_2(z) = 2z^5 + 0.007z^4 - 0.06z^3 + 0.02z^2 - 0.013z + 50.$$

Using Theorem 1.5, Theorem 1.6 ($a = b = 1$, $c = 2$, $d = 3$ and $j = 2$), Theorem 1.7 ($j = 2, k = 2$ and $t = 1$), Theorem 1.8 ($a = b = \frac{1}{4}$,$c = \frac{2}{3}$,$u = v = \frac{1}{3}$ and $w = \frac{3}{7}$) Theorem 2.1 ($k = 1$ and $t = 1$), Corollary 1.3 and Corollary 1.4, we obtain the corresponding regions for the polynomial $P_2(z)$ (see Table 1).

	The polynomial $P_1(z)$			The polynomial $P_2(z)$		
The annulus	r_1	r_2	Area of the annulus	r_1	r_2	Area of the annulus
D_6	1.2914	2.5665	15.4543	1.3503	2.6836	16.8971
D_7	1.1233	2.1385	10.4029	1.4640	2.4753	12.5155
D_9	1.1448	2.8950	22.2125	1.6235	2.2321	7.3722
D_{11}	1.2067	4.2328	51.7136	1.0445	4.5089	60
D_{13}	1.5328	2.1623	7.3078	1.6027	2.2610	7.99
D_{14}	1.4614	2.2673	9.4489	1.5281	2.3714	10.3059
D_{15}	**1.7533**	**1.8903**	**1.5679**	**1.7328**	**2.0912**	**4.3059**

Table 1. The regions for the polynomials $P_1(z)$ and $P_2(z)$.

From Table 1, it can be seen that Theorem 2.1 gives better bounds in terms of r_1 and r_2.

Finally, we give some comparisons for the new annuli obtained in Theorem 2.1 by different choices of the parameters k and t using the following examples.

Example 2.3. Let us consider the following polynomial

$$P_3(z) = z^6 + 0.09z^5 - 0.32z^4 - 0.13z^3 + 0.6z^2 + 0.4z + 10.$$

We fix the parameter k. We obtain some comparisons for the various values of the parameter t (see Table 2).

Example 2.4. Let us consider the following polynomial

$$P_4(z) = 100z^7 + 2z^6 + 9z^5 - 0.32z^4 - 0.13z^3 + 0.6z^2 + 0.4z + 40.$$

We fix the parameter t. We obtain the following comparison table for some values of the parameter k (see Table 2).

The polynomial $P_3(z)$					The polynomial $P_4(z)$				
k	t	r_1	r_2	Area of the annulus D_{15}	k	t	r_1	r_2	Area of annulus D_{15}
1	1	0.2403	9	254	**1**	**7**	**0.7561**	**2.62**	**19**
1	2	0.5434	4.2497	55	2	7	0.71	3.41	34
1	3	0.5898	3.91	46	3	7	0.18	11	380
1	**4**	**0.6022**	**3.8348**	**45**					

Table 2. The bounds of the region D_{15} for the polynomials $P_3(z)$ and $P_4(z)$.

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